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by

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SPACES WITH sn-NETWORK g-FUNCTIONS

TRAN VAN AN AND LUONG QUOC TUYEN

Abstract. In this paper, we introduce the concepts of an sn-network g-function, an sn-developable space, and a strongly sn-developable space as generalizations of a "weak base g-function," a "g-developable space," and a "strongly g-developable space," respectively. Then we give some characterizations of sn-symmetric spaces, Cauchy sn-symmetric spaces, sn-metrizable spaces, and Cauchy sn-symmetric spaces with σ-(P)-property sn-networks.

1. Introduction

In [11], Kyung Bai Lee introduced CWC-maps and g-developable spaces and gave some characterizations of g-developable spaces. Later, Zhi Min Gao [4] introduced the notion of weak base g-functions by means of weak bases to study the metrizability of a topological space. In 2006, Y. Tanaka and Y. Ge [18] introduced strongly g-developable spaces and gave some characterizations of g-developable spaces.

In this paper, we introduce the concepts of an sn-network g-function, an sn-developable space, and a strongly sn-developable space as generalizations of a "weak base g-function," a "g-developable space," and a "strongly g-developable space," respectively. Then we give some characterizations of sn-symmetric spaces, Cauchy sn-symmetric spaces, sn-metrizable spaces, and Cauchy sn-symmetric spaces with σ-(P)-property sn-networks.

Throughout this paper, all spaces are assumed to be T_1 and regular and $\mathbb{N}$ denotes the set of all natural numbers. Given two families $\mathcal{P}$ and

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$\mathcal{Q}$ of subsets of X, we denote $\mathcal{P} \bigwedge \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$ and $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$. For a sequence $\{x_n\}$ converging to x and $P \subset X$, we say that $\{x_n\}$ is *eventually* in P if $\{x\} \bigcup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$ and $\{x_n\}$ is *frequently* in P if some subsequence of $\{x_n\}$ is eventually in P.

Definition 1.1. For a cover $\mathcal{P}$ of a space X, let (P) be one of the following properties: point-finite, compact-finite, locally finite, point-countable, compact-countable, or locally countable. We say that $\mathcal{P}$ has the σ-(P)-*property* if $\mathcal{P}$ can be expressed as $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ where each $\mathcal{P}_n$ has the (P)-property.

Definition 1.2. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X such that for every $x \in X$, $\mathcal{P}_x$ is a network at x, and if $P_1, P_2 \in \mathcal{P}_x$, then $P \subset P_1 \cap P_2$ for some $P \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is a *weak base* [3], if for $G \subset X$, G is open in X if and only if for every $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$; $\mathcal{P}_x$ is said to be a *weak neighborhood base* at x.

(2) $\mathcal{P}$ is an *sn-network* [12], if each element of $\mathcal{P}_x$ is a sequential neighborhood of x for all $x \in X$; $\mathcal{P}_x$ is said to be an *sn-network* at x.

(3) X is *sn-first countable* [5] (*g-first countable, respectively* [17]), if there is a countable sn-network (a countable weak neighborhood base, respectively) at x in X for all $x \in X$.

(4) X is *sn-metrizable* [5] (*g-metrizable, respectively* [17]), if X has a σ-locally finite sn-network (weak base, respectively).

Definition 1.3 ([4])**.** A function $g : \mathbb{N} \times X \to \mathcal{P}(X)$ is a *weak base g-function* if it satisfies the following conditions:

(1) $x \in g(n, x)$ for all $x \in X$ and $n \in \mathbb{N}$;
(2) $g(n+1, x) \subset g(n, x)$ for all $n \in \mathbb{N}$;
(3) $\{g(n, x) : n \in \mathbb{N}, x \in X\}$ is a weak base for X.

Note that weak base g-functions were called CWC-maps and CWBC-maps in [11] and [16], respectively.

Definition 1.4. A function $g : \mathbb{N} \times X \to \mathcal{P}(X)$ is an *sn-network g-function* if it satisfies the following conditions:

(1) $x \in g(n, x)$ for all $x \in X$ and $n \in \mathbb{N}$;
(2) $g(n+1, x) \subset g(n, x)$ for all $n \in \mathbb{N}$;
(3) $\{g(n, x) : n \in \mathbb{N}, x \in X\}$ is an sn-network for X.

Remark 1.5. (1) Note that a weak base g-function is an sn-network g-function.

(2) If X is sequential, then g is an sn-network g-function if and only if g is a weak base g-function.

Let g be an sn-network g-function on X, let $\{x_n\}$ and $\{y_n\}$ be two sequences in X, and let $x \in X$. Consider the following conditions imposed on an sn-network g-function g for X.

(E) If $x_n \in g(n, x)$ for all $n \in \mathbb{N}$, then $x_n \to x$.
(F) If $x \in g(n, x_n)$ for all $n \in \mathbb{N}$, then $x_n \to x$.
(WF) If $x \in g(n, x_n)$ for all $n \in \mathbb{N}$, then $x_{n_k} \to x$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$.
(G) If $x, x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then $x_n \to x$.
(GP) Each $\{g(n, x) : x \in X\}$ has the (P)-property and if $x, x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then $x_n \to x$.
(H) If $x_n \to x$ and $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then $y_n \to x$.
(HLF) If $x_n \to x$ and $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then $y_n \to x$; and for each $x \in X$, there exists $U \in \tau$ such that $|\{U \cap g(n, y) : y \in X\}| < \omega$.
(GLF) Each $\{g(n, x) : n \in \mathbb{N}\}$ is locally finite and g satisfies (G).
(GPF) Each $\{g(n, x) : n \in \mathbb{N}\}$ is point-finite and g satisfies (G).

Definition 1.6. Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a sequence of covers of a space X.

(1) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *σ-strong network* for X [10] if $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for all $n \in \mathbb{N}$ and $\{\mathrm{St}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x for all $x \in X$.
(2) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *σ-(P)-strong network* for X if it is a σ-strong network and each $\mathcal{P}_n$ has the (P)-property.
(3) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *weak-development* [13] if $\{\mathrm{St}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a weak base at x for all $x \in X$.
(4) $\mathcal{P}$ is a *σ-(P)-strong network consisting of sn-covers* (*cs-covers*, *cfp-covers*, *cs^*-covers*, respectively) if each $\mathcal{P}_n$ is an sn-cover (cs-cover, cfp-cover, cs^*-cover, respectively).

Definition 1.7. (1) X is a *g-developable* space [11] if X has a weak base g-function satisfying (G).
(2) X is an *sn-developable* space if X has an sn-network g-function satisfying (G).
(3) X is a *strongly g-developable* space [18] if X is a sequential space with a σ-locally finite strong network consisting of cs-covers.
(4) X is a *strongly sn-developable* space if X has a σ-locally finite strong network consisting of cs-covers.

Remark 1.8. (1) Every g-developable space is an sn-developable space.

(2) Every strongly g-developable space is a strongly sn-developable space.
(3) If X is sequential, then
  (a) X is sn-developable if and only if it is g-developable;
  (b) X is strongly sn-developable if and only if it is strongly g-developable.

Definition 1.9 ([13])**.** Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is *uniform* if for each $x \in X$ and $\mathcal{G}$ is an infinite subfamily of $(\mathcal{P})_x$, then $\mathcal{G}$ is a network at x in X.
(2) $\mathcal{P}$ is a *uniform sn-network* (a *uniform weak base*, respectively) if $\mathcal{P}$ is both uniform and sn-network (weak base, respectively).
(3) $\mathcal{P}$ is *point-regular* if, for every $x \in U \in \tau$, the set $\{P \in (\mathcal{P})_x : P \not\subset U\}$ is finite.
(4) $\mathcal{P}$ is a *point-regular sn-network* (a *point-regular weak base*, respectively) if $\mathcal{P}$ is both point-regular and sn-network (weak base, respectively).

Definition 1.10 ([8])**.** Let d be a d-function on a space X.

(1) For each $x \in X$ and $n \in \mathbb{N}$, let $S_n(x) = \{y \in X : d(x,y) < 1/n\}$.
(2) For every $P \subset X$, put $d(P) = \sup\{d(x,y) : x, y \in P\}$.
(3) X is *symmetric* if $\{S_n(x) : n \in \mathbb{N}\}$ is a weak neighborhood base at x for all $x \in X$.
(4) X is *sn-symmetric* if $\{S_n(x) : n \in \mathbb{N}\}$ is an sn-network at x for all $x \in X$.

Definition 1.11. (1) A symmetric space (X,d) is called a *Cauchy symmetric* space ([19]) if every convergent sequence is d-Cauchy.
(2) An sn-symmetric space (X,d) is called a *Cauchy sn-symmetric* space [2] if every convergent sequence is d-Cauchy.

Remark 1.12. If X is a sequential space, then

(1) X is a symmetric space if and only if it is an sn-symmetric space;
(2) X is a Cauchy symmetric space if and only if it is a Cauchy sn-symmetric space.

Notation 1.13. Let $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-strong network for a space X. For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\alpha : \alpha \in \Lambda_n\}$ and endow Λ_n with the discrete topology. Then

$$M = \Big\{\alpha = (\alpha_n) \in \prod_{n\in\mathbb{N}} \Lambda_n : \{P_{\alpha_n}\} \text{ forms a network at some point } x_\alpha \in X\Big\}$$

is a metric space and the point x_α is unique in X for every $\alpha \in M$. Define $f : M \to X$ by $f(\alpha) = x_\alpha$. Let us call $(f, M, X, \mathcal{P}_n)$ a *Ponomarev system* following [15].

For some undefined or related concepts, we refer the reader to [10] and [13].

2. The Main Results

Theorem 2.1. *The following are equivalent for a space X:*

(1) *X is an sn-first countable space;*
(2) *there exists an sn-network g-function on X satisfying* (E)*;*
(3) *there exists an sn-network g-function on X.*

Proof. (1) $\Longrightarrow$ (2). Let $\mathcal{G} = \bigcup\{\mathcal{G}_x : x \in X\}$ be an sn-network for X, where each $\mathcal{G}_x = \{P_{n,x} : n \in \mathbb{N}\}$ is a countable sn-network at x. For each $n \in \mathbb{N}$ and $x \in X$, let

$$g(n,x) = \bigcap\{P_{i,x} : 1 \leq i \leq n\}.$$

Then g is an sn-network g-function on X. Assume that $\{x_n\}$ is a sequence in X and $x \in X$ such that $x_n \in g(n,x)$ for all $n \in \mathbb{N}$. Since $g(n,x)$ is a decreasing network at x, it implies that $x_n \to x$. Thus, g satisfies (E).

(2) $\Longrightarrow$ (3). Obvious.

(3) $\Longrightarrow$ (1). For each $x \in X$, put $\mathcal{G}_x = \{g(n,x) : n \in \mathbb{N}\}$. Then each $\mathcal{G}_x$ is a countable sn-network at x. Therefore, X is sn-first countable. □

Corollary 2.2. *The following are equivalent for a space X:*

(1) *X is a g-first countable space;*
(2) *there exists a weak base g-function on X satisfying* (E)*;*
(3) *there exists a weak base g-function on X.*

Theorem 2.3. *The following are equivalent for a space X:*

(1) *X is an sn-symmetric space;*
(2) *X has a σ-strong network $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ such that $\{St(x,\mathcal{G}_n) : n \in \mathbb{N}\}$ is an sn-network at x for all $x \in X$;*
(3) *there exists an sn-network g-function on X satisfying* (F)*;*
(4) *there exists an sn-network g-function on X satisfying* (WF).

Proof. (1) $\Longrightarrow$ (2). For each $n \in \mathbb{N}$, let $\mathcal{G}_n = \{P \subset X : d(P) < 1/n\}$. Then $\mathtt{St}(x,\mathcal{G}_n) = S_n(x)$ for all $n \in \mathbb{N}$ and $x \in X$. Therefore, $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a σ-strong network and $\{\mathtt{St}(x,\mathcal{G}_n) : n \in \mathbb{N}\}$ is an sn-network at x for all $x \in X$.

(2) $\Longrightarrow$ (3). Assume that (2) holds. For each $x \in X$ and $n \in \mathbb{N}$, let $g(n,x) = \mathtt{St}(x,\mathcal{G}_n)$. Then g is an sn-network g-function on X. Now, let $\{x_n\}$ be a sequence in X and $x \in X$ such that $x \in g(n,x_n)$ for all $n \in \mathbb{N}$. Then $x_n \in \mathtt{St}(x,\mathcal{G}_n)$ for all $n \in \mathbb{N}$. Thus, $x_n \to x$.

(3) $\Longrightarrow$ (4). Obvious.

(4) $\Longrightarrow$ (1). Let g be an sn-network g-function on X satisfying (WF). For each $x, y \in X$ with $x \neq y$, put $\delta(x,y) = \min\{n : x \notin g(n,y), y \notin g(n,x)\}$. Now, for each $x, y \in X$, denote

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1/\delta(x,y) & \text{if } x \neq y. \end{cases}$$

Then d is a d-function on X, and

(a) for each $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $S_{n_0}(x) \subset g(n,x)$. If not, there exists $i_0 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, there exists $x_n \in S_n(x) - g(i_0,x)$. Since $x_n \in S_n(x)$, $\delta(x, x_n) > n$. Thus, $x \in g(n, x_n)$ or $x_n \in g(n,x)$ for all $n \in \mathbb{N}$. This follows that $x_{n_k} \to x$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. On the other hand, since each $g(i_0, x)$ is a sequential neighborhood at x, $\{x_{n_k}\}$ is eventually in $g(i_0,x)$. This is a contradiction to $x_n \notin g(i_0,x)$ for all $n \in \mathbb{N}$.

(b) for each $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $g(n_0, x) \subset S_n(x)$. If not, there exists $i_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, there exists $x_n \in g(n,x) - S_{i_0}(x)$. Since $\{g(n,x) : n \in \mathbb{N}\}$ is a decreasing sn-network at x, it implies that $x_n \to x$. Thus, $\{x_n\}$ is eventually in $g(i_0, x)$. Pick $n \in \mathbb{N}$ such that $x_n \in g(i_0,x)$, then $\delta(x_n, x) > i_0$, so $x_n \in S_{i_0}(x)$. This is a contradiction to $x_n \notin S_{i_0}(x)$ for all $n \in \mathbb{N}$.

Then (a) and (b) imply that $\{S_n(x) : n \in \mathbb{N}\}$ is an sn-network at x for all $x \in X$, and X is sn-symmetric. □

Corollary 2.4. *The following are equivalent for a space X:*

(1) *X is a symmetric space;*
(2) *X has a weak-development;*
(3) *there exists a weak base g-function on X satisfying* (F)*;*
(4) *there exists a weak base g-function on X satisfying* (WF).

Theorem 2.5. *The following are equivalent for a space X:*

(1) *X is an sn-developable space;*
(2) *X is a Cauchy sn-symmetric space;*
(3) *X has a σ-strong network consisting of cs-covers;*
(4) *X has a σ-strong network consisting of sn-covers;*
(5) *X is a 1-sequence-covering and π-image of a metric space;*
(6) *X is a sequence-covering and π-image of a metric space.*

Proof. (1) $\Longrightarrow$ (2). Let g be an sn-network g-function satisfying (G). Then X is Cauchy sn-symmetric. In fact, for each $n \in \mathbb{N}$, put $\mathcal{G}_n = \{g(n,x) : x \in X\}$, and put $\delta(x,y) = \min\{n : x \notin \mathrm{St}(y, \mathcal{G}_n)\}$ for each

$x, y \in X$ with $x \neq y$. Next, for each $x, y \in X$, we denote

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1/\delta(x,y) & \text{if } x \neq y. \end{cases}$$

Then d is a d-function on X and $S_n(x) = \mathtt{St}(x, \mathcal{G}_n)$. Furthermore, we have

(a) $\{S_n(x) : n \in \mathbb{N}\}$ is a network at x for all $x \in X$. If not, there exist $x \in U \in \tau$ such that $S_n(x) \not\subset U$ for all $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, there exists $x_n \in S_n(x) - U$. Since $S_n(x) = \mathtt{St}(x, \mathcal{G}_n)$ for all $n \in \mathbb{N}$, it implies that for each $n \in \mathbb{N}$, there exists $y_n \in X$ such that $x, x_n \in g(n, y_n)$. By condition $(\mathtt{G})$, $x_n \to x$, implying that $\{x_n\}$ is eventually in U. This is a contradiction.

(b) Let $m, n \in \mathbb{N}$; we put $k = \max\{m, n\}$. Since $\mathcal{G}_{i+1}$ refines $\mathcal{G}_i$ and $S_i(x) = \mathtt{St}(x, \mathcal{G}_i)$ for all $i \in \mathbb{N}$, it implies that $S_k(x) \subset S_m(x) \cap S_n(x)$.

(c) Since $g(n, x)$ is a sequential neighborhood at x for all $x \in X$, and $g(n, x) \subset \mathtt{St}(x, \mathcal{G}_n) = S_n(x)$, it implies that each $S_n(x)$ is sequential neighborhood at x.

Then X is sn-symmetric. Now, let $\{x_i\}$ be a sequence in X, $x_i \to x$ and $\varepsilon > 0$. Take $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Since $g(n, x)$ is a sequential neighborhood at x, $\{x\} \bigcup \{x_i : i \geq m\} \subset g(n, x)$ for some $m \in \mathbb{N}$. Hence, $x_i \in \mathtt{St}(x_j, \mathcal{G}_n)$ for all $i, j \geq m$. This implies that $d(x_i, x_j) < 1/n < \varepsilon$ for all $i, j \geq m$. Therefore, X is Cauchy sn-symmetric.

(2) $\Longrightarrow$ (3). For each $n \in \mathbb{N}$, denote $\mathcal{G}_n = \{P \subset X : d(P) < 1/n\}$. Then $\mathtt{St}(x, \mathcal{G}_n) = S_n(x)$. Since X is Cauchy sn-symmetric, each $\mathcal{G}_n$ is a cs-cover. Therefore, $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a σ-strong network consisting of cs-covers.

(3) $\Longrightarrow$ (4). Let $\bigcup\{\mathcal{P}_i : i \in \mathbb{N}\}$ be a σ-strong network consisting of cs-covers. For each $x, y \in X$ with $x \neq y$, put $\delta(x, y) = \min\{n : y \notin \mathtt{St}(x, \mathcal{P}_n)\}$, and denote

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1/\delta(x,y) & \text{if } x \neq y. \end{cases}$$

Then d is a d-function on X, and $S_n(x) = \mathtt{St}(x, \mathcal{P}_n)$ for all $n \in \mathbb{N}$. We claim that for each $x \in X$ and $\varepsilon > 0$, there exists $k = k(x, \varepsilon) \in \mathbb{N}$ such that $d(x, y) < 1/k$ and $d(x, z) < 1/k$ imply $d(y, z) < \varepsilon$. Otherwise, there exist $x_0 \in X$, $\varepsilon_0 > 0$, and two sequences $\{y_n\}$ and $\{z_n\}$ in X such that $d(y_n, z_n) \geq \varepsilon_0$ whenever $d(x_0, y_n) < 1/n$ and $d(x_0, z_n) < 1/n$. Since $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ-strong network, $\{y_n\}$ and $\{z_n\}$ converge to x_0. Now, we choose $i \in \mathbb{N}$ such that $1/i < \varepsilon_0$. Since $\mathcal{P}_i$ is a cs-cover for X, $\{y_m, z_m\} \subset P$ for some $m \in \mathbb{N}$ and $P \in \mathcal{P}_i$. Thus, $y_m \in \mathtt{St}(z_m, \mathcal{P}_i)$, implying that $d(y_m, z_m) = 1/\delta(y_m, z_m) < 1/i < \varepsilon_0$. This is a contradiction.

Now, for each $x \in X$ and $n \in \mathbb{N}$, denote $k_{x,n} = k(x, 1/n)$ such that $d(y,z) < 1/n$ whenever $d(x,y) < 1/k_{x,n}$ and $d(x,z) < 1/k_{x,n}$. Without loss of generality, we can assume that $k_{x,n+1} > k_{x,n}$ for all $n \in \mathbb{N}$. Put $\mathcal{G}_n = \{S_{k_{x,n}}(x) : x \in X\}$ for every $n \in \mathbb{N}$. It is obvious that each $\mathcal{G}_n$ is an sn-cover and $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ for all $n \in \mathbb{N}$. Furthermore, $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a σ-strong network. If not, there exist $x \in U \in \tau$ such that $\mathtt{St}(x, \mathcal{G}_n) \not\subset U$ for all $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, there exists $x_n \in \mathtt{St}(x, \mathcal{G}_n) - U$. It follows that there exists $y_n \in X$ such that $x \in S_{k_{y_n,n}}(y_n)$ and $x_n \in S_{k_{y_n,n}}(y_n) - U$ for every $n \in \mathbb{N}$. Then $d(x, y_n) < 1/k_{y_n,n}$ and $d(x_n, y_n) < 1/k_{y_n,n}$. This implies that $d(x, x_n) < 1/n$. Thus, $x_n \to x$, a contradiction. Hence, $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a σ-strong network consisting of sn-covers.

(4) $\Longrightarrow$ (1). Let $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ be a σ-strong network consisting of sn-covers. Then, for each $n \in \mathbb{N}$ and $x \in X$, there is $i(n,x) \in \mathbb{N}$ such that $S_{i(n,x)}(x) \subset P$ for some $P \in \mathcal{G}_n$, and $i(n,x) < i(n+1,x)$. Now, for each $n \in \mathbb{N}$ and $x \in X$, we put $g(n,x) = S_{i(n,x)}(x)$. Then $g : \mathbb{N} \times X \to \mathcal{P}(X)$ is an sn-network g-function on X. Next, let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x \in X$ satisfying that $x, x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$. Then $x_n \to x$. In fact, let $x \in U \in \tau$. Since $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a σ-strong network, there exists $n_0 \in \mathbb{N}$ such that $\mathtt{St}(x, \mathcal{G}_n) \subset U$ for all $n \geq n_0$. Since $x \in g(n, y_n)$ for all $n \in \mathbb{N}$, it implies that $g(n, y_n) \subset U$ for all $n \geq n_0$. Thus, $x_n \in U$ for all $n \geq n_0$. Hence, $x_n \to x$.

(4) $\Longrightarrow$ (5). By [18, Lemma 2.2] and [9, Theorem 3.10].

(5) $\Longrightarrow$ (6). Obvious.

(6) $\Longrightarrow$ (3). By [10, Proposition 16(3b)]. $\square$

Corollary 2.6. *The following are equivalent for a space X:*

(1) *X is a g-developable space;*
(2) *X is a Cauchy symmetric space;*
(3) *X has a weak-development consisting of cs-covers;*
(4) *X has a weak-development consisting of sn-covers;*
(5) *X is a weak-open and π-image of a metric space.*

Theorem 2.7. *The following are equivalent for a space X:*

(1) *X is an sn-metrizable space;*
(2) *there exists an sn-network g-function on X satisfying* (HLF)*;*
(3) *X has a σ-locally finite strong network consisting of cfp-covers;*
(4) *X has a σ-locally finite strong network consisting of cs^*-covers;*

(5) *X is a compact-covering compact and mssc-image of a metric space;*
(6) *X is a sequentially-quotient π and mssc-image of a metric space;*
(7) *X is a 1-sequence-covering and mssc-image of a metric space.*

Proof. (1) $\Longrightarrow$ (2). Let $\mathcal{G} = \bigcup\{\mathcal{G}_n : n \in \mathbb{N}\} = \bigcup\{\mathcal{G}_x : x \in X\}$ be an sn-network, where each $\mathcal{G}_n$ is locally finite and each $\mathcal{G}_x$ is an sn-network at x. Without loss of generality, we can assume that each $\mathcal{G}_n$ is a discrete collection of closed subsets of X (see [5]). For each $n \in \mathbb{N}$ and $x \in X$, we put

$$h(n,x) = \begin{cases} P \in \mathcal{G}_n & \text{if } \mathcal{G}_n \cap \mathcal{G}_x \neq \emptyset, \\ X - \bigcup\{P \in \mathcal{G}_n : x \notin P\} & \text{if } \mathcal{G}_n \cap \mathcal{G}_x = \emptyset. \end{cases}$$

For each $x \in X$ and $n \in \mathbb{N}$, there exists $U \in \tau$ such that $x \in U$ and U meets at most only an element of $\mathcal{G}_n$. Since

$$\{U \cap h(n,x) : x \in X\} \subset \{U\} \cup \{U - P : P \in \mathcal{G}_n\} \cup \{U \cap P : P \in \mathcal{G}_n, U \cap P \neq \emptyset\},$$

it implies that $|\{U \cap h(n,x) : x \in X\}| < \omega$ for all $n \in \mathbb{N}$. Next, we shall show that every $h(n,x)$ is a sequential neighborhood at x. In fact, let $\{x_n\}$ be a sequence in X where $x_n \to x$. If $\mathcal{G}_n \cap \mathcal{G}_x \neq \emptyset$, then $h(n,x) = P \in \mathcal{G}_n \cap \mathcal{G}_x$, and $\{x_n\}$ is eventually in $h(n,x)$. If $\mathcal{G}_n \cap \mathcal{G}_x = \emptyset$, then $h(n,x) = X - \bigcup\{P \in \mathcal{G}_n : x \notin P\}$. Since $\mathcal{G}_n$ is discrete, $\bigcup\{P \in \mathcal{G}_n : x \notin P\}$ is closed. This implies that $h(n,x)$ is open. Therefore, each $h(n,x)$ is a sequential neighborhood at x.

Now, we put $g(n,x) = \bigcap\{h(k,x) : 1 \leq k \leq n\}$ for each $n \in \mathbb{N}$ and $x \in X$. It is easy to see that $g : \mathbb{N} \times X \to \mathcal{P}(X)$ is an sn-network g-function on X, and for each $x \in X$, there exists $U \in \tau$ such that $|\{U \cap g(n,x) : x \in X\}| < \omega$ for all $n \in \mathbb{N}$.

Next, let $\{x_i\}$ and $\{y_i\}$ be two sequences in X such that $x_i \to x \in X$, $x_i \in g(i, y_i)$ for all $i \in \mathbb{N}$, and $x \in V \in \tau$. Then there is $n \in \mathbb{N}$ such that $P \subset V$ for some $P \in \mathcal{G}_n \cap \mathcal{G}_x$, and $\{x\} \bigcup \{x_i : i \geq m\} \subset P$ for some $m \in \mathbb{N}$. Since $x_i \in g(i, y_i) \cap P$ for all $i \geq m$, $y_i \in P$ for all $i \geq m$. This implies that $\{y_n\}$ is eventually in P. Therefore, $y_n \to x$.

Then g is an sn-network g-function on X satisfying (HLF).

(2) $\Longrightarrow$ (3). Let g be an sn-network g-function on X satisfying (HLF). For each $n \in \mathbb{N}$ and $x \in X$, let

$$h(n,x) = \bigcap\{g(n,y) : x \in g(n,y)\} - \bigcup\{g(n,y) : x \notin g(n,y)\}.$$

Then $x \in h(n,x) \subset g(n,x)$. Put

$$\mathcal{H}_n = \{h(n,x) : x \in X\} \quad \text{and} \quad \mathcal{G}_n = \{\overline{h(n,x)} : x \in X\};$$

we have

(a) if $y \in h(n,x)$, then $x \in h(n,y)$. In fact, since $y \in h(n,x)$, it implies that $y \in g(n,z)$ if $x \in g(n,z)$ and $y \notin g(n,z)$ if $x \notin g(n,z)$. This follows that $x \in g(n,z)$ if and only if $y \in g(n,z)$. Therefore, $x \in h(n,y)$.

(b) $\mathcal{G}_n$ is locally finite. Let $x \in X$; then there exists $U \in \tau$ such that $|\{U \cap g(n,y) : y \in X\}| < \omega$. It implies that $|\{U \cap h(n,y) : y \in X\}| < \omega$. Firstly, we prove that for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is a partition of X. Indeed,

Case 1. if $\{x,y\} \subset g(n,z)$ for all $z \in X$, then

$$h(n,x) = h(n,y) = \bigcap\{g(n,z) : \{x,y\} \subset g(n,z)\};$$

Case 2. if there exists $z \in X$ such that $x \in g(n,z)$ and $y \notin g(n,z)$, then $h(n,x) \subset g(n,z)$ and $h(n,y) \cap g(n,z) = \emptyset$. Thus, $h(n,x) \cap h(n,y) = \emptyset$;

Case 3. if there exists $z \in X$ such that $x \notin g(n,z)$ and $y \in g(n,z)$, then $h(n,x) \cap g(n,z) = \emptyset$ and $h(n,y) \subset g(n,z)$. Thus, $h(n,x) \cap h(n,y) = \emptyset$.

Then $h(n,x) = h(n,y)$ or $h(n,x) \cap h(n,y) = \emptyset$ for all $x, y \in X$. Therefore, $\mathcal{H}_n$ is a partition of X.

Next, since each $\mathcal{H}_n$ is a partition of X, U meets only finitely many members $h(n,y)$. Thus, each $\mathcal{H}_n$ is locally finite. Therefore, each $\mathcal{G}_n$ is locally finite.

(c) $\{\mathtt{St}(x,\mathcal{G}_n) : n \in \mathbb{N}\}$ is a network at x for all $x \in X$. Let $x \in U \in \tau$; since X is regular, there exists $V \in \tau$ such that $x \in \overline{V} \subset U$. Then there exists $n_0 \in \mathbb{N}$ such that $\mathtt{St}(x,\mathcal{H}_{n_0}) \subset V$. If not, for each $n \in \mathbb{N}$, there exists $y_n \in \mathtt{St}(x,\mathcal{H}_n) - V$. Then, for each $n \in \mathbb{N}$, there exists z_n such that $x, y_n \in h(n,z_n)$. By (a), $z_n \in h(n,x) \subset g(n,x)$ for all $n \in \mathbb{N}$. Since $\{g(n,x) : n \in \mathbb{N}\}$ is a decreasing network at x, $z_n \to x$. On the other hand, since $y_n \in h(n,z_n)$, it follows from (a) that $z_n \in h(n,y_n) \subset g(n,y_n)$. By property (HLF) of g, it implies that $y_n \to x$. This is a contradiction. Thus, there exists $n_0 \in \mathbb{N}$ such that $\mathtt{St}(x,\mathcal{H}_{n_0}) \subset V$. From (b) this implies that $\mathtt{St}(x,\mathcal{G}_{n_0}) \subset U$.

Finally, for each $n \in \mathbb{N}$, put $\mathcal{Q}_n = \bigwedge\{\mathcal{G}_i : i \leq n\}$. Then, since each $\mathcal{G}_n$ is a locally finite closed cover, it follows that $\bigcup\{\mathcal{Q}_n : n \in \mathbb{N}\}$ is a σ-locally finite network consisting of cfp-covers.

(3) $\Longrightarrow$ (4). Obvious.

(4) $\Longrightarrow$ (1). Assume that (4) holds. Then X is an sn-first countable and $\aleph$-space. Therefore, X is sn-metrizable.

(3) $\Longrightarrow$ (5). By [18, Lemma 2.2].

(5) $\Longrightarrow$ (6). Obvious.

(6) $\Longrightarrow$ (1). By [6, Lemma 3.1] and [7, Theorem 5].

(1) $\Longrightarrow$ (7). If (1) holds, then X is a sequence-covering and $mssc$-image of a metric space by [7, Theorem 5]. Since X is sn-first countable, it follows from [1, Proposition 2.2] that X is a 1-sequence-covering and $mssc$-image of a metric space.

(7) $\Longrightarrow$ (1). Assume that (7) holds. Then X is an sn-first countable space. Furthermore, it follows from [7, Theorem 5] that X is an $\aleph$-space. Therefore, X is an sn-metrizable space. □

Corollary 2.8. *The following are equivalent for a space X:*

1. *X is a g-metrizable space;*
2. *there exists a weak base g-function on X satisfying* (HLF)*;*
3. *X is a weak-development consisting of locally finite cfp-covers;*
4. *X is a weak-development consisting of locally finite cs^*-covers;*
5. *X is a compact-covering quotient compact and mssc-image of a metric space;*
6. *X is a quotient π and mssc-image of a metric space;*
7. *X is a weak-open and mssc-image of a metric space.*

Theorem 2.9. *The following are equivalent for a space X:*

1. *X is a Cauchy sn-symmetric space with a σ-(P)-property sn-network;*
2. *X has a σ-(P)-strong network consisting of cs-covers;*
3. *X has a σ-(P)-strong network consisting of sn-covers;*
4. *there exists an sn-network g-function on X satisfying* (GP).

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3). By [2, Theorem 2.3].

(3) $\Longrightarrow$ (4). Let $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ be a σ-(P)-strong network consisting of sn-covers. Then, for each $n \in \mathbb{N}$ and $x \in X$, there is $i(n,x) \in \mathbb{N}$ such that $S_{i(n,x)}(x) \subset Q$ for some $Q \in \mathcal{G}_n$, and $i(n,x) < i(n+1,x)$. Now, we put $g(n,x) = S_{i(n,x)}(x)$ for every $n \in \mathbb{N}$ and $x \in X$. Then $g : \mathbb{N} \times X \to \mathcal{P}(X)$ is an sn-network g-function on X and each $\{g(n,x) : x \in X\}$ has (P)-property. Now, let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x \in X$ such that $x, x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$. Then $x_n \to x$. In fact, let $x \in U \in \tau$. Since $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a σ-strong network, there exists $n_0 \in \mathbb{N}$ such that $\mathtt{St}(x, \mathcal{G}_n) \subset U$ for all $n \geq n_0$. Since $x \in g(n, y_n)$ for all $n \in \mathbb{N}$, it implies that $g(n, y_n) \subset U$ for all $n \geq n_0$. Thus, $x_n \in U$ for all $n \geq n_0$, and $x_n \to x$. Therefore, g satisfies (GP).

(4) $\Longrightarrow$ (1). Let g be an sn-network g-function satisfying (GP). For each $n \in \mathbb{N}$, put $\mathcal{G}_n = \{g(n,x) : x \in X\}$. Then $\mathcal{G} = \bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is an sn-network with σ-(P)-property. Now, for each $x, y \in X$ with $x \neq y$, put

$\delta(x,y)=\min\{n : x \notin \mathtt{St}(y,\mathcal{G}_n)\}$, and denote

$$d(x,y)=\begin{cases}0 & \text{if } x=y,\\ 1/\delta(x,y) & \text{if } x\neq y.\end{cases}$$

Then d is a d-function on X and $S_n(x)=\mathtt{St}(x,\mathcal{G}_n)$ for all $n\in\mathbb{N}$. We shall show that $\{S_n(x) : n\in\mathbb{N}\}$ is a network at x for all $x\in X$. If not, there exist $x\in U\in\tau$ such that $S_n(x)\not\subset U$ for all $n\in\mathbb{N}$. Thus, for each $n\in\mathbb{N}$, there exists $x_n\in S_n(x)-U$. Since $S_n(x)=\mathtt{St}(x,\mathcal{G}_n)$ for all $n\in\mathbb{N}$, it implies that for each $n\in\mathbb{N}$, there exists $y_n\in X$ such that $x,x_n\in g(n,y_n)$. By condition (G), $x_n\to x$, a contradiction. This follows that X is sn-symmetric. Now, let $\{x_i\}$ be a sequence in X, $x_i\to x$, and $\varepsilon>0$. Take $n\in\mathbb{N}$ such that $1/n<\varepsilon$. Since $\{x\}\bigcup\{x_i : i\geq m\}\subset g(n,x)$ for some $m\in\mathbb{N}$, we have $x_i\in\mathtt{St}(x_j,\mathcal{G}_n)$ for all $i,j\geq m$. This implies that $d(x_i,x_j)<1/n<\varepsilon$ for all $i,j\geq m$. Therefore, X is Cauchy sn-symmetric with a σ-(P)-property sn-network. □

Corollary 2.10. *The following are equivalent for a space X:*

1. *X is a Cauchy symmetric space with a σ-(P)-property weak base;*
2. *X has a weak-development consisting of (P)-property cs-covers;*
3. *X has a weak-development consisting of (P)-property sn-covers;*
4. *there exists a weak base g-function g on X satisfying* (GP).

In case (P) is locally finite, we have the following.

Corollary 2.11. *The following are equivalent for a space X:*

1. *X is an sn-developable and sn-metrizable space;*
2. *X is a strongly sn-developable space;*
3. *X has a σ-locally finite strong network consisting of sn-covers;*
4. *there exists an sn-network g-function g on X satisfying* (GLF)*;*
5. *X is a 1-sequence-covering compact and mssc-image of a metric space;*
6. *X is a 1-sequence-covering compact and σ-image of a metric space;*
7. *X is a sequence-covering π and σ-image of a metric space.*

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4). By Theorem 2.9.

(3) $\Longrightarrow$ (5). Let $\bigcup\{\mathcal{G}_n : n\in\mathbb{N}\}$ be a σ-locally finite strong network consisting of sn-covers. Consider the Ponomarev system $(f,M,X,\mathcal{G}_n)$. By [18, Lemma 2.2], f is a sequence-covering and compact map. Thus, f is a 1-sequence-covering map by [1, Theorem 2.5]. Furthermore, since each $\mathcal{G}_n$ is locally finite, f is an $mssc$-map. Hence, (5) holds.

(5) $\Longrightarrow$ (6). By [15, Lemma 17].

(6) $\Longrightarrow$ (7). Obvious.

(7) $\Longrightarrow$ (1). Let $f : M \to X$ be a sequence-covering π- and σ-map and M be a metric space. By Theorem 2.5, X is an sn-developable space. This implies that X is an sn-first countable space by Theorem 2.1. On the other hand, since f is a sequence-covering and σ-map, this implies that X is an $\aleph$-space. Therefore, X is an sn-metrizable space. □

Corollary 2.12. *The following are equivalent for a space X:*

(1) *X is a g-developable and g-metrizable space;*
(2) *X is a strongly g-developable space;*
(3) *X has a weak-development consisting of locally finite sn-covers;*
(4) *there exists a weak base g-function g on X satisfying* (GLF)*;*
(5) *X is a weak-open compact-covering compact and $mssc$-image of a metric space;*
(6) *X is a weak-open compact-covering compact and σ-image of a metric space;*
(7) *X is a weak-open π- and σ-image of a metric space.*

Proof. By [1, Corollary 2.8] and Corollary 2.11, we only need to prove that (3) $\Longrightarrow$ (5). Let $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ be a weak-development consisting of locally finite sn-covers. We can assume that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ for all $n \in \mathbb{N}$. By using the proof of [18, Lemma 3.10], it follows that each $\mathcal{G}_n$ is a cfp-cover. Consider the Ponomarev system $(f, M, X, \mathcal{G}_n)$. By [18, Lemma 2.2], f is a sequence-covering compact-covering quotient and compact map. Thus, f is a weak-open map by [1, Corollary 2.9]. Furthermore, since each $\mathcal{G}_n$ is locally finite, f is an $mssc$-map. □

In case (P) is point-finite, by Theorem 2.9 and [13, Theorem 3.3.8], we have the following corollaries.

Corollary 2.13. *The following are equivalent for a space X:*

(1) *X has a uniform sn-network.*
(2) *X has a point-regular sn-network;*
(3) *there exists an sn-network g-function g on X satisfying* (GPF)*;*
(4) *X is a 1-sequence-covering and compact image of a metric space;*
(5) *X is a sequence-covering and compact image of a metric space.*

Corollary 2.14. *The following are equivalent for a space X:*

(1) *X has a uniform weak base;*
(2) *X has a point-regular weak base;*
(3) *there exists a weak base g-function g on X satisfying* (GPF)*;*
(4) *X is a weak-open and compact image of a metric space;*
(5) *X is a weak-open and compact image of a metric space.*

Example 2.15. Let $X = \mathbb{N} \cup \{p\}$ where $p \in \beta\mathbb{N} - \mathbb{N}$. Endow X with discrete topology. Then X is a metric space. Put $Y = \mathbb{N} \cup \{p\}$ and endow

Y with the subspace topology of $\beta\mathbb{N}$, then Y is not a k-space. Define $f : X \to Y$ by $f(x) = x$ for each $x \in X$. It is easy to see that f is a 1-sequence-covering and compact map. Hence, by Theorem 2.5, it follows that

(1) not every sn-developable space is g-developable;
(2) not every Cauchy sn-symmetric space is Cauchy symmetric;
(3) Not every sn-symmetric space is symmetric.

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