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SPLITTING OF COLLAPSING MAPS FOR FREE ABELIAN TOPOLOGICAL GROUPS

ALEXANDER DRANISHNIKOV

Abstract. We prove the following.

Theorem. *Suppose that X is a finite complex and Y is a connected subcomplex such that $H^{i+1}(X/Y; H_i(Y)) = 0$ for all $i > 0$. Then, for free abelian topological groups,*

$$\mathbb{A}(X) \cong \mathbb{A}(X/Y) \times \mathbb{A}(Y).$$

As a corollary, we obtain that $\mathbb{A}(\mathbb{C}P^2) = \mathbb{A}(S^2 \vee S^4)$, whereas $\mathbb{F}(\mathbb{C}P^2) \neq \mathbb{F}(S^2 \vee S^4)$ where $\mathbb{F}(X)$ denotes free topological group generated by X.

1. Introduction

The free topological group $\mathbb{F}(X)$ and the free abelian topological group $\mathbb{A}(X)$ generated by a topological space X were defined first by A. A. Markov [7] and [8] in the topological category and then by M. I. Graev [6] in the pointed topological category. Markov's and Graev's definitions are closely related. In this paper, we consider the latter. We consider these groups for finite CW complexes X. Note that there are natural embeddings $X \subset \mathbb{F}(X)$ and $X \subset \mathbb{A}(X)$ where the base point x_0 is identified with the unit.

The defining property of $\mathbb{A}(X)$ is the following: *For every continuous map $f : X \to G$ of a pointed compact metric space (X, e) to a topological abelian group G with $f(e) = 0 \in G$, there is a unique extension to a continuous homomorphism $\bar{f} : \mathbb{A}(X) \to G$.* For $\mathbb{F}(X)$, the defining property is similar (see [2]).

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In this paper, we answer the following apparently folklore question: Does there exist topological spaces X_1 and X_2 such that the free abelian topological groups $\mathbb{A}(X_1)$ and $\mathbb{A}(X_2)$ are isomorphic in the category of topological groups, but free topological groups $\mathbb{F}(X_1)$ and $\mathbb{F}(X_2)$ are not?

The submission of this note to the Proceedings of STDC-2018 is justified in part by the fact that shortly after the conference where I had the pleasure of talking to Arkady Leiderman who reminded me about that question. To the best of my knowledge, the question was not explicitly formulated in the published papers and the answer has not been known.

We recall the classical Dold–Thom theorem.

Theorem 1.1 ([4]). *The n^{th} homology group of a CW complex X is isomorphic to the n^{th} homotopy group of its free abelian topological group, $H_n(X) = \pi_n(\mathbb{A}(X))$.*

In view of John W. Milnor's theorem [9] (see Theorem 3.1 below), one could easily distinguish the topological groups $\mathbb{F}(X_1)$ and $\mathbb{F}(X_2)$ by their homotopy types. One the other hand, the Dold–Thom theorem [4] can be used to show that the topological groups $\mathbb{A}(X_1)$ and $\mathbb{A}(X_2)$ have homotopy types of the product of Eilenberg–McLane spaces and, hence, often they have the same homotopy type. Namely, they have the same homotopy type for CW complexes X_1 and X_2 with the same homology groups.

In [5], it is proven that for finite complexes $Y \subset X$ the collapsing map $\mathbb{A}(X) \to \mathbb{A}(X/Y)$ is a skew product of $\mathbb{A}(X/Y)$ and $\mathbb{A}(Y)$. In this paper, we present some sufficient conditions when this skew product is actually the direct product. This allows us to produce examples of nonhomeomorphic complexes X_1 and X_2 with isomorphic free abelian topological groups $\mathbb{A}(X_1)$ and $\mathbb{A}(X_2)$. Perhaps the simplest example obtained in this way that answers the above question in negative is $X_1 = S^2 \vee S^4$ and $X_2 = \mathbb{C}P^2$.

2. Splitting Theorem

It is proven in [5] that the free abelian topological group applied to the collapsing map $q : X \to X/Y$ of finite CW complexes produces a locally trivial bundle $\mathbb{A}(q) : \mathbb{A}(X) \to \mathbb{A}(X/Y)$ with the fiber $\mathbb{A}(Y)$.

We note that every locally trivial bundle $p : X \to Z$ with a path connected fiber F defines the i-homotopy, $i > 1$, local coefficient system $\mathcal{S}$ with the stalk $\pi_i(F)$. Every local coefficient system is defined by an action of the fundamental group $\pi_1(Z)$ on the stalk. Here, the action of the fundamental group $\pi_1(Z)$ on $\pi_i(F)$ is defined as follows. Given a loop $\phi : I \to Z$ based at z_0, we fix a lift $\bar{\phi} : I \to X$. Since F is path connected, we may assume that $\bar{\phi}$ is a loop. Let $\bar{\phi}(z_0) = x_0$. For an i-spheroid $f : (S^i, s_0) \to (F, x_0)$, we consider the homotopy $H : S^i \times I \to Z$, defined as $H(x,t) = \phi(t)$. Then consider a homotopy $\bar{H} : S^i \times I \to X$ covering H with $\bar{H}(x,0) = f(x) \in F = p^{-1}(z_0)$. We may assume that $\bar{H}(s_0,t) = \bar{\phi}(t)$ where $s_0 \in S^i$ is the base point. Then, by the definition, $[\phi]([f]) = [\bar{H}|_{S^i \times \{1\}}] \in \pi_i(F)$ is the result of the action of the element $[\phi] \in \pi_1(Z)$ on $[f] \in \pi_i(F)$. This definition does not depend on choices we made in representations of $[\phi]$, $[f]$, and $\bar{H}$.

Proposition 2.1. *Let $q : X \to X/Y$ be a collapsing map with a finite complex X and a connected subcomplex $Y \subset X$. Then the local coefficient system $\mathcal{S}$ defined by the fibration $\mathbb{A}(q) : \mathbb{A}(X)|_{(X/Y)} \to X/Y$ is trivial.*

Proof. By our convention, the base point e_0 of X is also the base point of Y. We show that the action of $\pi_1(X/Y)$ on $\pi_i(\mathbb{A}(Y))$ is trivial. Since A is connected, each $\alpha \in \pi_1(X/Y)$ can be represented by a loop $\phi : I \to X/Y$ that admits a lift $\bar{\phi} : I \to X$ to a loop. Let $f : (S^i, s_0) \to (\mathbb{A}(Y), e_0)$ be a spheroid. Let $H : S^i \times I \to X/Y$ be a homotopy as defined above with the base point $\{Y\} \in X/Y$. We define $\bar{H}(s,t) = f(s) + \bar{\phi}(t)$ where the sum is taken in $\mathbb{A}(X)$. Clearly, $\bar{H}$ is a lift of H. Note that $\bar{H}(s,1) = f(s) + e_0 = f(s)$. Thus, the action of α is trivial. $\square$

Next we consider the question: When does the short exact sequence of free topological abelian groups

$$0 \longrightarrow \mathbb{A}(Y) \xrightarrow{\mathbb{A}(i)} \mathbb{A}(X) \xrightarrow{\mathbb{A}(q)} \mathbb{A}(X/A) \to 0$$

split?

It splits naturally in the case when Y is a retract of X. If $r : X \to Y$ is a retraction, then the continuous homomorphism

$$(\mathbb{A}(q), \mathbb{A}(r)) : \mathbb{A}(X) \to \mathbb{A}(X/Y) \times \mathbb{A}(Y)$$

is an isomorphism of topological groups.

This fact also follows from O. G. Okunev's result [10] about parallel retractions which Okunev proved for nonabelian free topological groups as well.

It would be nice to have a complete answer to the above question. Here, we present the following sufficient condition for the splitting.

Theorem 2.2. *Suppose that X is a finite complex and Y is a connected subcomplex such that $H^{i+1}(X/Y; H_i(Y)) = 0$ for all $i > 0$. Then $\mathbb{A}(X) \cong \mathbb{A}(X/Y) \times \mathbb{A}(Y)$.*

Proof. We claim that the fibration $\mathbb{A}(q)$ admits a continuous section over X/Y. We construct such a section using the obstruction theory. By induction, we define a section $s_i : (X/Y)^{(i)} \to \mathbb{A}(X)$ of $\mathbb{A}(q)$ over the

i-skeleton. Since the fiber $\mathbb{A}(Y)$ is connected, such a section exists for $i=1$. For $i>1$, the obstruction to have a section s_{i+1} when a section s_i is already constructed lies in the cohomology group $H^{i+1}(X/Y;\pi_i(\mathbb{A}(Y))) = H^{i+1}(X/Y;H_i(Y)) = 0$ where $\pi_i(\mathbb{A}(Y))$ is generally a local coefficient system (see for example [3]). We use Proposition 2.1 to cover the case when $\pi_1(X/Y)\neq 0$.

Thus, there is a continuous section $s: X/Y \to \mathbb{A}(X)$. Then, by the defining property of free abelian topological group, s can be extended to a continuous homomorphism of topological groups $S:\mathbb{A}(X/Y)\to\mathbb{A}(X)$. Note that S is a section of $\mathbb{A}(q)$:

$$\mathbb{A}(q)(S(\sum n_j x_j)) = \mathbb{A}(q)(\sum n_j s(x_j)) = \sum n_j qs(x_j) = \sum n_j x_j.$$

We define a continuous group homomorphism $\phi:\mathbb{A}(X)\to\mathbb{A}(Y)$ by the formula $\phi(\mu)=\mu - S\mathbb{A}(q)(\mu)$. Then the homomorphism

$$(\mathbb{A}(q),\phi):\mathbb{A}(X)\to\mathbb{A}(X/Y)\times\mathbb{A}(Y)$$

is an isomorphism of topological groups with the inverse

$$\psi:\mathbb{A}(X/Y)\times\mathbb{A}(Y)\to\mathbb{A}(X)$$

defined as $\psi(w_1,w_2)=S(w_1)+w_2$. □

3. Applications

Theorem 3.1 (Milnor [9]; see also [1]). *For a CW complex X, the free topological group $\mathbb{F}(X)$ is homotopy equivalent to $\Omega\Sigma(X)$.*

Here $\Omega(X)$ denotes the space of loops in X based at $x_0\in X$ and

$$\Sigma(X) = X\times I/(X\times\partial I\cup\{x_0\}\times I)$$

denotes the reduced suspension of X. For a based space X the inclusion map $i:X\to\Omega\Sigma(X)$ is defined as $i(x)(t)=q(x,t)$ where $q:X\times I\to\Sigma(X)$ is the quotient map.

Theorem 3.2 (Freudenthal, see [3]). *Suppose that X is a k-connected CW complex. Then the inclusion $i:X\to\Omega\Sigma(X)$ induces an isomorphism of i-homotopy groups for $i\le 2k$ and an epimorphism for $i=2k+1$.*

Theorem 3.3. *There are finite complexes X_1 and X_2 such that $\mathbb{A}(X_1)=\mathbb{A}(X_2)$, but $\mathbb{F}(X_1)\neq\mathbb{F}(X_2)$.*

Proof. We take $X_1=S^4\vee S^2$ and $X_2=\mathbb{C}P^2$. Clearly, $\mathbb{A}(X_1)=\mathbb{A}(S^4)\times\mathbb{A}(S^2)$. Let $S^2\subset\mathbb{C}P^2$ be the 2-skeleton of $\mathbb{C}P^2$ for the natural CW complex structure on $\mathbb{C}P^2$. Then $\mathbb{C}P^2/S^2=S^4$. Note that the condition $H^{i+1}(S^4;H_i(S^2))=0$ of Theorem 2.2 is satisfied. Therefore, $\mathbb{A}(X_2)=\mathbb{A}(\mathbb{C}P^2)=\mathbb{A}(S^4)\times\mathbb{A}(S^2)=\mathbb{A}(X_1)$.

By Milnor's theorem,

$$\pi_3(\mathbb{F}(X_1)) = \pi_3(\Omega\Sigma(S^4 \vee S^2)) = \pi_4(S^5 \vee S^3) = \mathbb{Z}_2.$$

By the Freudenthal suspension theorem, the inclusion homomorphism

$$\pi_3(\mathbb{C}P^2) \to \pi_3(\Omega\Sigma\mathbb{C}P^2) = \pi_3(\mathbb{F}(\mathbb{C}P^2)) = \pi_3(\mathbb{F}(X_2))$$

is an epimorphism. Since $\pi_3(\mathbb{C}P^2) = 0$, we have $\pi_3(\mathbb{F}(X_2)) = 0$. Thus, the spaces $\mathbb{F}(X_1)$ and $\mathbb{F}(X_2)$ are not homotopy equivalent and, hence, not homeomorphic.

Note that in this example we do not use Proposition 2.1, since the spaces here are simply connected $\square$

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