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by

Elmas Irmak

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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EDGE PRESERVING MAPS OF THE CURVE GRAPHS IN LOW GENUS

ELMAS IRMAK

ABSTRACT. Let R be a compact, connected, orientable surface of genus g with n boundary components. Let $\mathcal{C}(R)$ be the curve graph of R. We prove that if $g = 0$, $n \geq 5$ or $g = 1$, $n \geq 3$, and $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ is an edge preserving map, then λ is induced by a homeomorphism of R, and this homeomorphism is unique up to isotopy.

1. INTRODUCTION

Let R be a compact, connected, orientable surface of genus g with n boundary components. The *mapping class group*, Mod_R, of R is defined to be the group of isotopy classes of orientation preserving self-homeomorphisms of R. The *extended mapping class group*, Mod_R^*, of R is defined to be the group of isotopy classes of all self-homeomorphisms of R. Abstract simplicial complexes on surfaces have been studied to get information about the algebraic structure of the extended mapping class groups of the surfaces. One of these complexes is the complex of curves. The vertex set of the complex of curves is the set of isotopy classes of nontrivial simple closed curves on R, where nontrivial means the curve does not bound a disk and it is not isotopic to a boundary component of R. A set of vertices forms a simplex in the complex of curves if its elements can be represented by pairwise disjoint simple closed curves on the surface. Let $\mathcal{C}(R)$ be the curve graph, the first skeleton of the complex of curves on R. A map on $\mathcal{C}(R)$ is edge preserving if it sends two vertices

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connected by an edge to two vertices connected by an edge. The main result is the following.

Theorem 1.1. *Let R be a compact, connected, orientable surface with $g = 0$ and $n \geq 5$ or $g = 1$ and $n \geq 3$. If $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ is an edge preserving map, then there exists a homeomorphism $h : R \to R$ such that $H(\alpha) = \lambda(\alpha)$ for every vertex α in $\mathcal{C}(R)$, where $H = [h]$ (i.e., λ is induced by h), and this homeomorphism is unique up to isotopy.*

Our main result completes the author's previous work given in [10] where she proves the statement of this theorem when $g \geq 2$ and $n \geq 0$. Results of this nature began with Nikolai V. Ivanov's famous result on automorphisms of the complex of curves given in [14], where Ivanov proves that every automorphism of the complex of curves is induced by a homeomorphism of R if the genus is at least two, and as an application, gives a classification of isomorphisms between any two finite index subgroups of the extended mapping class group of R. These results are extended for surfaces of genus zero and one by Mustafa Korkmaz in [15] and independently by Feng Luo in [16]. In [7], [8], and [9], the author proves that the superinjective simplicial maps of the complex of curves on a compact, connected, orientable surface are induced by homeomorphisms if the genus is at least two, and, using this result, she presents a classification of injective homomorphisms from finite index subgroups of the extended mapping class group to the extended mapping class group. These results are extended to lower genus cases by Jason Behrstock and Dan Margalit in [2] and Robert W. Bell and Margalit in [3]. We remind the reader that superinjective simplicial maps are simplicial maps that preserve geometric intersection zero and nonzero properties. After these results, Kenneth J. Shackleton, in [17], proves that locally injective simplicial maps of the complex of curves are induced by homeomorphisms.

Javier Aramayona and Christopher J. Leininger [1] prove that there is an exhaustion of the complex of curves by a sequence of finite rigid sets. Elmas Irmak and Luis Paris [12] prove that superinjective simplicial maps of the two-sided curve complex are induced by homeomorphisms on compact, connected, nonorientable surfaces when the genus is at least 5. In [13], they also present a classification of injective homomorphisms from finite index subgroups of mapping class group to the whole mapping class group on these surfaces. In this paper, we use some techniques given by Irmak and Paris in [12] and some techniques given by Aramayona and Leininger in [1].

In [6] Jésus Hernández Hernández proves that if S_1 and S_2 are orientable surfaces of finite topological type such that S_1 has genus at least 3 and the complexity of S_1 is an upper bound of the complexity of S_2, and

$\theta : \mathcal{C}(S_1) \to \mathcal{C}(S_2)$ is an edge-preserving map, then S_1 is homeomorphic to S_2 and θ is induced by a homeomorphism. In [10] the author gives a new proof of this result for edge preserving maps of $\mathcal{C}(R)$ when $g \geq 2$ and $n \geq 0$ by first proving the result on the nonseparating curve graph. Since superinjective simplicial maps are edge preserving, this improved the results of the author given in [7], [8], and [9]. We also note that edge preserving maps of the curve graphs are used to get information about the maps of Hatcher–Thurston graphs; see [5] and [10]. Automorphisms of the Hatcher–Thurston complex are classified by Irmak and Korkmaz in [11].

In this paper, the author proves the remaining cases on the edge preserving maps of the curve graphs when $g = 0$ and $n \geq 5$ or $g = 1$ and $n \geq 3$. We note that when $g = 0$ and $n \in \{1, 2, 3\}$, the curve graph is empty. For the other cases, when $g = 0$ and $n = 4$ or $g = 1$ and $n \in \{0, 1, 2\}$, the statement is not true. When $g = 0$ and $n = 4$ or $g = 1$ and $n \in \{0, 1\}$, the curve graph is represented by the Farey graph (see Figure 1) (by putting edges between vertices that have geometric intersection two in $g = 0$ and $n = 4$ case and by putting edges between vertices that have geometric intersection one in the other two cases). It is easy to see that there are edge preserving maps of the Farey graph that are not induced by homeomorphisms of the corresponding surfaces in these cases. When $g = 1$ and $n = 2$, the curve graph is isomorphic to the curve graph of the surface M with $g = 0$ and $n = 5$; see [16, Lemma 2.1]. There are automorphisms of the curve graph of M switching vertices that correspond to nonseparating and separating curves on the surface with $g = 1$ and $n = 2$. So the statement is not true for $g = 1$ and $n = 2$.

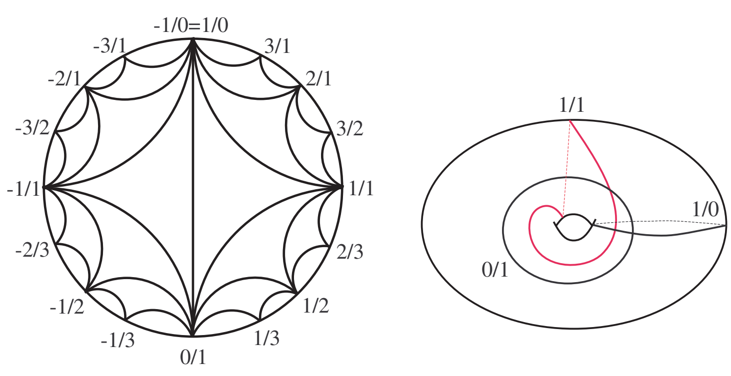

FIGURE 1. Farey graph

2. Edge Preserving Maps of $\mathcal{C}(R)$ When $g = 1$ and $n \geq 3$

In this section we will always assume that $g = 1$ and $n \geq 3$ and that $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ is an edge preserving map.

We first give some definitions. Let P be a set of pairwise disjoint nontrivial simple closed curves on R. The set P is called a *pair of pants decomposition of* R if R_P (the surface obtained from R by cutting along P) is the disjoint union of genus zero surfaces with three boundary components, *pairs of pants*. A pair of pants of a pants decomposition is the image of one of these connected components under the quotient map $q : R_P \to R$. Let a and b be two distinct elements in a pair of pants decomposition P on R. Then a is called *adjacent* to b with respect to P if and only if there exists a pair of pants in P which has a and b on its boundary.

Lemma 2.1. *If $\mathcal{A}$ is a set of vertices in $\mathcal{C}(R)$ where every pair has geometric intersection zero, then λ restricted to $\mathcal{A}$ is injective.*

Proof. Let $\mathcal{A}$ be a set of vertices in $\mathcal{C}(R)$ where every pair has geometric intersection zero. Let α and β be distinct elements in $\mathcal{A}$. Since $i(\alpha, \beta) = 0$, there is an edge between α and β. Since λ is edge preserving, there is an edge between $\lambda(\alpha)$ and $\lambda(\beta)$. So $\lambda(\alpha) \neq \lambda(\beta)$. Hence, λ restricted to $\mathcal{A}$ is injective. □

Lemma 2.2. *Let P be a pants decomposition on R. A set of pairwise disjoint representatives of $\lambda([P])$ is a pants decomposition on R.*

Proof. The proof follows from Lemma 2.1. □

Lemma 2.3. *Let α_1 and α_2 be two vertices of $\mathcal{C}(R)$. If $i(\alpha_1, \alpha_2) = 1$, then $i(\lambda(\alpha_1), \lambda(\alpha_2)) \neq 0$.*

Proof. Let a and b be minimally intersecting representatives of α_1 and α_2, respectively. We complete a and b to a curve configuration $\{a, b, c, d, e\}$ as shown in Figure 2. Then we complete $\{a, c, e\}$ to a pants decomposition P on R. Let P' be a set of pairwise disjoint representatives of $\lambda([P])$. The set P' is a pants decomposition on R. We see that $i([b], [x]) = 0$ for all $x \in P \setminus \{a\}$ and there is an edge between $[b]$ and $[x]$ for all $x \in P \setminus \{a\}$. Since λ is edge preserving, we have $i(\lambda([b]), \lambda([x])) = 0$ for all $x \in P \setminus \{a\}$ and there is an edge between $\lambda([b])$ and $\lambda([x])$ for all $x \in P \setminus \{a\}$. This implies that either $i(\lambda([a]), \lambda([b])) \neq 0$ or $\lambda([a]) = \lambda([b])$. With a similar argument, we can see that either $i(\lambda([d]), \lambda([b])) \neq 0$ or $\lambda([d]) = \lambda([b])$. If $\lambda([a]) = \lambda([b])$, then we could not have $i(\lambda([d]), \lambda([b])) \neq 0$, and $\lambda([d]) = \lambda([b])$ since λ is edge preserving. Hence, $i(\lambda([a]), \lambda([b])) \neq 0$. □

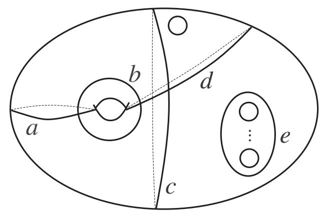

FIGURE 2. Intersection one

Lemma 2.4. *Let* $\{y, c_1, c_2, \cdots c_{n-1}\}$ *be the curves shown in Figure 3. Then we have* $i(\lambda([y]), \lambda([c_i])) \neq 0$ *for all* $i = 1, 2, \cdots n-1$.

Proof. To see that $i(\lambda([y]), \lambda([c_1])) \neq 0$, we complete y to a pants decomposition P using all the unlabeled curves given in Figure 3(i). Let P' be a set of pairwise disjoint representatives of $\lambda([P])$. The set P' is a pants decomposition on R. We see that $i([c_1], [x]) = 0$ for all $x \in P \setminus \{y\}$ and there is an edge between $[c_1]$ and $[x]$ for all $x \in P \setminus \{y\}$. Since λ is edge preserving, we have $i(\lambda([c_1]), \lambda([x])) = 0$ for all $x \in P \setminus \{y\}$ and there is an edge between $\lambda([c_1])$ and $\lambda([x])$ for all $x \in P \setminus \{y\}$. This implies that either $i(\lambda([c_1]), \lambda([y])) \neq 0$ or $\lambda([c_1]) = \lambda([y])$. Let a be the curve shown in Figure 3(i). Since $i([a]), [y]) = 1$, by Lemma 2.3 we know $i(\lambda([a]), \lambda([y])) \neq 0$. But since $i([a]), [c_1]) = 0$, we have $i(\lambda([a]), \lambda([c_1])) = 0$. So $\lambda([c_1])$ cannot be equal to $\lambda([y])$. Hence, $i(\lambda([y]), \lambda([c_1])) \neq 0$. With similar arguments, we see that $i(\lambda([y]), \lambda([c_i])) \neq 0$ for all $i = 2, 3, \cdots, n-1$ (see Figure 3(ii)–(iv).) □

Lemma 2.5. *Let* $P = \{a, c_1, c_2, c_3, \cdots, c_{n-1}\}$ *where the curves are as shown in Figure 3. Let* P' *be a pair of pants decomposition of* R *such that* $\lambda([P]) = [P']$. *If* $x, y \in P$ *and* x *is adjacent to* y *with respect to* P, *then* $\lambda([x])$ *and* $\lambda([y])$ *have representatives in* P' *which are adjacent to each other with respect to* P'.

Proof. We see that a is adjacent to c_1 with respect to P. To see that $\lambda([a])$ and $\lambda([c_1])$ have representatives in P' which are adjacent to each other with respect to P', it is enough to find a curve p_1 (shown in Figure 3(v)) which intersects only a and c_1 and not any other curve in P and to control that (i) $i(\lambda([p_1]), \lambda([a])) \neq 0$; (ii) $i(\lambda([p_1]), \lambda([c_1])) \neq 0$; and (iii) $i(\lambda([p_1]), \lambda([x])) = 0$ for every $x \in P \setminus \{a, c_1\}$.

(i) Since a and p_1 have geometric intersection one, by using Lemma 2.3, we see that $i(\lambda([p_1]), \lambda([a])) \neq 0$.
(ii) To see that $i(\lambda([p_1]), \lambda([c_1])) \neq 0$, we consider the following: Let $Q = (P \setminus \{a\}) \cup \{b\}$ where the curve b is as shown in Figure 3(v).

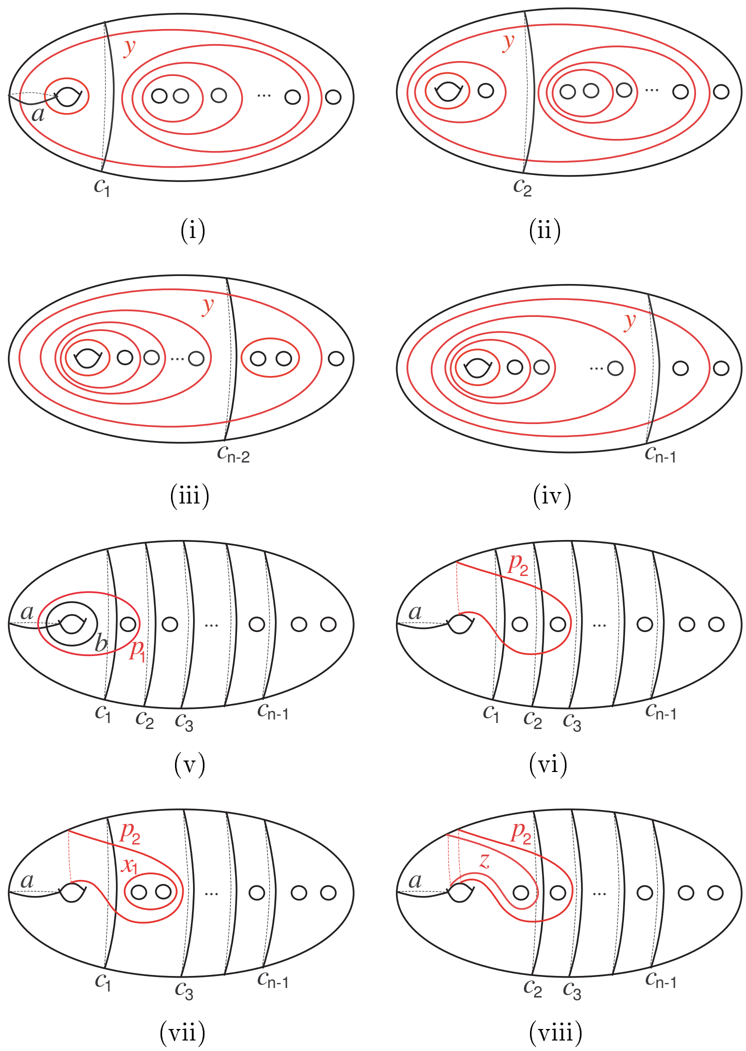

FIGURE 3. Adjacency

Then Q is a pants decomposition on R and $i(\lambda([p_1]), \lambda([x])) = 0$ for every $x \in Q \setminus \{c_1\}$. So either $i(\lambda([p_1]), \lambda([c_1])) \neq 0$ or $\lambda([p_1]) = \lambda([c_1])$. Since $i([a]), [p_1]) = 1$, by Lemma 2.3, $i(\lambda([a]), \lambda([p_1])) \neq$

0. But since $i([a]),[c_1]) = 0$, we have $i(\lambda([a]),\lambda([c_1])) = 0$. So $\lambda([p_1])$ cannot be equal to $\lambda([c_1])$. Hence, $i(\lambda([p_1]),\lambda([c_1])) \neq 0$.

(iii) Since λ is edge preserving, $i(\lambda([p_1]),\lambda([x])) = 0$ for every $x \in P \setminus \{a, c_1\}$.

This gives us that $\lambda([a])$ and $\lambda([c_1])$ have representatives in P' which are adjacent to each other with respect to P'.

To see that $\lambda([c_1])$ and $\lambda([c_2])$ have representatives in P' which are adjacent to each other with respect to P', it is enough to find a curve p_2 shown in Figure 3(vi) which intersects only c_1 and c_2 and not any other curve in P and to control that (i) $i(\lambda([p_2]),\lambda([c_1])) \neq 0$; (ii) $i(\lambda([p_2]),\lambda([c_2])) \neq 0$; and (iii) $i(\lambda([p_2]),\lambda([x])) = 0$ for every $x \in P \setminus \{c_1, c_2\}$.

(i) To see that $i(\lambda([p_2]),\lambda([c_1])) \neq 0$, we consider the following: Let $Q = (P \setminus \{c_2\}) \cup \{x_1\}$ where the curve x_1 is as shown in Figure 3(vii). Then Q is a pants decomposition on R and $i(\lambda([p_2]),\lambda([x])) = 0$ for every $x \in Q \setminus \{c_1\}$. So either $i(\lambda([p_2]),\lambda([c_1])) \neq 0$ or $\lambda([p_2]) = \lambda([c_1])$. Since $i([b]),[p_2]) = 1$, $i(\lambda([b]),\lambda([p_2])) \neq 0$ by Lemma 2.3. But since $i([b]),[c_1]) = 0$, we have $i(\lambda([b]),\lambda([c_1])) = 0$. So $\lambda([p_2])$ cannot be equal to $\lambda([c_1])$. Hence, $i(\lambda([p_2]),\lambda([c_1])) \neq 0$.

(ii) To see that $i(\lambda([p_2]),\lambda([c_2])) \neq 0$, we consider $T = (P \setminus \{c_1\}) \cup \{z\}$ where the curve z is as shown in Figure 3(viii). Then T is a pants decomposition on R and $i(\lambda([p_2]),\lambda([x])) = 0$ for every $x \in T \setminus \{c_2\}$. So either $i(\lambda([p_2]),\lambda([c_2])) \neq 0$ or $\lambda([p_2]) = \lambda([c_2])$. Since $i([b]),[p_2]) = 1$, by Lemma 2.3 $i(\lambda([b]),\lambda([p_2])) \neq 0$. But since $i([b]),[c_2]) = 0$, we have $i(\lambda([b]),\lambda([c_2])) = 0$. So $\lambda([p_2])$ cannot be equal to $\lambda([c_2])$. Hence, $i(\lambda([p_2]),\lambda([c_2])) \neq 0$.

(iii) Since λ is edge preserving, $i(\lambda([p_2]),\lambda([x])) = 0$ for every $x \in P \setminus \{c_1, c_2\}$.

This gives us that $\lambda([c_1])$ and $\lambda([c_2])$ have representatives in P' which are adjacent to each other with respect to P'.

To see that $\lambda([c_2])$ and $\lambda([c_3])$ have representatives in P' which are adjacent to each other with respect to P', it is enough to find a curve p_3 shown in Figure 4(i) which intersects only c_2 and c_3 and not any other curve in P and to control that (i) $i(\lambda([p_3]),\lambda([c_2])) \neq 0$; (ii) $i(\lambda([p_3]),\lambda([c_3])) \neq 0$; and (iii) $i(\lambda([p_3]),\lambda([x])) = 0$ for every $x \in P \setminus \{c_2, c_3\}$.

(i) To see that $i(\lambda([p_3]),\lambda([c_2])) \neq 0$, we consider $U = (P \setminus \{c_3\}) \cup \{x_2\}$ where the curve x_2 is as shown in Figure 4(ii). We see that U is a pants decomposition on R and $i(\lambda([p_3]),\lambda([x])) = 0$ for every $x \in U \setminus \{c_2\}$. So either $i(\lambda([p_3]),\lambda([c_2])) \neq 0$ or $\lambda([p_3]) = \lambda([c_2])$. By Lemma 2.4, we have $i(\lambda([y]),\lambda([c_2])) \neq 0$. Since $i(([y]),[p_3]) =$

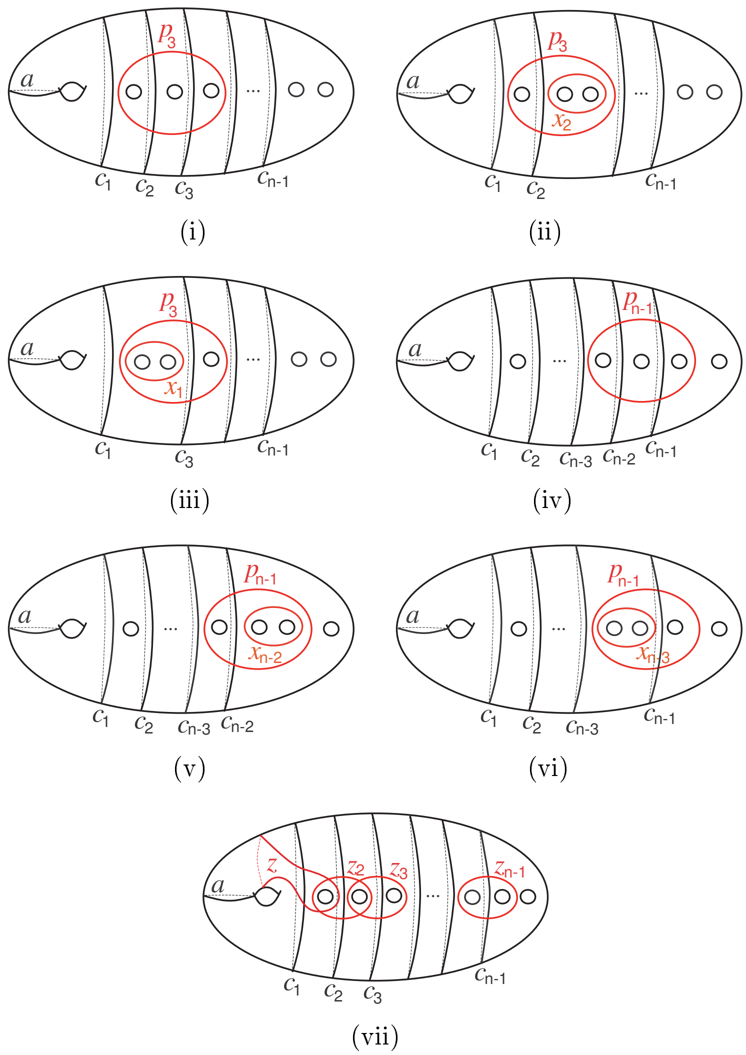

FIGURE 4. Adjacency, Nonadjacency

0, we have $i(\lambda([y]), \lambda([p_3])) = 0$. So $\lambda([p_3]) \neq \lambda([c_2])$. Hence, $i(\lambda([p_3]), \lambda([c_2])) \neq 0$.

(ii) To see that $i(\lambda([p_3]), \lambda([c_2])) \neq 0$, we consider $V = (P \setminus \{c_2\}) \cup \{x_1\}$ where the curve x_1 is as shown in Figure 4(iii). We see that V is a pants decomposition on R and $i(\lambda([p_3]), \lambda([x])) = 0$ for every

$x \in V \setminus \{c_3\}$. So either $i(\lambda([p_3]), \lambda([c_3])) \neq 0$ or $\lambda([p_3]) = \lambda([c_3])$. By Lemma 2.4, we have $i(\lambda([y]), \lambda([c_3])) \neq 0$. Since $i(([y]), [p_3]) = 0$, we have $i(\lambda([y]), \lambda([p_3])) = 0$. So $\lambda([p_3]) \neq \lambda([c_3])$. Hence, $i(\lambda([p_3]), \lambda([c_3])) \neq 0$.

(iii) Since λ is edge preserving, $i(\lambda([p_3]), \lambda([x])) = 0$ for every $x \in P \setminus \{c_2, c_3\}$.

This gives us that $\lambda([c_2])$ and $\lambda([c_3])$ have representatives in P' which are adjacent to each other with respect to P'.

The proof of the statement that $\lambda([c_i])$ and $\lambda([c_{i+1}])$ have representatives in P' which are adjacent to each other with respect to P' for $i = 2, 3, \cdots, n-1$ is similar to the proof of this last case (see Figure 4(iv)–(vi)). □

Lemma 2.6. *Let $P = \{a, c_1, c_2, c_3, \cdots, c_{n-1}\}$ where the curves are as shown in Figure 4(vii). Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. If $x, y \in P$ and x is not adjacent to y with respect to P, then $\lambda([x])$ and $\lambda([y])$ have representatives in P' which are not adjacent to each other with respect to P'.*

Proof. Consider the curves z and z_i given in Figure 4(vii). We will first show that (i) $i(\lambda([z]), \lambda([c_1])) \neq 0$ and (ii) $i(\lambda([z_i]), \lambda([c_i])) \neq 0$ for all $i = 2, 3, 4, \cdots, n-1$.

(i) To see that $i(\lambda([z]), \lambda([c_1])) \neq 0$, we observe that $i(\lambda([z]), \lambda([x])) = 0$ for every $x \in P \setminus \{c_1\}$. So either $i(\lambda([z]), \lambda([c_1])) \neq 0$ or $\lambda([z]) = \lambda([c_1])$. Since $i([b], [z]) = 1$, we have $i(\lambda([b]), \lambda([z)) \neq 0$ by Lemma 2.3. But since $i([b]), [c_1]) = 0$, we have $i(\lambda([b]), \lambda([c_1])) = 0$. So $\lambda([z])$ cannot be equal to $\lambda([c_1])$. Hence, $i(\lambda([z]), \lambda([c_1])) \neq 0$.

(ii) To see that $i(\lambda([z_2]), \lambda([c_2])) \neq 0$, we observe that $i(\lambda([z_2]), \lambda([x])) = 0$ for every $x \in P \setminus \{c_2\}$. So either $i(\lambda([z_2]), \lambda([c_2])) \neq 0$ or $\lambda([z_2]) = \lambda([c_2])$. We have $i(\lambda([y]), \lambda([c_2])) \neq 0$ by Lemma 2.4. But since $i([y], [z_2]) = 0$, we have $i(\lambda([y]), \lambda([z_2])) = 0$. So $\lambda([z_2])$ cannot be equal to $\lambda([c_2])$. Hence, $i(\lambda([z_2]), \lambda([c_2])) \neq 0$. Similarly, we see that $i(\lambda([z_i]), \lambda([c_i])) \neq 0$ for all $i = 3, 4, \cdots, n-1$.

To see that if $x, y \in P$ and x is not adjacent to y with respect to P, then $\lambda([x])$ and $\lambda([y])$ have representatives in P' which are not adjacent to each other with respect to P', it is enough to find two disjoint curves w and t such that w intersects only x nontrivially and not the other curves in P, that t intersects only y nontrivially and not the other curves in P, and that $i(\lambda([w]), \lambda([x])) \neq 0$; $i(\lambda([t]), \lambda([y])) \neq 0$; $i(\lambda([w]), \lambda([q])) = 0$ for all $q \in P \setminus \{x\}$; $i(\lambda([t]), \lambda([q])) = 0$ for all $q \in P \setminus \{y\}$; $i(\lambda([t]), \lambda([w])) = 0$. For the pair a and c_i, when $i = 2, 3, \cdots, n-1$, the curves b and z_i would satisfy this where the curve b is as shown in Figure 3(v). For the pair c_1 and c_i, when $i = 3, 4, \cdots, n-1$, the curves z and z_i would satisfy this. For the pair c_2 and c_i, when $i = 4, 5, \cdots, n-1$, the curves z_2 and z_i would satisfy this. Similarly, we see that nonadjacency is preserved for every nonadjacent pair in P. □

Lemma 2.7. *If α_1 and α_2 are two vertices of $\mathcal{C}(R)$ with $i(\alpha_1, \alpha_2) = 1$, then $i(\lambda(\alpha_1), \lambda(\alpha_2)) = 1$.*

Proof. Let a and b be representatives of α_1 and α_2, respectively. We will complete a and b to a curve configuration $\{a, b, c, d, e, f\}$ as shown in Figure 5(i). We can let $c_1 = c$ and $c_2 = e$ and complete $\{a, c, e\}$ to a pants decomposition P as in Lemma 2.5, and using that adjacency and nonadjacency are preserved with respect to P' by Lemma 2.5 and Lemma 2.6, we can see that $\lambda([c])$ has a representative c' which is a separating curve that separates the surface into two pieces and one of these is a torus T with one boundary component and $\lambda([a])$ has a nonseparating representative, say a', in T. Let b', d', e', and f' be minimally intersecting representatives of $\lambda([b])$, $\lambda([d])$, $\lambda([e])$, and $\lambda([f])$, respectively, such that all the curves a', b', c', d', e', and f' minimally intersect each other. By Lemma 2.3, we know that $i([a'], [b']) \neq 0$ and $i([b'], [d']) \neq 0$.

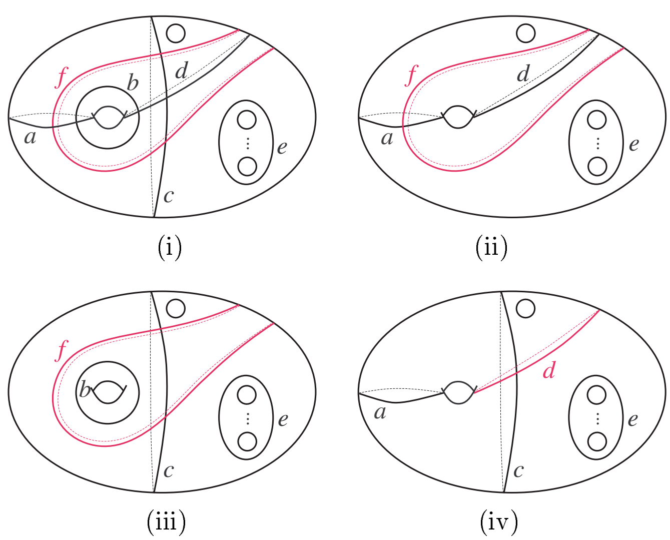

FIGURE 5. Intersection one

We will prove that $i([f'],[a']) \neq 0$, $i([f'],[c']) \neq 0$, and $i([d'],[c']) \neq 0$. To see $i([f'],[a']) \neq 0$, let $U = (P \setminus \{c_1\}) \cup \{d\}$. Then U is a pants decomposition on R and $i(\lambda([f]),\lambda([x])) = 0$ for every $x \in U \setminus \{a\}$; see Figure 5(i). So either $i(\lambda([f]),\lambda([a])) \neq 0$ or $\lambda([f]) = \lambda([a])$. By Lemma 2.3, $i(\lambda([a]),\lambda([b])) \neq 0$. Since $i(([f]),[b]) = 0$, we have $i(\lambda([f]),\lambda([b])) = 0$. So $\lambda([f])$ cannot be equal to $\lambda([a])$. Hence, $i(\lambda([f]),\lambda([a])) \neq 0$.

To see that $i([f'],[c']) \neq 0$, let $V = (P \setminus \{a\}) \cup \{b\}$. Then V is a pants decomposition on R and $i(\lambda([f]),\lambda([x])) = 0$ for every $x \in V \setminus \{c\}$; see Figure 5(iii). So either $i(\lambda([f]),\lambda([c])) \neq 0$ or $\lambda([f]) = \lambda([c])$. By the above paragraph, we know that $i(\lambda([f]),\lambda([a])) \neq 0$. Since $i(([a]),[c]) = 0$, we have $i(\lambda([a]),\lambda([c])) = 0$. So $\lambda([f])$ cannot be equal to $\lambda([c])$. Hence, $i(\lambda([f]),\lambda([c])) \neq 0$.

To see that $i([d'],[c']) \neq 0$, we observe that $i(\lambda([d]),\lambda([x])) = 0$ for every $x \in P \setminus \{c\}$; see Figure 5(iv). So either $i(\lambda([d]),\lambda([c])) \neq 0$ or $\lambda([d]) = \lambda([c])$. By Lemma 2.3, we know that $i(\lambda([b]),\lambda([d])) \neq 0$. Since $i(([b]),[c]) = 0$, we have $i(\lambda([b]),\lambda([c])) = 0$. So $\lambda([d])$ cannot be equal to $\lambda([c])$. Hence, $i(\lambda([d]),\lambda([c])) \neq 0$.

The above intersection information implies that there is an arc of d', say γ_1, in T that starts and ends at c' (the boundary of T) such that γ_1 is disjoint from a'. Also, there is an arc of f', say γ_2, in T that is disjoint from γ_1 and starts and ends at c'. Then, since b' is disjoint from $\gamma_2 \cup c'$ and b' intersects a' by Lemma 2.3, we see that $i(a',b') = 1$. $\square$

If $f : R \to R$ is a homeomorphism, then we will use the same notation for f and $[f]$. Let $\mathcal{C} = \{a_1, a_2, \cdots, a_n, b, m_1, m_2, \cdots, m_n, r_1, r_2, \cdots, r_n, v_2, v_3, \cdots, v_n\}$ where the curves are as shown in Figure 6.

Lemma 2.8. *There exists a homeomorphism $h : R \to R$ such that $h([x]) = \lambda([x])$ for all $x \in \mathcal{C}$.*

Proof. We will consider all the curves in $\mathcal{C}$ as shown in Figure 6. Let $a_i' \in \lambda([a_i])$, $b' \in \lambda([b])$, $m_i' \in \lambda([m_i])$, $r_i' \in \lambda([r_i])$, and $v_j' \in \lambda([v_j])$ where $i = 1, 2, \cdots, n$, $j = 2, 3, \cdots, n$ are minimally intersecting representatives.

By using Lemma 2.7 and that λ is edge preserving, we see that a regular neighborhood of $a_1' \cup a_2' \cup \cdots \cup a_n' \cup b$ is a torus with n boundary components as shown in Figure 7. So there exists a homeomorphism h such that $h([x]) = \lambda([x])$ for all $x \in \{a_1, a_2, \cdots, a_n, b\}$. This implies that if two nonseparating curves x and y and a boundary component of R bound a pair of pants, then $\lambda([x])$ and $\lambda([y])$ have representatives x' and y' such that x', y', and a boundary component of R bound a pair of pants on R.

We will now show that $h([m_i]) = \lambda([m_i])$ for all $i = 1, 2, \cdots, n$. The curve m_1 is the unique nontrivial curve up to isotopy that is disjoint from

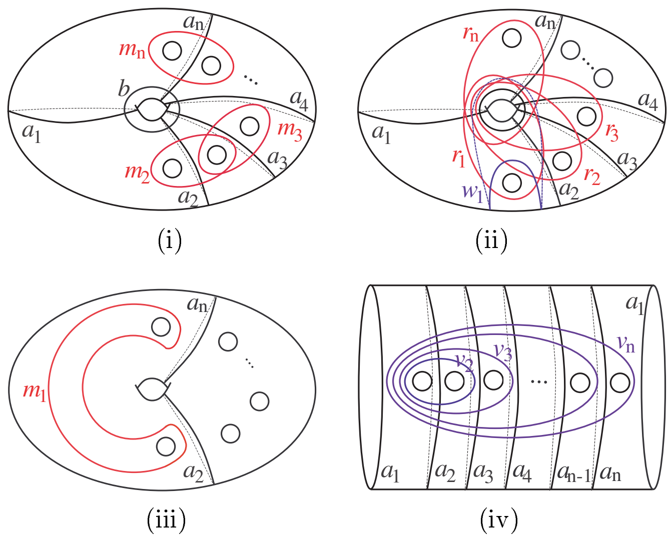

FIGURE 6. Curves in $\mathcal{C}$

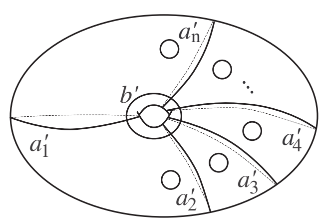

FIGURE 7. Curves

all the curves in $\{a_2, a_3, \cdots, a_n, b\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([m_1]) = \lambda([m_1])$. The curve m_2 is the unique nontrivial curve up to isotopy that is disjoint from all the curves in $\{a_3, a_4, \cdots, a_n, a_1, b\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([m_2]) = \lambda([m_2])$. Similarly, we have $h([m_i]) = \lambda([m_i])$ for all $i = 3, 4, \cdots, n$.

The curve $v_2 = m_2$, so $h([v_2]) = \lambda([v_2])$. The curve v_3 is the unique nontrivial curve up to isotopy that is disjoint from all the curves in $\{a_4, a_5, \cdots, a_n, a_1, b, m_2, m_3\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([v_3]) = \lambda([v_3])$. The curve v_4 is the unique nontrivial curve up to isotopy that is disjoint from all the curves in $\{a_5, a_6, \cdots, a_n, a_1, b, m_2, m_3, m_4\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([v_4]) = \lambda([v_4])$. Similarly, we have $h([v_i]) = \lambda([v_i])$ for all $i = 5, 6, \cdots, n$.

Consider the curve w_1 as shown in the Figure 6(ii). There exists a homeomorphism $\phi : R \to R$ of order two such that the map ϕ_* induced by ϕ on $\mathcal{C}(R)$ sends the isotopy class of each curve in $\{a_1, a_2, \cdots, a_n, m_1, m_2, \cdots, m_n\}$ to itself and switches $[r_1]$ and $[w_1]$. We can see that $\lambda([r_1]) \neq \lambda([w_1])$ as follows: Consider the curve y we had in Lemma 2.4. We will first prove that $i(\lambda([w_1]), \lambda([y])) \neq 0$. We complete y to a pants decomposition P on R such that $i([w_1], [x]) = 0$ for every $x \in P \setminus \{y\}$; see Figure 8(i). Then we will have $i(\lambda([w_1]), \lambda([x])) = 0$ for every $x \in P \setminus \{y\}$. So either $i(\lambda([w_1]), \lambda([y])) \neq 0$ or $\lambda([w_1]) = \lambda([y])$. By Lemma 2.4, we know that $i(\lambda([y]), \lambda([c_{n-1}])) \neq 0$; see Figure 8(ii). We also see that $i(\lambda([w_1]), \lambda([c_{n-1}])) = 0$. So $\lambda([w_1]) \neq \lambda([y])$. Hence, $i(\lambda([w_1]), \lambda([y])) \neq 0$. Since $i(\lambda([y]), \lambda([r_1])) = 0$ and $i(\lambda([w_1]), \lambda([y])) \neq 0$, we see that $\lambda([r_1]) \neq \lambda([w_1])$.

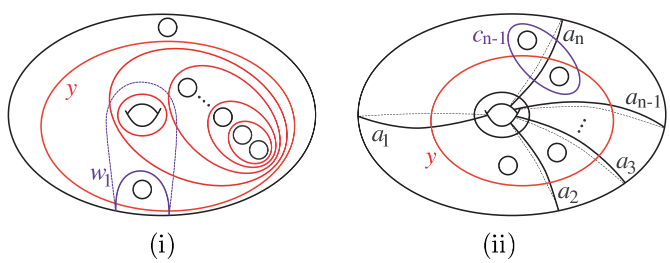

FIGURE 8. Curves

There are only two nontrivial curves, namely r_1 and w_1, up to isotopy that are disjoint from each of $m_3, m_4, \cdots, m_n$, bound a pair of pants with b and a boundary component of R, and intersect each of $a_1, a_2, \cdots, a_n$ once. Since we know that $h([x]) = \lambda([x])$ for all these curves, λ preserves these properties by Lemma 2.7, and $\lambda([r_1]) \neq \lambda([w_1])$; by replacing λ with $\lambda \circ \phi_*$ if necessary, we can assume that we have $h([r_1]) = \lambda([r_1])$ and $h([w_1]) = \lambda([w_1])$. To get the proof of the lemma, it is enough to prove the result for this λ. The curve r_2 is the unique nontrivial curve up to isotopy that is disjoint from each of $m_4, m_5, \cdots, m_n, m_1, w_1$, bounds a pair of pants with b and a boundary component of R, and intersects each of $a_1, a_2, \cdots, a_n$ once. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ preserves these properties, we see that $h([r_2]) = \lambda([r_2])$. Similarly, we get $h([r_i]) = \lambda([r_i])$ for all $i = 3, 4, \cdots, n$. Hence, $h([x]) = \lambda([x])$ for all $x \in \mathcal{C}$. $\square$

Consider the curves given in Figure 6(i). Let t_x be the Dehn twist about x. Let σ_i be the half twist along m_i. The mapping class group Mod_R can be generated by $\{t_x : x \in \{a_1, a_2, \cdots, a_n, b\}\} \cup \{\sigma_2, \sigma_3, \cdots, \sigma_n\}$; see [4, Corollary 4.15]. Let $G = \{t_x : x \in \{a_1, a_2, \cdots, a_n, b\}\} \cup \{\sigma_2, \sigma_3, \cdots, \sigma_n\}$. Let $h : R \to R$ be a homeomorphism which satisfies the statement of Lemma 2.8. We know $h([x]) = \lambda([x])$ for all $x \in \mathcal{C}$. We will follow the techniques given by Irmak and Paris [13] to obtain the homeomorphism we want. We will say that a subset $\mathcal{A} \subset \mathcal{C}(R)$ has *trivial stabilizer* if we have the following: $h \in Mod_R^*$ and $h([x]) = [x]$ for every vertex $x \in A$ implies that h is the identity.

Lemma 2.9. *For all $f \in G$, there exists a set $L_f \subset \mathcal{C}(R)$ such that $\lambda([x]) = h([x])$ for all $x \in L_f \cup f(L_f)$. The set L_f can be chosen to have trivial stabilizer.*

Proof. We have $h([x]) = \lambda([x])$ for all $x \in \mathcal{C}$ by Lemma 2.8. Let $f \in G$. For $f = t_b$, let $L_f = \{a_1, a_2, \cdots, a_n, b, r_1\}$. The set L_f has trivial stabilizer. We know $\lambda([x]) = h([x])$ for all $x \in L_f$. We need to check the equation for $t_b(a_i)$; the other curves in L_f are fixed by t_b. We will first check the equation for $t_b(a_n)$. Consider the curves given in Figure 9. The curve s_1 is the unique nontrivial curve up to isotopy that is disjoint from all the curves in $\{a_1, r_n, v_{n-1}\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([s_1]) = \lambda([s_1])$. The curve $t_b(a_n)$ is the unique nontrivial curve up to isotopy that is disjoint from all the curves in $\{m_1, s_1, v_{n-1}\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([t_b(a_n)]) = \lambda([t_b(a_n)])$.

The curve $t_b(a_1)$ is the unique nontrivial curve up to isotopy that is disjoint from $t_b(a_n)$ and v_n and that intersects each of a_1 and b nontrivially once. Since we know that $h([x]) = \lambda([x])$ for all these curves and that λ is edge preserving and preserves intersection one, we have $h([t_b(a_1)]) = \lambda([t_b(a_1)])$. The curve $t_b(a_2)$ is the unique nontrivial curve up to isotopy that is disjoint from all the curves in $\{t_b(a_n), m_1, m_3, m_4, \cdots, m_n\}$ and that intersects each of a_2 and b nontrivially once. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving and preserves intersection one, we have $h([t_b(a_2)]) = \lambda([t_b(a_2)])$. Similarly, we get $h([t_b(a_i)]) = \lambda([t_b(a_i)])$ for all $i = 3, 4, \cdots, n-1$. This proves the statement of the lemma for $f = t_b$.

For $f = t_{a_2}$, let $L_f = \{a_1, a_2, \cdots, a_n, b, r_2\}$. The set L_f has trivial stabilizer. We know $\lambda([x]) = h([x])$ for all $x \in L_f$. We just need to check the equation for $t_{a_2}(b)$ and $t_{a_2}(r_2)$ since the other curves in L_f are fixed by t_{a_2}. Consider the curves given in Figure 9(v). The curve s_2 is the

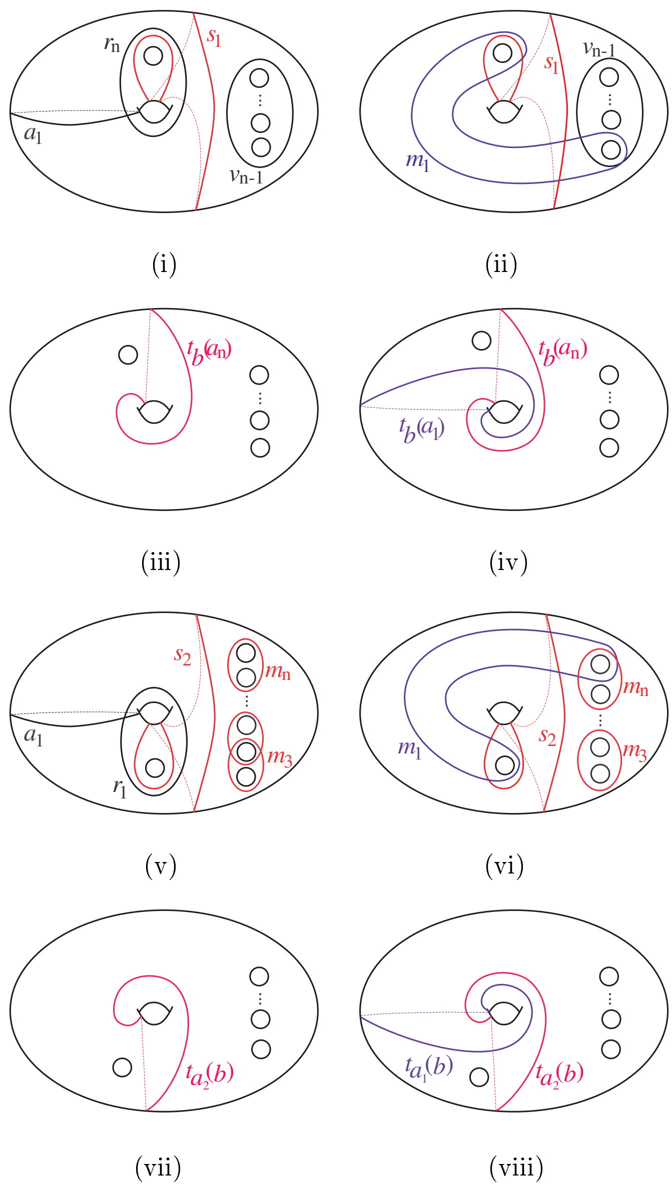

FIGURE 9. Twists

unique nontrivial curve up to isotopy that is disjoint from all the curves in $\{a_1, r_1, m_3, m_4, \cdots, m_n\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([s_2]) = \lambda([s_2])$. The curve $t_{a_2}(b)$ is the unique nontrivial curve up to isotopy that is disjoint from all the curves in $\{m_1, s_2, m_3, m_4, \cdots, m_n\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([t_{a_2}(b)]) = \lambda([t_{a_2}(b)])$. The curve $t_{a_1}(b)$ is the unique nontrivial curve up to isotopy that is disjoint from $t_{a_2}(b)$ and v_n. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([t_{a_1}(b)]) = \lambda([t_{a_1}(b)])$. Similarly, we get $h([t_{a_i}(b)]) = \lambda([t_{a_i}(b)])$ for all $i = 3, 4, \cdots, n$. The curve $t_{a_2}(r_2)$ is the unique nontrivial curve up to isotopy that is disjoint from each of $m_1, m_2, m_4, m_5, \cdots, m_n, t_{a_1}(b)$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([t_{a_2}(r_2)]) = \lambda([t_{a_2}(r_2)])$. This proves the statement of the lemma for $f = t_{a_2}$.

Similarly, for $f = t_{a_j}$ when $j \in \{1, 3, 4, \cdots, n\}$, let $L_f = \{a_1, a_2, \cdots, a_n, b, r_j\}$. The set L_f has trivial stabilizer. We know $\lambda([x]) = h([x])$ for all $x \in L_f$. We just need to check the equation for $t_{a_j}(b)$ and $t_{a_j}(r_j)$ since the other curves in L_f are fixed by t_{a_j}. In the above paragraph we already obtained that $h([t_{a_j}(b)]) = \lambda([t_{a_j}(b)])$. When $j < n$, the curve $t_{a_j}(r_j)$ is the unique nontrivial curve up to isotopy that is disjoint from each of $m_1, m_2, \cdots, m_j, m_{j+2}, m_{j+3}, \cdots, m_n, t_{a_1}(b)$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([t_{a_j}(r_j)]) = \lambda([t_{a_j}(r_j)])$ when $j < n$. The curve $t_{a_n}(r_n)$ is the unique nontrivial curve up to isotopy that is disjoint from each of $m_2, m_3, \cdots, m_n, t_{a_1}(b)$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([t_{a_n}(r_n)]) = \lambda([t_{a_n}(r_n)])$. Hence, we obtain the statement of the lemma for $f = t_{a_j}$ for all $j \in \{1, 2, \cdots, n\}$.

For $f = \sigma_i$, where $i \in \{2, 3, \cdots, n\}$, we let $L_f = \{a_1, a_2, \cdots, a_n, b, r_o\}$ where $r_o \in \{r_1, r_2, \cdots, r_n\}$ such that r_o is disjoint from m_i. We know that $\lambda([x]) = h([x])$ for all $x \in L_f$. We just need to check that $h([\sigma_i(a_i)]) = \lambda([\sigma_i(a_i)])$ for each i since the other curves in L_f are fixed by σ_i. For $i = 2$, we use the curve u_1 shown in Figure 10(i). The curve u_1 is the unique nontrivial curve up to isotopy that is disjoint from $a_3, a_4, \cdots, a_n, b, r_1$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([u_1]) = \lambda([u_1])$. The curve $\sigma_2(a_2)$, which is shown as j_1 in Figure 10(ii), is the unique curve up to isotopy disjoint from $a_1, a_3, a_4, \cdots, a_n, u_1$ which intersects b once and is nonisotopic to a_3. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ preserves these properties, we see that $h([\sigma_2(a_2)]) = \lambda([\sigma_2(a_2)])$. For $i = 3$, we use the curve u_2 shown in Figure 10(iii). The curve u_2 is the unique nontrivial curve up to isotopy that is disjoint from $a_4, a_5, \cdots, a_n, b, r_1, r_2$.

Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([u_2]) = \lambda([u_2])$. The curve $\sigma_3(a_3)$, which is shown as j_2 in Figure 10(iv), is the unique curve up to isotopy disjoint from $a_1, a_2, a_4, a_5, \cdots, a_n, u_2$ which intersects b once and is nonisotopic to a_4. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ preserves these properties, we see that $h([\sigma_3(a_3)]) = \lambda([\sigma_3(a_3)])$. Similarly, we get $h([\sigma_i(a_i)]) = \lambda([\sigma_i(a_i)])$ for each $i = 4, 5, \cdots, n$. ☐

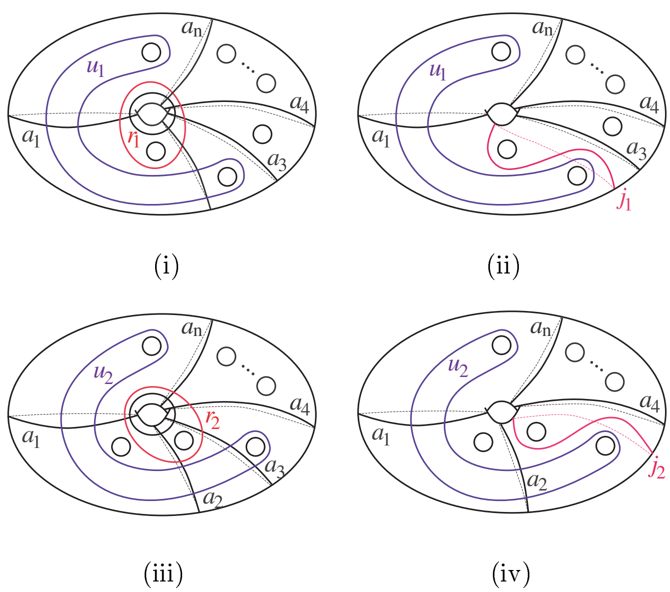

FIGURE 10. Twists, Half-twists

Theorem 2.10. *There exists a homeomorphism $h : R \to R$ such that $H(\alpha) = \lambda(\alpha)$ for every vertex α in $\mathcal{C}(R)$ where $H = [h]$, and this homeomorphism is unique up to isotopy.*

Proof. Let $f \in G$. There exists $L_f \subset \mathcal{C}(R)$ which satisfies the statement of Lemma 2.9. Consider $\mathcal{C}$ given in Lemma 2.8. Let $\mathcal{X} = \mathcal{C} \cup \left(\bigcup_{f \in G}(L_f \cup f(L_f))\right)$. For each vertex x in the curve complex, there exist $r \in Mod_R$ and a vertex y in the set $\mathcal{X}$ such that $r(y) = x$. By following the construction given in [12], we let $\mathcal{X}_1 = \mathcal{X}$ and $\mathcal{X}_k = \mathcal{X}_{k-1} \cup (\bigcup_{f \in G}(f(\mathcal{X}_{k-1}) \cup f^{-1}(\mathcal{X}_{k-1})))$ when $k \geq 2$. We observe that $\mathcal{C}(R) = \bigcup_{k=1}^{\infty} \mathcal{X}_k$. We will prove that $h([x]) = \lambda([x])$ for all $x \in \mathcal{X}_k$ for each $k \geq 1$. We will give the proof by induction on k. By using Lemma 2.8 and Lemma 2.9, we see that $h([x]) = \lambda([x])$ for each $x \in \mathcal{X}_1$. Assume that $h([x]) = \lambda([x])$ for all $x \in \mathcal{X}_{k-1}$ for some $k \geq 2$. Let $f \in G$. There exists a homeomorphism h_f of R such that $h_f([x]) = \lambda([x])$ for all $x \in f(\mathcal{X}_{k-1})$. We have $f(L_f) \subset \mathcal{X}_{k-1} \cap f(\mathcal{X}_{k-1})$. This implies that we have $h_f = h$ since $f(L_f)$ has trivial stabilizer. Similarly, there exists a homeomorphism h'_f of R such that $h'_f([x]) = \lambda([x])$ for all $x \in f^{-1}(\mathcal{X}_{k-1})$. We have $L_f \subset \mathcal{X}_{k-1} \cap f^{-1}(\mathcal{X}_{k-1})$. This implies that we have $h'_f = h$ since L_f has trivial stabilizer. So $h([x]) = \lambda([x])$ for each $x \in \mathcal{X}_k$. Hence, by induction, $h([x]) = \lambda([x])$ for each $x \in \mathcal{X}_k$ for all $k \geq 1$. Since $\mathcal{C}(R) = \bigcup_{k=1}^{\infty} \mathcal{X}_k$, we have $h([x]) = \lambda([x])$ for every vertex $[x] \in \mathcal{C}(R)$. It is easy to see that this homeomorphism is unique up to isotopy. □

3. Edge Preserving Maps of $\mathcal{C}(R)$ When $g = 0$ and $n \geq 5$

In this section, we will always assume that $g = 0$, $n \geq 5$, and $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ is an edge preserving map. As in the second section, we have the following two lemmas.

Lemma 3.1. *The map λ is injective on every set of vertices in $\mathcal{C}(R)$ if each pair in the set has geometric intersection zero.*

Lemma 3.2. *Let P be a pants decomposition on R. A set of pairwise disjoint representatives of $\lambda([P])$ is a pants decomposition on R.*

Let $\mathcal{C}_1 = \{a_1, a_2, a_3, \cdots, a_{n-3}, b_1, b_2, b_3, \cdots, b_{n-3}, c\}$ where the curves are as shown in Figure 11(i). Let $P = \{a_1, a_2, a_3, \cdots, a_{n-3}\}$. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. For all i, let a'_i be the representative of $\lambda([a_i])$ in P' and let b'_i be the representative of $\lambda([b_i])$ such that every pair in $P' \cup \{b'_1, b'_2, b'_3, \cdots, b'_{n-3}\}$ intersects minimally. Let c' be the representative of $\lambda([c])$ that intersects the elements of $P' \cup \{b'_1, b'_2, b'_3, \cdots, b'_{n-3}\}$ minimally.

Lemma 3.3. *We have $i([a'_i], [b'_i]) \neq 0$ for all $i = 1, 2, \cdots, n-3$.*

Proof. We will first show that $i([a'_1], [b'_1]) \neq 0$. We see that $i([b_1], [x]) = 0$ for all $x \in P \setminus \{a_1\}$ and there is an edge between $[b_1]$ and $[x]$ for all $x \in P \setminus \{a_1\}$. Since λ is edge preserving, we have $i(\lambda([b_1]), \lambda([x])) = 0$ for all $x \in P \setminus \{a_1\}$ and there is an edge between $\lambda([b_1])$ and $\lambda([x])$ for all $x \in P \setminus \{a_i\}$. This implies that either $i(\lambda([b_1]), \lambda([a_1])) \neq 0$ or $\lambda([b_1]) = \lambda([a_1])$. With a similar argument, we can see that $i(\lambda([c]), \lambda([a_1])) \neq 0$ or $\lambda([c]) = \lambda([a_1])$. If $\lambda([b_1]) = \lambda([a_1])$, then we could not have $i(\lambda([c]), \lambda([a_1])) \neq 0$ or $\lambda([c]) = \lambda([a_1])$ since $\lambda([c])$ and $\lambda([b_1])$ are connected by an edge. So $i(\lambda([b_1]), \lambda([a_1])) \neq 0$.

To see that $i(\lambda([b_2]), \lambda([a_2])) \neq 0$, we observe that $i([b_2], [x]) = 0$ for all $x \in P \setminus \{a_2\}$ and there is an edge between $[b_2]$ and $[x]$ for all $x \in P \setminus \{a_2\}$.

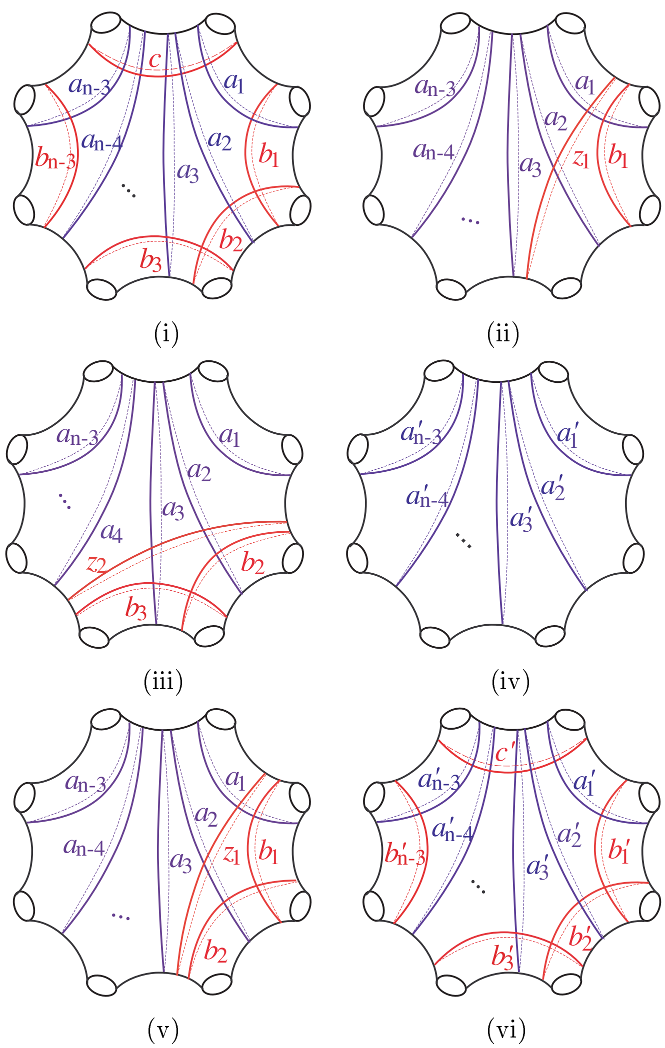

FIGURE 11. Curves in $\mathcal{C}_1$

Since λ is edge preserving, we have $i(\lambda([b_2]), \lambda([x])) = 0$ for all $x \in P \setminus \{a_2\}$ and there is an edge between $\lambda([b_2])$ and $\lambda([x])$ for all $x \in P \setminus \{a_2\}$. This implies that either $i(\lambda([b_2]), \lambda([a_2])) \neq 0$ or $\lambda([b_2]) = \lambda([a_2])$. Since $i(\lambda([b_1]), \lambda([a_1])) \neq 0$ and there is a homeomorphism sending the pair (a_1, b_1) to (b_1, b_2), we can see that $i(\lambda([b_1]), \lambda([b_2])) \neq 0$. If $\lambda([b_2]) = \lambda([a_2])$, then we could not have $i(\lambda([b_1]), \lambda([b_2])) \neq 0$ since $\lambda([b_1])$ and

$\lambda([a_2])$ are connected by an edge. So $i(\lambda([b_2]), \lambda([a_2])) \neq 0$. With similar arguments, we get $i(\lambda([b_i]), \lambda([a_i])) \neq 0$ for all $i = 1, 2, \cdots, n-3$. □

Lemma 3.4. *The curves a'_i and a'_{i+1} are adjacent to each other with respect to P' for all $i = 1, 2, \cdots, n-4$.*

Proof. We will first prove that a'_1 and a'_2 are adjacent to each other with respect to P'. Let z_1 be the curve shown in Figure 11(ii). The set $Q = (P \setminus \{a_1\}) \cup \{b_1\}$ is a pants decomposition on R. We see that $i([z_1], [x]) = 0$ for all $x \in Q \setminus \{a_2\}$ and there is an edge between $[z_1]$ and $[x]$ for all $x \in Q \setminus \{a_2\}$. Since λ is edge preserving, we have $i(\lambda([z_1]), \lambda([x])) = 0$ for all $x \in Q \setminus \{a_2\}$ and there is an edge between $\lambda([z_1])$ and $\lambda([x])$ for all $x \in Q \setminus \{a_2\}$. This implies that either $i(\lambda([z_1]), \lambda([a_2])) \neq 0$ or $\lambda([z_1]) = \lambda([a_2])$. Since $i(\lambda([z_1]), \lambda([b_2])) = 0$ and $i(\lambda([a_2]), \lambda([b_2])) \neq 0$ by Lemma 3.3, we cannot have $\lambda([z_1]) = \lambda([a_2])$. So $i(\lambda([z_1]), \lambda([a_2])) \neq 0$. Since $i(\lambda([a_2]), \lambda([b_2])) \neq 0$ by Lemma 3.3 and there is a homeomorphism sending the pair (a_2, b_2) to (z_1, a_1), we can see that $i(\lambda([z_1]), \lambda([a_1])) \neq 0$. Since $i(\lambda([z_1]), \lambda([a_1])) \neq 0$, $i(\lambda([z_1]), \lambda([a_2])) \neq 0$, and $i(\lambda([z_1]), \lambda([x])) = 0$ for all $x \in P \setminus \{a_1, a_2\}$, we see that a'_1 and a'_2 are adjacent to each other with respect to P'.

Consider the curve z_2 given in Figure 11(iii). The set $T = (P \setminus \{a_2\}) \cup \{b_2\}$ is a pants decomposition on R. We see that $i([z_2], [x]) = 0$ for all $x \in T \setminus \{a_3\}$ and there is an edge between $[z_2]$ and $[x]$ for all $x \in T \setminus \{a_3\}$. Since λ is edge preserving, we have $i(\lambda([z_2]), \lambda([x])) = 0$ for all $x \in T \setminus \{a_3\}$ and there is an edge between $\lambda([z_2])$ and $\lambda([x])$ for all $x \in T \setminus \{a_3\}$. This implies that either $i(\lambda([z_2]), \lambda([a_3])) \neq 0$ or $\lambda([z_2]) = \lambda([a_3])$. Since $i(\lambda([z_2]), \lambda([b_3])) = 0$ and $i(\lambda([a_3]), \lambda([b_3])) \neq 0$ by Lemma 3.3, we cannot have $\lambda([z_2]) = \lambda([a_3])$. So $i(\lambda([z_2]), \lambda([a_3])) \neq 0$. The set $V = (P \setminus \{a_3\}) \cup \{b_3\}$ is a pants decomposition on R. We see that $i([z_2], [x]) = 0$ for all $x \in V \setminus \{a_2\}$ and there is an edge between $[z_2]$ and $[x]$ for all $x \in V \setminus \{a_2\}$. Since λ is edge preserving, we have $i(\lambda([z_2]), \lambda([x])) = 0$ for all $x \in V \setminus \{a_2\}$ and there is an edge between $\lambda([z_2])$ and $\lambda([x])$ for all $x \in V \setminus \{a_2\}$. This implies that either $i(\lambda([z_2]), \lambda([a_2])) \neq 0$ or $\lambda([z_2]) = \lambda([a_2])$. Since $i(\lambda([z_2]), \lambda([b_2])) = 0$ and $i(\lambda([a_2]), \lambda([b_2])) \neq 0$ by Lemma 3.3, we cannot have $\lambda([z_2]) = \lambda([a_2])$. So $i(\lambda([z_2]), \lambda([a_2])) \neq 0$. Since $i(\lambda([z_2]), \lambda([a_2])) \neq 0$, $i(\lambda([z_2]), \lambda([a_3])) \neq 0$, and $i(\lambda([z_2]), \lambda([x])) = 0$ for all $x \in P \setminus \{a_2, a_3\}$, we see that a'_2 and a'_3 are adjacent to each other with respect to P'. Similarly, a'_i and a'_{i+1} are adjacent to each other with respect to P' for all $i = 1, 2, \cdots, n-4$. □

Lemma 3.5. *If $x, y \in P$ and x is not adjacent to y with respect to P, then $\lambda([x])$ and $\lambda([y])$ have representatives in P' which are not adjacent to each other with respect to P'.*

Proof. It is enough to find two disjoint curves w and t such that w intersects only x nontrivially and not the other curves in P; t intersects only y nontrivially and not the other curves in P; and $i(\lambda([w]), \lambda([x])) \neq 0$; $i(\lambda([t]), \lambda([y])) \neq 0$; $i(\lambda([w]), \lambda([q])) = 0$ for all $q \in P \setminus \{x\}$; $i(\lambda([t]), \lambda([q])) = 0$ for all $q \in P \setminus \{y\}$; $i(\lambda([t]), \lambda([w])) = 0$. By using Lemma 3.3, we can see that for the pair a_i and a_j that are not adjacent to each other with respect to P, the curves b_i and b_j would satisfy the above properties. So we see that nonadjacency is preserved for every nonadjacent pair in P. □

Lemma 3.6. *There exists a homeomorphism $h : R \to R$ such that $h([x]) = \lambda([x])$ for all $x \in P = \{a_1, a_2, \cdots, a_{n-3}\}$.*

Proof. The proof follows from Lemma 3.4 and Lemma 3.5; see Figure 11(iv). □

Lemma 3.7. *We have the following: $i([a'_1], [b'_1]) = 2$; $i([a'_{n-3}], [b'_{n-3}]) = 2$; $i([a'_{n-3}], [c']) = 2$; $i([c'], [a'_1]) = 2$; and $i([b'_i], [b'_{i+1}]) = 2$ for all $i = 1, 2, \cdots, n-4$.*

Proof. We will give the proof when $n \geq 6$. The proof is similar when $n = 5$. We will first show that $i([a'_1], [b'_1]) = 2$. Consider the curves given in Figure 11(v). By Lemma 3.6, we have $h([x]) = \lambda([x])$ for all $x \in \{a_1, a_2, \cdots, a_{n-3}\}$. Let a'_i be as shown in Figure 11(iv). Let M' be the connected component of $R_{a'_3}$ (cut surface along a'_3) bounded by a'_3 and four boundary components of R containing a'_1. Let z'_1 be a representative of $\lambda([z_1])$ which intersects minimally with all the elements in $\{a'_1, a'_2, a'_3, b'_1, b'_2\}$. By Lemma 3.3, we have $i([a'_1], [b'_1]) \neq 0$. Since there exists a homeomorphism sending the pair (a_1, b_1) to (b_1, b_2) and $i([a'_1], [b'_1]) \neq 0$, by using similar curve configurations, we see that $i([b'_1], [b'_2]) \neq 0$.

By Lemma 3.3, we have $i([a'_2], [b'_2]) \neq 0$. In the proof of Lemma 3.4, we showed that $i([z'_1], [a'_1]) \neq 0$ and $i([z'_1], [a'_2]) \neq 0$.

By using the intersection information for each pair of curves in $\{a'_1, a'_2, a'_3, b'_1, b'_2, z'_1\}$ and using that λ is edge preserving, we can see that the curves $a'_1, b'_2, z'_1, b'_1, a'_2$ form a pentagon in $C(R)$; see [15]. Since a'_1 is a curve that separates a pair of pants and there is a homeomorphism sending a_1 to b_2, by using similar curve configurations, we can see that b'_2 is a curve that separates a pair of pants. Since a'_2 is a curve that separates a genus zero surface with four boundary components and there is a homeomorphism sending a_2 to z_1, we see that z'_1 is a curve that separates a genus zero surface with four boundary components on R. Using all this information about these curves and [15, Theorem 3.2], we get $i([a'_1], [b'_1]) = 2$. Since, for each of the remaining pairs (x, y) in the statement of the lemma, there exists a homeomorphism sending the pair (a_1, b_1) to (x, y) and $i([a_1'], [b_1']) = 2$, by using similar curve configurations, we get $i([x], [y]) = 2$. □

If $f : R \to R$ is a homeomorphism, then we will use the same notation for f and $[f]$. Recall that $\mathcal{C}_1 = \{a_1, a_2, a_3, \cdots, a_{n-3}, b_1, b_2, b_3, \cdots, b_{n-3}, c\}$ where the curves are as shown in Figure 11(i). Let $\mathcal{C}_2 = \{w_1, w_2, \cdots, w_n, r_1, r_2, \cdots, r_n\}$ where the curves are as shown in Figure 12.

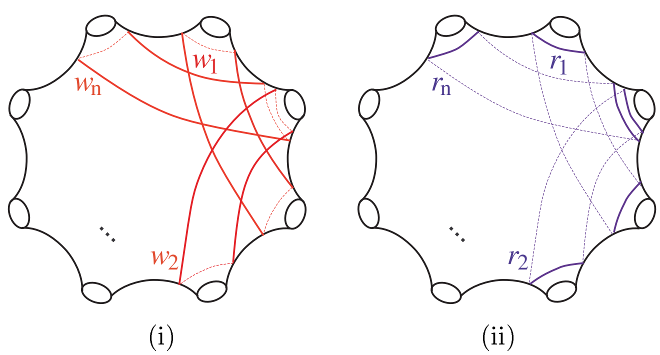

FIGURE 12. Curves in $\mathcal{C}_2$

Lemma 3.8. *There exists a homeomorphism $h : R \to R$ such that $h([x]) = \lambda([x])$ for all $x \in \mathcal{C}_1$.*

Proof. The proof follows from Lemma 3.6, Lemma 3.7, and the fact that λ is edge preserving; see Figure 11(vi). □

Lemma 3.9. *There exists a homeomorphism $h : R \to R$ such that $h([x]) = \lambda([x])$ for all $x \in \mathcal{C}_1 \cup \mathcal{C}_2$.*

Proof. Let $h : R \to R$ be a homeomorphism which satisfies the statement of Lemma 3.8. We will give the proof when $n \geq 6$. The proof for $n = 5$ is similar. Consider the curves in $\mathcal{C}_2$ given in Figure 12. There exists a homeomorphism $\phi : R \to R$ of order two such that the map ϕ_* induced by ϕ on $\mathcal{C}(R)$ sends the isotopy class of each curve in $\mathcal{C}_1$ to itself and switches $[r_1]$ and $[w_1]$. Since there is a homeomorphism sending the pair (a_1, b_1) to (a_1, r_1), by using Lemma 3.7, we see that $i(\lambda[a_1], \lambda[r_1]) = 2$. Similarly, we have $i(\lambda[a_1], \lambda[w_1]) = 2$; $i(\lambda[b_1], \lambda[r_1]) = 2$; and $i(\lambda[b_1], \lambda[w_1]) = 2$. The curves r_1 and w_1 are the only nontrivial curves up to isotopy disjoint from a_2, and intersect each of a_1 and b_1 nontrivially twice in the four-holed sphere cut by a_2. Since we know that $h([x]) = \lambda([x])$ for all these curves,

λ preserves these properties; by replacing λ with $\lambda \circ \phi_*$, if necessary, we can assume that we have $h([w_1]) = \lambda([w_1])$.

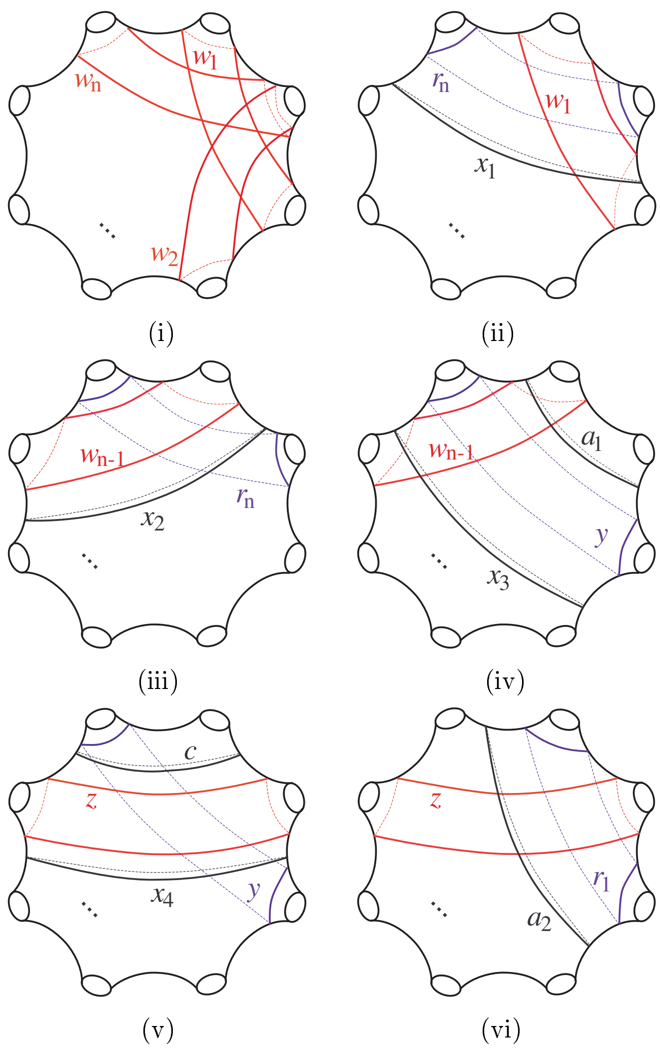

FIGURE 13. Curves

Consider the curves given in Figure 13. The curve x_1 is the unique nontrivial curve up to isotopy that is disjoint from all the curves in $\{c, a_1, b_2, b_3, \cdots, b_{n-3}\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([x_1]) = \lambda([x_1])$. The curve r_n is the unique nontrivial curve up to isotopy that is nonisotopic to and disjoint from each curve in $\{x_1, w_1, b_2, b_3, \cdots, b_{n-3}\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ preserves these properties, we have $h([r_n]) = \lambda([r_n])$. The curve x_2 is the unique nontrivial curve up to isotopy that is disjoint from all the curves in $\{c, a_{n-3}, b_1, b_2, \cdots, b_{n-4}\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([x_2]) = \lambda([x_2])$. The curve w_{n-1} is the unique nontrivial curve up to isotopy that is nonisotopic to and disjoint from each curve in $\{x_2, r_n, b_1, b_2, \cdots, b_{n-4}\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ preserves these properties, we have $h([w_{n-1}]) = \lambda([w_{n-1}])$.

The curve x_3 is the unique nontrivial curve up to isotopy that is disjoint from all the curves in $\{c, a_1, b_1, b_3, b_4, \cdots, b_{n-3}\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([x_3]) = \lambda([x_3])$. The curve y is the unique nontrivial curve up to isotopy that is nonisotopic to and disjoint from all the curves in $\{a_1, x_3, w_{n-1}, b_3, b_4, \cdots, b_{n-3}\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ preserves these properties, we have $h([y]) = \lambda([y])$. The curve x_4 is the unique nontrivial curve up to isotopy that is disjoint from all the curves in $\{c, a_1, b_2, b_3, \cdots, b_{n-4}, a_{n-3}\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ is edge preserving, we have $h([x_4]) = \lambda([x_4])$. The curve z is the unique nontrivial curve up to isotopy that is nonisotopic to and disjoint from all the curves in $\{c, y, x_4, b_2, b_3, \cdots, b_{n-4}\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ preserves these properties, we have $h([z]) = \lambda([z])$. The curve r_1 is the unique nontrivial curve up to isotopy that is nonisotopic to and disjoint from each curve in $\{a_2, z, b_3, b_4, \cdots, b_{n-3}, a_{n-3}\}$. Since we know that $h([x]) = \lambda([x])$ for all these curves and λ preserves these properties, we have $h([r_1]) = \lambda([r_1])$. Hence, we have $h([w_1]) = \lambda([w_1])$ and $h([r_1]) = \lambda([r_1])$. Similarly, we get $h([w_i]) = \lambda([w_i])$ for all $i = 2, 3, \cdots, n$ and $h([r_i]) = \lambda([r_i])$ for all $i = 2, 3, \cdots, n$. □

We will use the notation h_x for the half twist along x. Consider the curves in Figure 11(i). The group Mod_R can be generated by $\{h_x : x \in \{a_1, b_1, b_2, \cdots, b_{n-3}, a_{n-3}, c\}\}$; see [4, Corollary 4.15]. Let $G = \{h_x : x \in \{a_1, b_1, b_2, \cdots, b_{n-3}, a_{n-3}, c\}\}$. Let $h : R \to R$ be a homeomorphism which satisfies the statement of Lemma 3.9. We know $h([x]) = \lambda([x])$ for all $x \in \mathcal{C}_1 \cup \mathcal{C}_2$. We will follow the techniques given by Irmak and Paris [13] to obtain the homeomorphism we want.

Lemma 3.10. *For all $f \in G$, there exists a set $L_f \subset \mathcal{C}(R)$ such that $\lambda([x]) = h([x])$ for all $x \in L_f \cup f(L_f)$. The set L_f can be chosen to have trivial stabilizer.*

Proof. We have $h([x]) = \lambda([x])$ for all $x \in \mathcal{C}_1 \cup \mathcal{C}_2$ by Lemma 3.9. Let $f \in G$. For $f = h_{b_1}$, let $L_f = \{a_1, b_1, b_2, \cdots, b_{n-3}, a_{n-3}, c, w_{n-1}\}$. The set L_f has trivial stabilizer. We know $\lambda([x]) = h([x])$ for all $x \in L_f$. We will check the equation for $h_{b_1}(a_1)$ and $h_{b_1}(b_2)$ since the other curves in L_f are fixed by h_{b_1}. Consider the curves given in Figure 14(i),(ii). We see that $w_1 = h_{b_1}(a_1)$ and $r_2 = h_{b_1}(b_2)$. So, by Lemma 3.9, we have $\lambda([h_{b_1}(a_1)]) = h([h_{b_1}(a_1)])$ and $\lambda([h_{b_1}(b_2)]) = h([h_{b_1}(b_2)])$. So, when $f = h_{b_1}$, we have $\lambda([x]) = h([x])$ for all $x \in L_f \cup f(L_f)$.

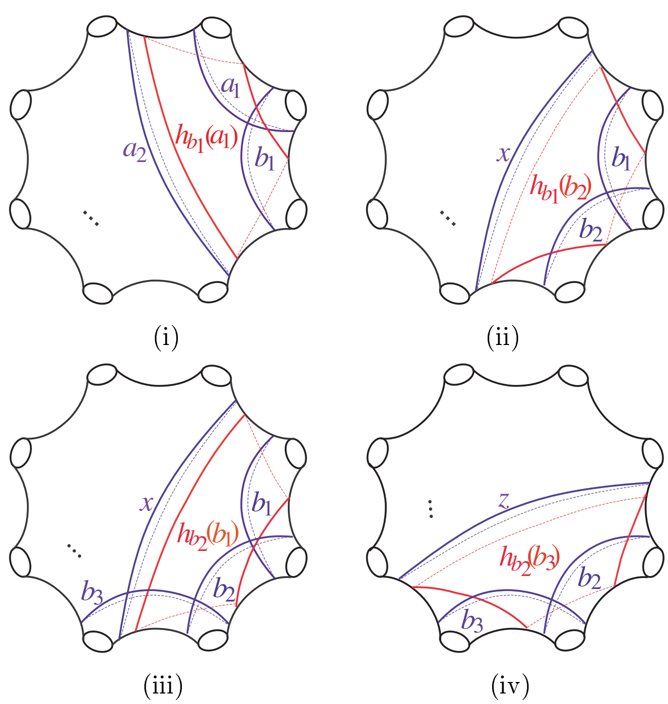

FIGURE 14. Half-twists

For $f = h_{b_2}$, let $L_f = \{a_1, b_1, b_2, \cdots, b_{n-3}, a_{n-3}, c, w_n\}$. The set L_f has trivial stabilizer. We know $\lambda([x]) = h([x])$ for all $x \in L_f$. We will check the equation for $h_{b_2}(b_1)$ and $h_{b_2}(b_3)$ since the other curves in L_f are fixed by h_{b_2}. Consider the curves given in Figure 14(iii),(iv). We see that

$w_2 = h_{b_2}(b_1)$ and $r_3 = h_{b_2}(b_3)$. So, by Lemma 3.9, we have $\lambda([h_{b_2}(b_1)]) = h([h_{b_2}(b_1)])$ and $\lambda([h_{b_2}(b_3)]) = h([h_{b_2}(b_3)])$. So, when $f = h_{b_2}$, we have $\lambda([x]) = h([x])$ for all $x \in L_f \cup f(L_f)$.

For $f \in G \setminus \{h_{b_1}, h_{b_2}\}$, similarly we let $L_f = \{a_1, b_1, b_2, \cdots, b_{n-3}, a_{n-3}, c, w_f\}$ where $w_f \in \{w_1, w_2, \cdots, w_n\}$ and w_f is fixed by f. Similar to the previous cases, we have $\lambda([x]) = h([x])$ for all $x \in L_f \cup f(L_f)$. □

Theorem 3.11. *There exists a homeomorphism $h : R \to R$ such that $H(\alpha) = \lambda(\alpha)$ for every vertex α in $\mathcal{C}(R)$ where $H = [h]$, and this homeomorphism is unique up to isotopy.*

Proof. The proof is similar to the proof of Theorem 2.10 using Lemma 3.9 and Lemma 3.10. □

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Department of Mathematics; University of Michigan; Ann Arbor, MI 48105

Email address: eirmak@umich.edu