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TOPOLOGY PROCEEDINGS

Volume 54, 2019

Pages 193–197

http://topology.nipissingu.ca/tp/

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Electronically published on March 7, 2019

Topology Proceedings

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E-mail: topolog@auburn.edu
ISSN: (Online) 2331-1290, (Print) 0146-4124

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http://topology.nipissingu.ca/tp/

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E-Published on March 7, 2019

A FAMILY OF GENERALIZED INVERSE LIMITS HOMEOMORPHIC TO "THE MONSTER"

FARUQ MENA AND ROBERT P. ROE

Abstract. We show that two generalized inverse limit spaces that one might suspect are not homeomorphic are in fact homeomorphic.

1. Introduction and Definitions

We are interested in the family of upper semi-continuous functions $f_a : [0,1] \to [0,1]$ and the corresponding inverse limits $X_a = \varprojlim\{[0,1], f_a\}$, where the graph $\gamma(f_a)$ of f_a is the union of the line segments from $(0,0)$ to $(a,1)$ to $(1,a)$ to $(1,0)$ for $a \in [0,1]$. For $a \in (0,1)$, f_a is a generalized upper semi-continuous (usc) Markov function and it follows from results of Iztok Banič and Tjaša Lunder [1] that if $a, b \in (0,1)$, then X_a is homeomorphic to X_b. But for $a \in (0,1)$, X_a and X_1 are not homeomorphic since the first contains the topologist's sine curve as a subcontinuum and the second is the harmonic fan. The functions f_a where $a \neq 0$, and f_0 do not satisfy the hypothesis of Banič and Lunder's theorem so we may ask, are $X_{\frac{1}{2}}$ and X_0 homeomorphic? In his master's thesis, Christopher David Jacobsen [4] studies $X_{\frac{1}{2}}$ where he shows that it contains $2^{\aleph_0}$ arc components and each arc component is dense. The space X_0 is often referred to as "the monster," a name reportedly coined by Banič.

Several other authors also have results showing when families of functions have homeomorphic inverse limits. For example, W. T. Ingram and William S. Mahavier [3] have shown that if f and g are usc functions which are topologically conjugate, then the corresponding inverse limit spaces are homeomorphic. Michel Smith and Scott Varagona [6] have shown that

2010 *Mathematics Subject Classification.* 54C60, 54E45, 54F15.
Key words and phrases. generalized inverse limit, inverse limits, set valued functions, upper semi-continuous.

N-type usc functions which follow the same pattern have homeomorphic inverse limits. Again, f_a (where $a \in (0,1)$), and f_0 do not satisfy the hypothesis of their theorem.

James P. Kelly and Jonathan Meddaugh [5] examine when it is the case that a sequence of usc functions f_i converging to a usc function f implies that $\varprojlim\{[0,1], f_i\}$ converges to $\varprojlim\{[0,1], f\}$. If we let $a_i \in (0,1)$ with $a_i \to 0$, then X_{a_i} are all homeomorphic by Banić and Lunder's theorem but, again, the functions f_{a_i} and f_0 do not satisfy Kelly and Meddaugh's hypothesis. Thus, it seems somewhat surprising that it is the case that $X_{\frac{1}{2}}$ (and, hence, X_a for $a \in (0,1)$) and X_0 are homeomorphic, as we show in our theorem.

A topological space X is a continuum if it is a non-empty, compact, connected, metric space. A continuum subset of the space X is called a subcontinuum of X. Let X and Y be topological spaces; a function $f : X \to 2^Y$ is usc at x provided that for all open sets V in Y which contain $f(x)$, there exists an open set U in X with $x \in U$ such that if $t \in U$, then $f(t) \subseteq V$. If a function $f : X \to 2^Y$ is usc at x for each $x \in X$, we say that f is usc Let X and Y be compact metric spaces and $f : X \to 2^Y$ a function. It is well known that f is usc if and only if the graph of f, $\gamma(f) = \{(x,y) : x \in X \text{ and } y \in f(x)\}$, is closed in $X \times Y$. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of continua and for each $i \in \mathbb{N}$, let $f_i : X_{i+1} \to 2^{X_i}$ be a usc function. The inverse limit of $\{X_i, f_i\}$ is denoted as $\varprojlim\{X_i, f_i\}$ and defined by $\varprojlim\{X_i, f_i\} = \{(x_i)_{i=1}^{\infty} : x_i \in f_i(x_{i+1}), x_i \in X_i \text{ for all } i \in \mathbb{N}\}$.

2. Main Theorem

Theorem 2.1. *X_0 is homeomorphic to $X_{\frac{1}{2}}$.*

Proof. Let $f : [0,1] \longrightarrow 2^{[0,1]}$ be given by $f(x) = 2x$ for $x \in [0, \frac{1}{2}]$, $f(x) = \frac{3}{2} - x$ for $x \in [\frac{1}{2}, 1]$, and $f(1) = [0, \frac{1}{2}]$.

$$f(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}] \\ \frac{3}{2} - x & \text{if } x \in [\frac{1}{2}, 1) \\ [0, \frac{1}{2}] & \text{if } x = 1. \end{cases}$$

Let $g : [0,1] \longrightarrow 2^{[0,1]}$ be given by

$$g(x) = \begin{cases} [0,1] & \text{if } x = 0 \\ 1 - x & \text{if } x \in (0,1]. \end{cases}$$

Let $A = \{(a_1, ..., a_i, ...) : a_i \in \{0,1\} \text{ and } a_i = 1 \Rightarrow a_{i+1} = 0\}$. Let $B = \{(b_1, ..., b_i, ...) : b_i \in \{0, (\frac{1}{2})^n\} \text{ and } b_i = 0 \Rightarrow b_{i+1} \in \{0,1\} \text{ and } b_i = (\frac{1}{2})^n \Rightarrow b_{i+1} \in \{(\frac{1}{2})^{n+1}, 1\}\}$. It is clear that A and B are subsets of

$\varprojlim\{[0,1],g\}$ and $\varprojlim\{[0,1],f\}$, respectively. Two points x and y in A are said to be adjacent if there is $n \in \mathbb{N} \cup \{\infty\}$ such that

- $\pi_i(x) = \pi_i(y)$ for $i \geq n+1$;
- $\pi_{n+1}(x) = 0 = \pi_{n+1}(y)$;
- $\pi_i(x) = 1 - \pi_i(y)$ for $i \leq n$.

Define $r_{xy}^A : [0,1] \to \varprojlim\{[0,1],g\}$ by $r_{xy}(t) = (t, 1-t, t, ..., 1-t, t, 0, x_{n+2}, ...)$. We say r_{xy}^A is a straight line in $\varprojlim\{[0,1],g\}$ connecting x and y. Notice that any two distinct straight lines can only intersect at endpoints.

Two points z and w in B are said to be adjacent if there is $n \in \mathbb{N} \cup \{\infty\}$ and a positive integer m such that

- $\pi_i(z) = \pi_i(w)$ for $i \geq n+1$;
- $\pi_{n+1}(z) = 1 = \pi_{n+1}(w)$;
- $\pi_n(z) = \frac{1}{2}^{m-1}$;
- $\pi_n(w) = \frac{1}{2}^{m}$;
- $\pi_i(w) = 2\pi_{i+1}(w)$ for $n-m \leq i < n$;
- $\pi_i(w) = \frac{3}{2} - \pi_{i+1}(w)$ for $1 \leq i < n-m$;
- $\pi_i(z) = 2\pi_i(w)$ for $n-m \leq i < n+1$;
- $\pi_i(z) = \frac{3}{2} - \pi_{i+1}(w)$ for $1 \leq i < n-m$.

Define $r_{zw}^B : [\frac{1}{2}^m, \frac{1}{2}^{m-1}] \to \varprojlim\{[0,1],f\}$, where $r_{zw}^B(t) = (\frac{3}{2} - x_2, ..., \frac{3}{2} - x_{n-m}, x_{n-m}, ..., 4t, 2t, t, 1, x_{n+2}, ...)$ where $x_{n-m} = 2^{n-m}t$ and $\frac{1}{2} \leq 2^{n-m}t \leq 1$. As before, we say r_{zw}^B is a straight line in $\varprojlim\{[0,1],f\}$ connecting z and w. Again, any two distinct straight lines can only intersect at endpoints.

Define $H : B \longrightarrow A$ such that $H(b_1, b_2, ...) = (h_1(b_1), h_2(b_2), , ..)$, $h_i(b_i) = 1$ for $b_i = \frac{1}{2}$, and $h_i(b_i) = 0$ otherwise. Define $S : A \longrightarrow B$ such that $S(a_1, a_2, ...) = (s_1(a_1), s_2(a_2), , ..)$, where

$$s_1(a_1) = \begin{cases} \frac{1}{2} & \text{if } a_1 = 1 \\ 1 & \text{if } a_1 = 0 \text{ and } a_2 = 1 \\ 0 & \text{if } a_1 = a_2 = 0, \end{cases}$$

and if $s_k(a_k)$ has been defined for $1 \leq k < i$, let

$$s_i(a_i) = \begin{cases} \frac{1}{2} & \text{if } a_i = 1 \\ 1 & \text{if } a_i = 0 \text{ and } a_{i+1} = 1 \\ \frac{1}{2}s_{i-1}(a_{i-1}) & \text{otherwise.} \end{cases}$$

From the definitions of S, it can be seen that S is one-to-one and onto. Since all component functions s_i are continuous, S is continuous; hence, S is a homeomorphism between A and B. Further, one can see $H = S^{-1}$. Let a and c be adjacent points in A and let r_{ac}^A be a straight line in $\varprojlim\{[0,1],g\}$ so there is n such that $\pi_i(a)=\pi_i(c)$ for all $i\geq n+1$, $\pi_{n+1}(a)=\pi_{n+1}(c)=0$, and one of $\pi_n(a)$ and $\pi_n(c)$ is zero and the other is 1. Suppose without loss of generality, $\pi_n(a)=0$ and $\pi_n(c)=1$. We wish to show that there is a unique corresponding straight line $r^B_{S(a)S(c)}$ in $\varprojlim\{[0,1],f\}$ connecting $S(a)$ and $S(c)$. By the definition of S, $s_n(a_n)=\frac{1}{4}$, $s_{n-1}(a_{n-1})=s_n(c_n)=\frac{1}{2}$, $s_{j-1}(a_{j-1})=\frac{3}{2}-s_j(a_j)$ for all $j<n$, and $s_{j-1}(c_{j-1})=\frac{3}{2}-s_j(c_j)$ for all $j\leq n$. Let $l=\min\{k:k>n+1 \text{ and } a_k=1\}$. So there is a positive integer m such that $l=n+m$. Since $a_{l-1}=c_{l-1}=0$ and $a_l=c_l=1$, $s_{l-1}(a_{l-1})=s_{l-1}(c_{l-1})=1$ and $s_l(a_l)=s_l(c_l)=\frac{1}{2}$. Also, $s_{l-2}(a_{l-2})=s_{n+m-2}(a_{n+m-2})=\frac{1}{2^m}$ and $s_{l-2}(c_{l-2})=s_{n+m-2}(c_{n+m-2})=\frac{1}{2^{m-1}}$. Hence, $S(a)$ and $S(c)$ are adjacent in B; therefore, r^A_{ac} is homeomorphic to the corresponding straight line $r^B_{S(a)S(c)}$.

Let p and q be adjacent points in B and let r^B_{pq} be a straight line in $\varprojlim\{[0,1],f\}$ so there is n such that $\pi_{n+1}(p)=\pi_{n+1}(q)=1$, $\pi_n(p)=\pi_n(q)$ for all $i\geq n+1$, and one of $\pi_n(p)$ and $\pi_n(q)$ is $\frac{1}{2^m}$ and the other is $\frac{1}{2^{m+1}}$ $m\geq 1$.

Suppose without loss of generality, $\pi_n(p)=\frac{1}{2^m}$ and $\pi_n(q)=\frac{1}{2^{m+1}}$. So $\pi_{n-m+2}(p)=\frac{1}{4}$, $\pi_{n-m+2}(q)=\frac{1}{8}$ by the definition of H, $h(\pi_i(p))$ and $h(\pi_i(q))$ are equal to zero for $n-m+2<i\leq n+1$; therefore, $n-m+2$ is the least positive integer such that the images of $h(\pi_{n-m+2}(p))$ and $h(\pi_{n-m+2}(q))$ are zero and $h(\pi_{n-m+1}(p))=1$ and $h(\pi_{n-m+1}(q))=0$. This means that $H(p)$ and $H(q)$ are adjacent points in A. Thus, the set of straight lines in $\varprojlim\{[0,1],g\}$ is mapped one-to-one and onto the set of straight lines in $\varprojlim\{[0,1],f\}$.

Hence, S (or H) can be piecewise linearly extended to a homeomorphism between $\varprojlim\{[0,1],g\}$ and $\varprojlim\{[0,1],f\}$, completing the proof of the theorem. □

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