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ENDPOINTS OF INVERSE LIMITS FOR A FAMILY OF SET-VALUED FUNCTIONS

LORI ALVIN AND JAMES P. KELLY

Abstract. We study the endpoints of inverse limits of set-valued functions. In a previous article (2016), one of the authors studied this topic using R. H. Bing's definition of endpoints (most often associated with chainable continua), and showed that if a set-valued function F has its inverse equal to the union of continuous, single-valued functions, then a point $\mathbf{p} = (p_0, p_1, \ldots)$ is an endpoint of $\varprojlim F$ if and only if $\pi_{[0,n]}(\mathbf{p})$ is an endpoint of $\pi_{[0,n]}(\varprojlim F)$ for infinitely many $n \in \mathbb{N}$. The question was posed whether this same result would hold if instead we used A. Lelek's definition of endpoint (most often associated with dendroids).

We present an example giving a negative answer to this question. We go on to give characterizations for the sets of endpoints for a family of set-valued functions. These functions have graphs which consist of a symmetric tent map and a straight line connecting the critical point to either $(0,1)$, $(\frac{1}{2},1)$, or $(1,1)$. The endpoints of inverse limits of tent maps are well-studied, but we show that the addition of the straight line fundamentally alters the set of endpoints.

1. Introduction

Suppose that $F : [0,1] \to 2^{[0,1]}$ is an upper semi-continuous set-valued function and that $\mathbf{p} \in \varprojlim F$. Assume the following definition of an endpoint of a continuum: p is an *endpoint* of the continuum X if for any two subcontinua $H, K \subseteq X$ which both contain p, either $H \subseteq K$ or $K \subseteq H$. (This definition is given by R. H. Bing in [4] and is primarily used in the context of arc-like continua.) Using this definition, it is shown in [7, Theorem 1.2] that $\mathbf{p}$ is an endpoint of $\varprojlim F$ provided that for infinitely many

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$n \in \mathbb{N}$, $(p_0, p_1, \ldots, p_{n-1})$ is an endpoint of

$$\Gamma_n = \left\{ \mathbf{x} \in \prod_{i=0}^{n-1} X : x_{i-1} \in F(x_i) \text{ for all } 1 \leq i < n \right\}.$$

Additionally, if the function F has its inverse equal to the union of mappings, then $\mathbf{p}$ is an endpoint of $\varprojlim F$ if and only if $(p_0, p_1, \ldots, p_{n-1})$ is an endpoint of Γ_n for all (infinitely many) $n \in \mathbb{N}$ [7, Theorem 1.3].

There are several other definitions for what it means for a point to lie at the "end" of a continuum, including those of A. Lelek [8] and Harlan C. Miller [9]. Lelek's definition states that a point p is an endpoint of the continuum X if p is an endpoint of any arc in X which contains p. Miller's definition states that a point p is a terminal point in a continuum X if every irreducible continuum in X containing p is irreducible between p and some other point. It is observed in [7] that if either Lelek's or Miller's definition is used for set-valued functions satisfying the union of mappings property, then if $\mathbf{p}$ is an endpoint of $\varprojlim F$, then $(p_0, p_1, \ldots, p_{n-1})$ is an endpoint of Γ_n for all $n \in \mathbb{N}$. However, it remains an open question as to whether $(p_0, p_1, \cdots, p_{n-1})$ being an endpoint of Γ_n for all (infinitely many) $n \in \mathbb{N}$ implies that $\mathbf{p}$ is an endpoint of $\varprojlim F$ (see [7, Quesion 3.1]).

In this paper, we show that if we assume Lelek's definition of an endpoint, then there exists a set-valued function whose inverse is equal to the union of mappings and $(p_0, p_1, \ldots, p_{n-1})$ is an endpoint of Γ_n for all $n \in \mathbb{N}$, but $\mathbf{p}$ is not an endpoint of $\varprojlim F$ (Example 3.1). This gives a negative answer to [7, Question 3.1].

We go on in sections 4, 5, and 6 to explore the collection of endpoints (using Lelek's definition) of inverse limits for a family of set-valued functions that are tent maps with a sticker attached to the critical point. The inverse limits of tent maps have been thoroughly studied, and much is known about their inverse limits and, in particular, their collections of endpoints; see [1], [2], [3], [5]. However, by adding to the graph a straight line connecting the critical point to either (1,1), (0,0), or $(\frac{1}{2}$,1), we completely alter the set of endpoints of the inverse limit.

2. Preliminaries

We begin this section with some definitions and terminology that will be used throughout the paper. A *continuum* is a non-empty, compact, connected metric space. An *arc* is any space which is homeomorphic to the closed interval $[0, 1]$. Clearly, an arc A is a continuum, and if we let $h : [0, 1] \to A$ be a homeomorphism, then we refer to $h(0)$ and $h(1)$ as *endpoints* of A. We would like to extend the concept of an endpoint of an arc to that of an endpoint of a continuum, and several definitions of endpoints have been used. The most common definition of an endpoint, given by Bing, is that p is an *endpoint of the continuum* X if for any two subcontinua of X both containing p, one of them contains the other; we note that this definition does not align with our intuitive understanding of an endpoint in the case of a triod. Thus, if the continuum contains any triods, we must use a different definition.

In this paper, unless otherwise stated, we use Lelek's definition of an endpoint: A point $p \in X$ is an *endpoint of* X if and only if p is an endpoint of every arc in X that contains p. (More generally, if X is a compact metric space, we say p is an *endpoint of* X if it is an endpoint of its connected component in X.)

Given a continuum X, we define 2^X to be the space consisting of all non-empty, compact subsets of X. Given a function $F : X \to 2^Y$, we define its *graph* to be the set $\Gamma(F) = \{(x, y) : y \in F(x)\}$. It is known that F is upper semi-continuous if and only if $\Gamma(F)$ is closed in $X \times Y$ [6].

Let $\mathbf{X}$ be a sequence of continua and let $\mathbf{F}$ be a sequence of upper semi-continuous functions such that for all $i \in \mathbb{N}$, $F_i : X_i \to 2^{X_{i-1}}$; then the pair $\{\mathbf{X},\mathbf{F}\}$ is called an *inverse sequence*. The *inverse limit of the inverse sequence* $\{\boldsymbol{X},\boldsymbol{F}\}$ is the set

$$\varprojlim \mathbf{F} = \left\{ \mathbf{x} \in \prod_{i=0}^{\infty} X_i : x_{i-1} \in F_i(x_i) \text{ for all } i \in \mathbb{N} \right\}.$$

Given the inverse sequence $\{\mathbf{X},\mathbf{F}\}$, we define $\Gamma_1 = X_0$, and for $n \geq 2$, we define

$$\Gamma_n = \left\{ \mathbf{x} \in \prod_{i=0}^{n-1} X : x_{i-1} \in F_i(x_i) \text{ for all } 1 \leq i < n \right\}.$$

Also, for each $n \geq 0$, we define projection mappings

$$\pi_n \colon \varprojlim \mathbf{F} \to X_n \quad \text{and} \quad \pi_{[0,n-1]} \colon \varprojlim \mathbf{F} \to \Gamma_n$$

by $\pi_n(\mathbf{x}) = x_n$ and $\pi_{[0,n-1]}(\mathbf{x}) = (x_0, x_1, \cdots, x_{n-1})$, respectively.

Given a continuum X and an upper semi-continuous function $F : X \to 2^X$, there is a naturally induced inverse sequence $\{\mathbf{X},\mathbf{F}\}$ where for all $i \in \mathbb{N}$, $X_{i-1} = X$ and $F_i = F$. In this case, we simply denote the associated inverse limit of F by $\varprojlim F$.

We will also consider spaces of forward orbits. Let $\mathbf{X}$ be a sequence of spaces and $\mathbf{f}$ be a sequence of continuous functions such that for all $i \geq 0$, $f_i \colon X_i \to X_{i+1}$. Then we define

$$\varinjlim \mathbf{f} = \left\{ \mathbf{x} \in \prod_{i=0}^{\infty} X_i : x_i = f_i(x_{i-1}) \text{ for all } i \geq 0 \right\}.$$

Clearly, $\varinjlim \mathbf{f}$ is homeomorphic to X_0. Just as with inverse limits, in the case where each X_i is the same space X and each f_i is the same function f, we write $\varinjlim f$.

The following result is given in [7] using Bing's definition of endpoint.

Theorem 2.1 ([7, Theorem 1.3]). *Let $\{\boldsymbol{X},\boldsymbol{F}\}$ be an inverse sequence. Suppose that for each $i \in \mathbb{N}$ there exists a collection*

$$\{f_\alpha^{(i)} : X_{i-1} \to X_i\}_{\alpha \in A_i}$$

of continuous functions such that

$$\Gamma(F_i^{-1}) = \bigcup_{\alpha \in A_i} \Gamma(f_\alpha^{(i)}).$$

Then for every $\mathbf{p} \in \varprojlim \boldsymbol{F}$, the following are equivalent using Bing's definition of an endpoint.

(1) $\mathbf{p}$ *is an endpoint of* $\varprojlim \boldsymbol{F}$.
(2) $(p_0, \ldots, p_{n-1})$ *is an endpoint of* Γ_n *for infinitely many* $n \in \mathbb{N}$.
(3) $(p_0, \ldots, p_{n-1})$ *is an endpoint of* Γ_n *for all* $n \in \mathbb{N}$.

In this paper, we show that this theorem fails to hold true if we assume Lelek's definition of an endpoint. All of our examples in this paper make the assumption that the function $F : [0,1] \to 2^{[0,1]}$ is such that there is a pair of continuous functions $f, g : [0,1] \to [0,1]$ such that

$$\Gamma(F^{-1}) = \Gamma(f) \cup \Gamma(g).$$

Then $\varprojlim F = \varinjlim (f \cup g)$. In order to demonstrate the difference in using Lelek's definition we provide the definition of a *branch point* of $\varprojlim F$.

Note that if $\mathbf{x} \in \varprojlim F$, then $\mathbf{x} = (x_0, x_1, x_2, \ldots)$ is such that $x_0 \in [0,1]$ and $x_{i+1} \in \{f(x_i), g(x_i)\}$ for all $i \geq 0$. Let

$$\Sigma = \{\mathbf{h} = (h_1, h_2, h_3, \ldots) : h_i \in \{f, g\} \text{ for all } i \in \mathbb{N}\}$$

and

$$\Sigma_n = \{(h_1, h_2, \ldots, h_{n-1}) : h_i \in \{f, g\}\}.$$

Given $\mathbf{h} = (h_1, h_2, \ldots, h_{n-1}) \in \Sigma_n$, we define

$$\mathbf{h}(x_0) = (x_0, h_1(x_0), h_2(h_1(x_0)), \ldots, h_{n-1}(h_{n-2}(\ldots(h_1(x_0)))))$$

and

$$\mathbf{h}([0,1]) = \{\mathbf{h}(x_0) : x_0 \in [0,1]\}.$$

Thus, $\Gamma_n = \bigcup_{\mathbf{h} \in \Sigma_n} \mathbf{h}([0,1])$. A point $(x_0, \ldots, x_{n-1}) \in \Gamma_n$ is called a *branch point of* Γ_n if there exist $\mathbf{h}, \mathbf{j} \in \Sigma_n$ such that

$$(x_0, \ldots, x_{n-1}) \in \mathbf{h}([0,1]) \cap \mathbf{j}([0,1]),$$

$(x_0, \ldots, x_{n-1})$ is on the boundary of $\mathbf{h}([0,1]) \cap \mathbf{j}([0,1])$, and $(x_0, \ldots, x_{n-1})$ is not an endpoint of $\mathbf{h}([0,1])$ or $\mathbf{j}([0,1])$. That is, intuitively $(x_0, \ldots, x_{n-1})$ is precisely the point where the arcs $\mathbf{h}([0,1])$ and $\mathbf{j}([0,1])$ diverge. Similarly, we will refer to the point $\mathbf{x} = (x_0, x_1, \ldots)$ as a *branch point of* $\varprojlim F$ if $(x_0, \ldots, x_{n-1})$ is a branch point of Γ_n for some n. In this case, there will exist two sequences of functions $\mathbf{h}, \mathbf{j} \in \Sigma$ such that $\mathbf{x}$ is on the boundary of the intersection of the arcs $\varinjlim \mathbf{h} = \varinjlim(h_1, h_2, \ldots)$ and $\varinjlim \mathbf{j} = \varinjlim(j_1, j_2, \ldots)$ and $\mathbf{x}$ is not an endpoint of either arc. Note that if $\mathbf{h} = (f, f, f, \ldots)$, we simply write $\varinjlim \mathbf{h} = \varinjlim f$.

Finally, in order to show that Theorem 2.1 does not hold if we assume Lelek's definition of an endpoint, our functions $F : [0,1] \to 2^{[0,1]}$ are set-valued functions constructed from symmetric tent maps. We define a *symmetric tent map* $T_s : [0,1] \to [0,1]$ by $T_s(x) = \min\{sx, s(1-x)\}$ where $s \in [0,2]$; here, s is referred to as the *slope* of the tent map.

3. A Counterexample

In this section, we provide an example of a set-valued function F with $\Gamma(F^{-1}) = \Gamma(f) \cup \Gamma(g)$, such that there exists a point $\mathbf{p} = (p_0, p_1, p_2, \ldots)$ with the property that $(p_0, \ldots, p_{n-1})$ is an endpoint of Γ_n for all $n \in \mathbb{N}$, but $\mathbf{p}$ is not an endpoint of $\varprojlim F$. This shows that Theorem 2.1 does not hold if we assume Lelek's definition of an endpoint.

Example 3.1. Let $F : [0,1] \to 2^{[0,1]}$ be the set-valued function obtained by attaching the line segment connecting the points $(\frac{1}{2}, \frac{1+\sqrt{5}}{4})$ and $(1,1)$ to the tent map with slope $s = \frac{1+\sqrt{5}}{2}$. See Figure 1. We will show that the point $\mathbf{p} = (1,1,1,1,\ldots)$ is such that $(1,1,\ldots,1) \in \Gamma_n$ is an endpoint of Γ_n for all $n \in \mathbb{N}$, but $\mathbf{p}$ is not an endpoint of $\varprojlim F$.

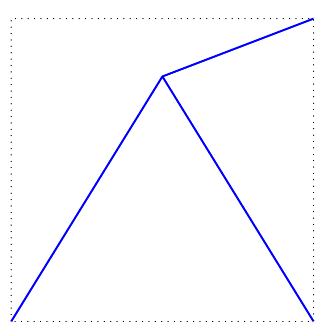

FIGURE 1. The set-valued function $F : [0,1] \to 2^{[0,1]}$

In this example, we may think of the inverse of F as the union of two mappings as is seen in figures 2 and 3.

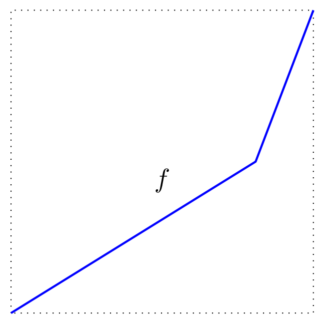

FIGURE 2. $f : [0,1] \to [0,1]$

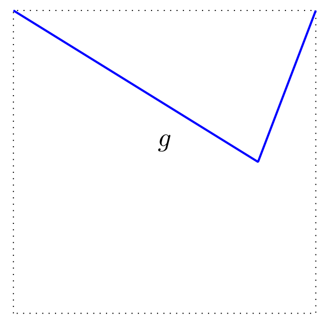

FIGURE 3. $g : [0,1] \to [0,1]$

Note that 1 is an endpoint of $\Gamma_1 = [0,1]$ and $(1,1)$ is an endpoint of Γ_2. In general, $(1,1,\ldots,1)$ is always an endpoint of Γ_n. We have drawn Γ_2 in Figure 4 and Γ_3 in Figure 5 for the reader.

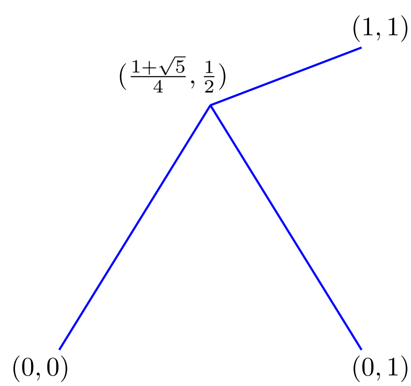

FIGURE 4. Γ_2

Observe that Γ_2 is the union of the arcs $(x_0, f(x_0))$ and $(x_0, g(x_0))$ where $x_0 \in [0,1]$; further, these two arcs precisely coincide on $(x_0, f(x_0)) = (x_0, g(x_0))$ where $x_0 \in [\frac{1+\sqrt{5}}{4}, 1]$. Note that $(\frac{1+\sqrt{5}}{4}, \frac{1}{2})$ is a branch point of Γ_2 because it is the point where the two arcs $(x_0, f(x_0))$ and $(x_0, g(x_0))$ diverge; for ease, we will say that it is the point where the arcs associated with (f) and (g) diverge. The points $(\frac{1+\sqrt{5}}{4}, \frac{1}{2}, \frac{\sqrt{5}-1}{4})$, $(\frac{1+\sqrt{5}}{4}, \frac{1}{2}, \frac{5-\sqrt{5}}{4})$, $(\frac{3\sqrt{5}-3}{4}, \frac{1+\sqrt{5}}{4}, \frac{1}{2})$, and $(\frac{\sqrt{5}-1}{4}, \frac{1+\sqrt{5}}{4}, \frac{1}{2})$ are branch points for Γ_3; they represent the places where the arcs associated with (f,f), (f,g), (g,f), and

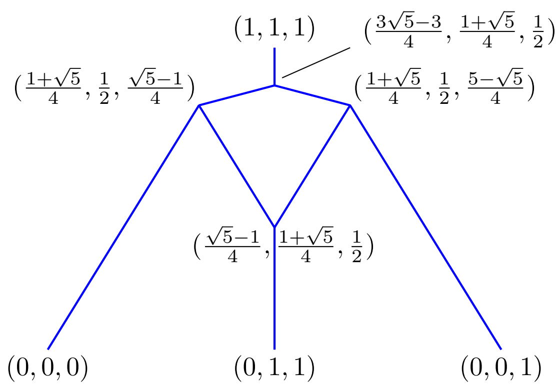

FIGURE 5. Γ_3

(g,g) diverge. In $\varprojlim F$, every branch point occurs precisely at those points $\mathbf{x}$ where $x_i = \frac{1+\sqrt{5}}{4}$ for some i.

Observe that $\varprojlim F$ contains $\varinjlim f \cup \varinjlim g$, where $\varinjlim f$ is an arc with endpoints $(0,0,0,\ldots)$ and $(1,1,1,\ldots)$ and $\varinjlim g$ is an arc with endpoints $(0,1,1,1,\ldots)$ and $(1,1,1,\ldots)$. We will show that $(1,1,1,\ldots)$ is the only point common to $\varinjlim f$ and $\varinjlim g$.

Let $\mathbf{y}$ be a point in $\varinjlim g$ with $\mathbf{y} \neq (1,1,1,\ldots)$. Then $g^n(y_0) \to m$, where m is the fixed point of the tent map with slope $\frac{1+\sqrt{5}}{2}$; we can thus choose an $N \in \mathbb{N}$ such that whenever $n \geq N$, $d(y_n, m) < m/2$. Similarly, let $\mathbf{x}$ be a point in $\varinjlim f$ with $\mathbf{x} \neq (1,1,1,\ldots)$. Then $f^n(x_0) \to 0$ and there exists an $N' \in \mathbb{N}$ such that whenever $n \geq N'$, $d(x_n, 0) < m/2$. Thus, for all $n \geq \max(N, N')$, $d(x_n, y_n) > 0$, and hence $\mathbf{x} \neq \mathbf{y}$. We thus conclude that the only point in $\varinjlim f \cap \varinjlim g$ is $(1,1,1,\ldots)$. Hence, $\varinjlim f \cup \varinjlim g$ is an arc with endpoints $(0,0,0,\ldots)$ and $(0,1,1,1,\ldots)$ that contains $(1,1,1,\ldots)$; therefore, $(1,1,1,\ldots)$ is not an endpoint of $\varprojlim F$.

The only possible endpoints of $\varprojlim F$ are those endpoints of the form

$$(0,0,0,\ldots),\ (1,1,1,\ldots),\ \text{or}\ (0,\ldots,0,1,1,\ldots),$$

because the endpoints of Γ_n are always of the form

$$(0,0,\ldots,0),\ (1,1,\ldots,1),\ \text{or}\ (0,\ldots,0,1,\ldots,1).$$

A similar argument to the above will show that none of the points of the form $(0,\ldots,0,1,1,\ldots)$ are endpoints of $\varprojlim F$. Therefore, $(0,0,0,\ldots)$ is the only possible endpoint for $\varprojlim F$. We conclude this example by showing that $(0,0,0,\ldots)$ is indeed an endpoint of $\varprojlim F$.

Note that $(0,0,0,\ldots)$ is an endpoint of $\varinjlim f$ and that

$$B = \left\{ (x_0, x_1, x_2, \ldots) : x_0 \in \left[0, \frac{1+\sqrt{5}}{2}\right], x_{i+1} = f(x_i) \right\} \subseteq \varinjlim f$$

is an arc in $\varprojlim F$ with no branch points other than

$$\left(\frac{1+\sqrt{5}}{2}, f\left(\frac{1+\sqrt{5}}{2}\right), f^2\left(\frac{1+\sqrt{5}}{2}\right), \ldots\right).$$

The only sequence of points that can converge to $(0,0,0,\ldots)$ in $\varprojlim F$ are all eventually either in B or in a subsequence of the arcs $\varinjlim(f, g, g, \ldots)$, $\varinjlim(f, f, g, g, \ldots)$, $\varinjlim(f, f, f, g, g, \ldots)$, $\ldots$, which limits onto $\varinjlim f$. Thus, $(0,0,0,\ldots)$ is an endpoint of every arc containing it and, hence, $(0,0,0,\ldots)$ is an endpoint of $\varprojlim F$.

Recall that the collection of endpoints of the inverse limit of the tent map T with slope $s = \frac{1+\sqrt{5}}{2}$ is the set of points

$$\left\{ (0,0,0,\ldots), \left(\frac{1}{2}, \frac{1+\sqrt{5}}{4}, \frac{\sqrt{5}-1}{4}, \ldots\right), \left(\frac{1+\sqrt{5}}{4}, \frac{\sqrt{5}-1}{4}, \frac{1}{2}, \ldots\right), \left(\frac{\sqrt{5}-1}{4}, \frac{1}{2}, \frac{1+\sqrt{5}}{4}, \ldots\right) \right\}.$$

Therefore, adding the arc connecting the points $(\frac{1}{2}, \frac{1+\sqrt{5}}{4})$ and $(1,1)$ fundamentally changes the overall structure of the inverse limit by removing all of the endpoints except the one at $(0,0,0,\ldots)$. We also note that in this case, no additional endpoints were added by attaching the arc. This may be considered a rather surprising result, as somehow the arc attached to the critical point changes the "core" of the inverse limit. In the remainder of this paper, we explore the collection of endpoints for the family of symmetric tent maps with various arcs attached to the critical point and analyze similarities and differences since both the slope and the attached arc are varied.

4. Functions F_s Obtained via Attaching onto the Graph of T_s the Straight Line from the Critical Point to $(1,1)$

In this section, we consider set-valued functions obtained from the symmetric tent map T_s by attaching an arc from the critical point to $(1,1)$; we refer to the family of such functions by F_s, where s indicates the slope of the associated tent map T_s. We observe how the collection of endpoints varies as the parameter s is changed.

4.1. F_s with $1 < s < 2$.

The function in Example 3.1 is not unique; every set-valued function F_s obtained from a symmetric tent map by attaching the arc connecting the critical point and $(1, 1)$ will also have $(0, 0, 0, \ldots)$ as the only endpoint of $\varprojlim F_s$ as long as the slope s of the symmetric tent map is such that $1 < s < 2$. This is because every symmetric tent map with $s > 1$ has two repelling fixed points. When the arc connecting the critical point and $(1, 1)$ is attached to the tent map, 1 is an attracting fixed point for the forward iteration of F_s. Hence, when we decompose the inverse of the set-valued function into two continuous maps (as was done in figures 2 and 3), 1 will be a repelling fixed point of both f and g, and the other fixed points in both f and g will be attracting. The same arguments as in Example 3.1 will show that the only endpoint of $\varprojlim F_s$ is $(0, 0, 0, \ldots)$ when $s > 1$.

4.2. F_s with $s = 1$.

Now consider the function F_s where $s = 1$; the graph of F_1 is included in Figure 6. We give a description of $\varprojlim F_1$ and show that it has countably many endpoints including $(0, 0, 0, \ldots)$ which is the limit of the rest of the endpoints.

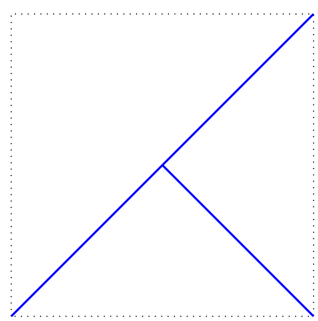

Figure 6. The set-valued function $F_1 : [0, 1] \to 2^{[0,1]}$

For ease, we decompose F_1^{-1} into the union of two mappings, as seen in figures 7 and 8.

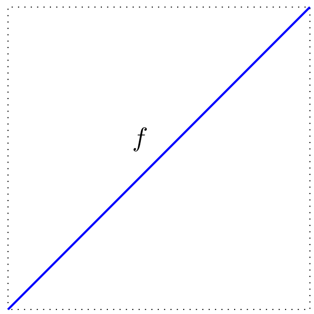

FIGURE 7. $f : [0,1] \to [0,1]$

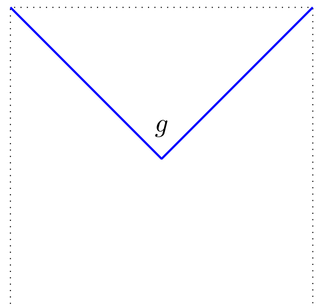

FIGURE 8. $g : [0,1] \to [0,1]$

Note that f is just the identity on $[0,1]$, whereas we may think of $g(x) = \max\{1-x, x\}$. This implies that when considering $\varinjlim(f \cup g) = \varprojlim F_1$, every point $x_n \in [\frac{1}{2}, 1]$ is such that $f(x_n) = g(x_n) = x_n$. Further, if $x_n \in [0, \frac{1}{2}]$, then $f(x_n) = x_n$ and $g(x_n) = 1 - x_n$. Hence, every point in $\varprojlim F_1$ is of the form $(a, a, a, \ldots)$, $(a, a, \ldots, a, 1-a, 1-a, 1-a, \ldots)$, or $(1-a, 1-a, 1-a, \ldots)$ where $a \in [0, \frac{1}{2}]$.

Therefore, we may think of Γ_2 as being the union of the following three arcs:

(1) the arc of the form (a, a), where $a \in [0, \frac{1}{2}]$;
(2) the arc of the form $(a, 1-a)$, where $a \in [0, \frac{1}{2}]$; and
(3) the arc of the form $(1-a, 1-a)$, where $a \in [0, \frac{1}{2}]$.

Similarly, we may think of Γ_3 as the union of the following four arcs:

(1) the arc of the form (a, a, a), where $a \in [0, \frac{1}{2}]$;
(2) the arc of the form $(a, a, 1-a)$, where $a \in [0, \frac{1}{2}]$;
(3) the arc of the form $(a, 1-a, 1-a)$, where $a \in [0, \frac{1}{2}]$; and
(4) the arc of the form $(1-a, 1-a, 1-a)$, where $a \in [0, \frac{1}{2}]$.

Inductively, we may think of Γ_n as the union of $n+1$ arcs all joined at the point $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. We conclude that, in general, the endpoints of $\varprojlim F_1$ will be all points of the form $(1, 1, \ldots)$, $(0, 0, \ldots)$, or $(0, \ldots, 0, 1, 1, \ldots)$. That is, $\varprojlim F_1$ will have a countable collection of endpoints. Additionally, $\varprojlim F_1$ will be a fan with vertex $(\frac{1}{2}, \frac{1}{2}, \ldots)$, as drawn in Figure 9.

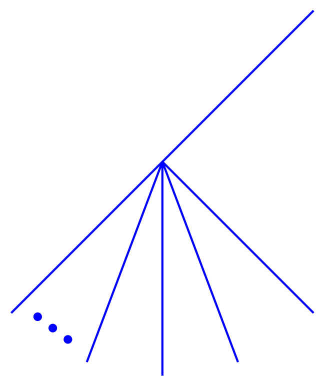

FIGURE 9. Inverse limit of F_1

4.3. F_s WITH $s < 1$.

We finally consider the function F_s where $s < 1$; the graph of an example of a function F_s with $s < 1$ is included in Figure 10. We give a description of $\varprojlim F_s$ and show that it has countably many endpoints and that for any $0 < s < t < 1$, $\varprojlim F_s$ and $\varprojlim F_t$ are homeomorphic.

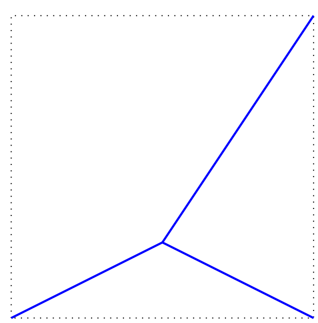

FIGURE 10. The set-valued function $F_s : [0,1] \to 2^{[0,1]}, s < 1$

Once again, we decompose F_s^{-1} into the union of two mappings, as seen in figures 11 and 12.

Referring to the critical point of T_s as $(\frac{1}{2}, \frac{s}{2})$, we note that the graphs of f and g are identical on the domain $[\frac{s}{2}, 1]$, where $\frac{s}{2} < \frac{1}{2}$. Observe that $\varinjlim f$ is the arc with endpoints $(0,0,0,\ldots)$ and $(1,1,1,\ldots)$, whereas $\varinjlim g$ is the arc with endpoints $(0,1,1,\ldots)$ and $(1,1,1,\ldots)$. Further, observe that if $x_n \in [\frac{s}{2}, 1]$, then $x_i \in [\frac{s}{2}, 1]$ for all $i \geq n$, and it will not matter whether we apply f or g to obtain x_{i+1}. Therefore, without loss of

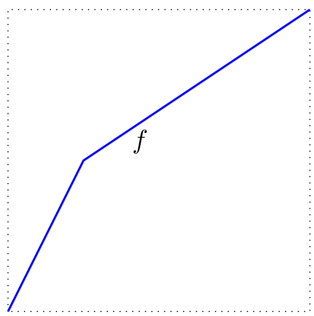

FIGURE 11.
$f : [0,1] \to [0,1]$

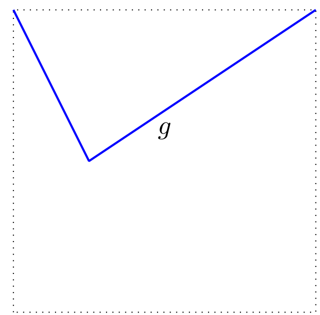

FIGURE 12.
$g : [0,1] \to [0,1]$

generality, we can assume that once g has been applied to a point x_i, it will be applied for all subsequent iterates. We observe that the point $(\frac{s}{2}, \frac{1}{2}, g(\frac{1}{2}), g^2(\frac{1}{2}), \ldots)$ is a branch point of $\varprojlim F_s$. That is, the arc connecting $(\frac{s}{2}, \frac{1}{2}, g(\frac{1}{2}), g^2(\frac{1}{2}), \ldots)$ and $(1,1,1,\ldots)$ in $\varprojlim F_s$ is exactly the intersection of $\varinjlim f$ and $\varinjlim g$, and it is at that point where the two arcs diverge. Similarly, the point $(f^{-1}(\frac{s}{2}), \frac{s}{2}, \frac{1}{2}, g(\frac{1}{2}), \ldots)$ is a branch point of $\varprojlim F_s$; the intersection of $\varinjlim f$ and $\varinjlim(f, g, g, \ldots)$ is precisely an arc connecting $(f^{-1}(\frac{s}{2}), \frac{s}{2}, \frac{1}{2}, g(\frac{1}{2}), \ldots)$ and $(1,1,1,\ldots)$, where $\varinjlim(f, g, g, \ldots)$ is the arc with endpoints $(1,1,1,\ldots)$ and $(0,0,1,1,1,\ldots)$. We may proceed in this manner comparing $\varinjlim f$ and $\varinjlim(f, \ldots, f, g, g, \ldots)$ and noting that due to the location of branch points in $\varprojlim F_s$, every point of the form $(0,0,0,\ldots)$, $(1,1,1,\ldots)$, or $(0,\ldots,0,1,1,\ldots)$ will be an endpoint of $\varprojlim F_s$. That is, there is a countable collection of endpoints for $\varprojlim F_s$ when $s < 1$. Further, $\varprojlim F_s$ is homeomorphic to $\varprojlim F_t$ whenever $s, t \in (0,1)$; their inverse limit, a comb, is drawn in Figure 13.

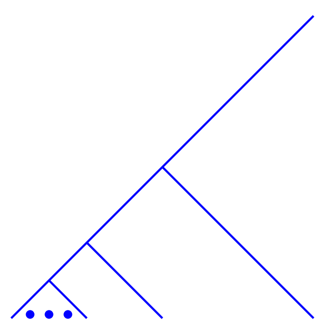

FIGURE 13. Inverse limit of F_s for $s < 1$

5. Functions G_s Obtained via Attaching onto the Graph of T_s the Straight Line from the Critical Point to $(0,1)$

In this section, we consider set-valued functions obtained from the symmetric tent map T_s by attaching an arc from the critical point to $(0,1)$; we refer to the family of such functions by G_s where s indicates the slope of the associated tent map T_s. We observe how the collection of endpoints varies as the parameter s is changed.

5.1. G_s with $1 < s < 2$.

We first consider set-valued functions of the form G_s $(1 < s < 2)$ where G_s is the union of the symmetric tent map T_s and the arc connecting the critical point and $(0,1)$. See Figure 14. We show that the set of endpoints of $\varprojlim G_s$ is an uncountable subset of $\{0,1\}^\omega$.

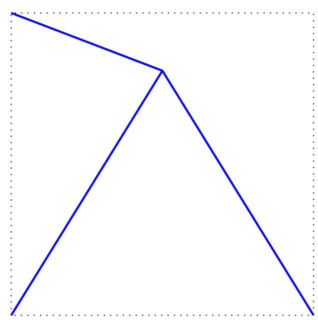

Figure 14. The set-valued function $G_s : [0,1] \to 2^{[0,1]}$, $1 < s < 2$

We may again think of the inverse of G_s as the union of two mappings as is seen in figures 15 and 16.

We note that

$$f(x) = \begin{cases} \frac{x}{s} & x < \frac{s}{2} \\ \frac{1-x}{2-s} & x \geq \frac{s}{2} \end{cases}$$

and

$$g(x) = \begin{cases} 1 - \frac{x}{s} & x < \frac{s}{2} \\ \frac{1-x}{2-s} & x \geq \frac{s}{2} \end{cases}.$$

In this case, f has an attracting fixed point at 0 and g has an attracting fixed point at $\frac{s}{s+1}$. We emphasize that if $x \geq \frac{s}{2}$, then $f(x) = g(x)$. Further, $g(x) > \frac{s}{2}$ if and only if $x < \frac{2s-s^2}{2}$, whereas $f(x) < \frac{s}{2}$ for all x. Therefore, viewing $\varprojlim G_s = \varinjlim f \cup g$, note that any point $\mathbf{x} =$

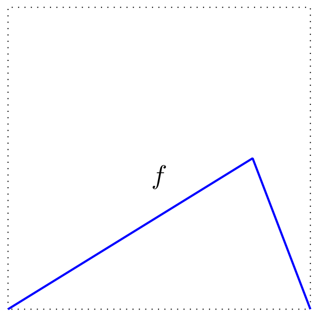

FIGURE 15.
$f : [0,1] \to [0,1]$

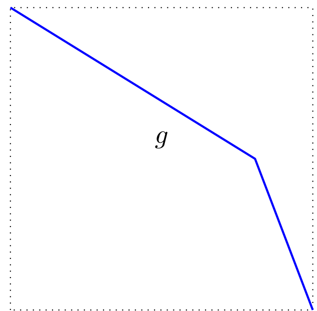

FIGURE 16.
$g : [0,1] \to [0,1]$

$(x_0, x_1, x_2, \ldots) \in \varprojlim G_s$ is such that if $x_i \in [\frac{s}{2}, 1]$, then $x_i = g(x_{i-1})$ and $x_{i+1} = f(x_i) = g(x_i)$.

Consider all points in the set $P = \{\mathbf{p} \in \{0,1\}^\omega : p_i p_{i+1} \neq 11\}$. Then $(p_0, p_1, \ldots, p_{n-1})$ will be an endpoint of Γ_n for all $n \in \mathbb{N}$, and every endpoint of Γ_n will be of the form $(p_0, p_1, \ldots, p_{n-1})$ where $p_i p_{i+1} \neq 11$; therefore, P is the set of all potential endpoints for $\varprojlim G_s$. We claim that the collection of endpoints of $\varprojlim G_s$ will depend upon the parameter s and the number of consecutive 0s that appear in the point $\mathbf{p} \in P$.

Note that $\varprojlim G_s$ contains $\varinjlim g \cup \varinjlim(g, f, g, f, \ldots)$, where $\varinjlim g$ is an arc with endpoints $(0, 1, 0, 1, \ldots)$ and $(1, 0, 1, 0, \ldots)$, and $\varinjlim(g, f, g, f, \ldots)$ is an arc with endpoints $(0, 1, 0, 1, \ldots)$ and $(1, 0, 0, 1, 0, 1, 0, \ldots)$. If $0 < x_0 < \frac{2s-s^2}{2}$, then $g^2(x_0) = f(g(x_0)) = \frac{x_0}{2s-s^2} > x_0$. Further, as $\frac{s}{s+1}$ is an attracting fixed point of g, there will exist an even n such that $g^n(x_0) > \frac{2s-s^2}{2}$. Thus, $g^{n+1}(x_0) \neq f(g^n(x_0))$. Let $\mathbf{x} \in \varinjlim g$ with $\mathbf{x} \neq (0, 1, 0, 1, \ldots)$ and $\mathbf{y} \in \varinjlim(g, f, g, f, \ldots)$ with $\mathbf{y} \neq (0, 1, 0, 1, \ldots)$. Without loss of generality, suppose that $x_0 = y_0 < \frac{2s-s^2}{2}$. Then there will exist some n such that $x_n \neq y_n$. That is, $\mathbf{x} \neq \mathbf{y}$. Hence, the only point in $\varinjlim g \cap \varinjlim(g, f, g, f, \ldots)$ is $(0, 1, 0, 1, \ldots)$, which implies $(0, 1, 0, 1, \ldots)$ is not an endpoint of $\varprojlim G_s$. In general, any point with tail $010101\ldots$ will not be an endpoint of $\varprojlim G_s$. A similar argument will show that for every $1 < s < 2$ there exists an $N_s \in \mathbb{N}$ such that if the number of consecutive 0s in the tail of $\mathbf{p} \in P$ is always less than N_s, then $\mathbf{p}$ is not an endpoint of $\varprojlim G_s$.

If $\mathbf{p} \in P$ has only finitely many 1s, then $\mathbf{p}$ will be an endpoint of $\varprojlim G_s$. This is because if we think of G_s^{-1} as $f \cup g$, then there exists an $N \in \mathbb{N}$ such that $p_i = 0$ for all $i \geq N$, and $p_{i+1} = f(p_i)$. The only branch points in $\varprojlim G_s$ occur at points $\mathbf{x}$ where $x_n = \frac{s}{2}$ for some $n \in \mathbb{N}$, and since $f(x_i) < \frac{1}{2}$ for all $x_i \in [0,1]$, there will exist an arc in $\varprojlim G_s$ with endpoints $\mathbf{p}$ and a branch point that contains no other branch points of $\varprojlim G_s$. Hence, every arc in $\varprojlim G_s$ that contains $\mathbf{p}$ will have $\mathbf{p}$ as an endpoint. This argument can be modified to show that for every $s > 1$ there exists an $N_s \in \mathbb{N}$ such that if the number of consecutive 0s in the tail of $\mathbf{p} \in P$ is always greater than N_s, then $\mathbf{p}$ will be an endpoint of $\varprojlim G_s$. We do note that if $\mathbf{p} \in \varprojlim G_s$ contains fewer than N_s consecutive 0s infinitely often in its tail, then $\mathbf{p}$ may or may not be an endpoint of $\varprojlim G_s$. This will depend upon the pattern of 1s and 0s immediately following each such occurrence. In many ways, it is difficult to completely classify this collection of endpoints, since the characterization completely depends upon the slope s and the pattern of blocks of consecutive 0s.

5.2. G_s WITH $s = 1$.

Now consider the function G_s where $s = 1$; the graph of G_1 is included in Figure 17. We show that the set of endpoints of $\varprojlim G_1$ forms a Cantor set which is a proper subset of $\{0,1\}^\omega$.

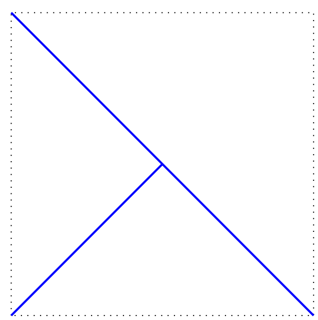

FIGURE 17. The set-valued function $G_1 : [0,1] \to 2^{[0,1]}$

Once again, the decomposition of G_1^{-1} into the union of two mappings is seen in figures 18 and 19.

Observe that f is just the tent map T_1 and g is the function $g(x) = 1 - x$. This implies that when considering $\varprojlim G_1 = \varinjlim f \cup g$, every point $x_n \in [\frac{1}{2}, 1]$ is such that $f(x_n) = g(x_n) = 1 - x_n$. Further, if $x_n \in [0, \frac{1}{2}]$, then $f(x_n) = x_n$ and $g(x_n) = 1 - x_n$. Hence, every point in $\varprojlim G_1$ is of the form $\mathbf{p} = (p_1, p_2, p_3, \ldots)$ where every $p_i \in \{a, 1-a\}$ where $a \in [0, \frac{1}{2}]$, and if $p_n = 1 - a$, then $p_{n+1} = a$.

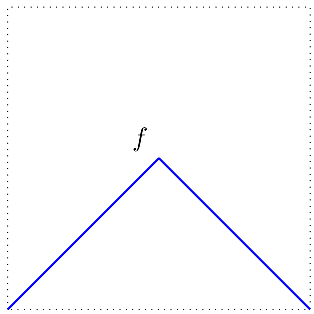

FIGURE 18.
$f : [0, 1] \to [0, 1]$

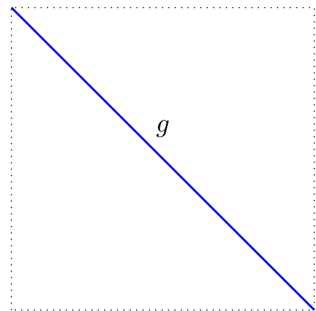

FIGURE 19.
$g : [0, 1] \to [0, 1]$

Therefore, we may think of Γ_2 as being the union of three arcs:

(1) the arc of the form (a, a), where $a \in [0, \frac{1}{2}]$;
(2) the arc of the form $(a, 1-a)$, where $a \in [0, \frac{1}{2}]$; and
(3) the arc of the form $(1-a, a)$, where $a \in [0, \frac{1}{2}]$.

Similarly, we may think of Γ_3 as the union of the following five arcs:

(1) the arc of the form (a, a, a), where $a \in [0, \frac{1}{2}]$;
(2) the arc of the form $(a, a, 1-a)$, where $a \in [0, \frac{1}{2}]$;
(3) the arc of the form $(a, 1-a, a)$, where $a \in [0, \frac{1}{2}]$;
(4) the arc of the form $(1-a, a, a)$, where $a \in [0, \frac{1}{2}]$; and
(5) the arc of the form $(1-a, a, 1-a)$, where $a \in [0, \frac{1}{2}]$.

Inductively, if we let $B_0 = 1$, $B_1 = 2$, and $B_n = B_{n-1} + B_{n-2}$ for $n \geq 2$, then Γ_n is the union of B_n arcs all joined at $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. We conclude that, in general, the endpoints of $\varprojlim G_1$ will be all points which are sequences of 0s and 1s such that no two 1s appear consecutively. That is, $\varprojlim G_1$ will have a Cantor set of endpoints. Additionally, $\varprojlim G_1$ will be a Cantor fan. This is similar to F_1; however, recall that F_1 had countably many endpoints.

5.3. G_s WITH $s < 1$.

Now consider the function G_s where $s < 1$; the graph of G_s is included in Figure 20. Just as in subsection 4.3, we show that for any $0 < s < t < 1$, $\varprojlim G_s$ is homeomorphic to $\varprojlim G_t$. In addition, we show that the set of endpoints of $\varprojlim G_s$ is the same as the set of endpoints of $\varprojlim G_1$.

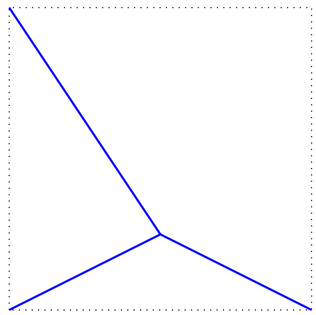

FIGURE 20. The set-valued function $G_1 : [0,1] \to 2^{[0,1]}$

Figures 21 and 22 show the decomposition of G_s^{-1} into the union of two mappings.

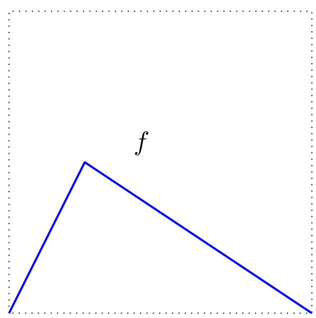

FIGURE 21.
$f : [0,1] \to [0,1]$

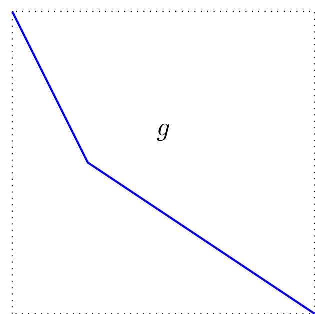

FIGURE 22.
$g : [0,1] \to [0,1]$

We again have that

$$f(x) = \begin{cases} \frac{x}{s} & x < \frac{s}{2} \\ \frac{1-x}{2-s} & x \geq \frac{s}{2} \end{cases}$$

and

$$g(x) = \begin{cases} 1 - \frac{x}{s} & x < \frac{s}{2} \\ \frac{1-x}{2-s} & x \geq \frac{s}{2} \end{cases}.$$

In this case, 0 is a repelling fixed point of f, and $\frac{1}{2-s+1}$ is an attracting fixed point for both f and g. Referring to the critical point of T_s as $(\frac{1}{2}, \frac{s}{2})$, we note that the graphs of f and g are identical on the domain $[\frac{s}{2}, 1]$, where $\frac{s}{2} < \frac{1}{2}$. Further, observe that if $x_n \in [\frac{s}{2}, 1]$, then $x_{n+1} \in [0, \frac{1}{2}]$, and it will not matter whether we apply f or g to obtain x_{n+1}. That is, we will have branch points at $\mathbf{x} \in \varprojlim G_s$ whenever $x_n = \frac{s}{2}$ for some $n \in \mathbb{N}$. We again have that the set $P = \{\mathbf{p} \in \{0,1\}^\omega : p_i p_{i+1} \neq 11\}$ is the collection of all potential endpoints for $\varprojlim G_s$; indeed, the set P is precisely the collection of endpoints for $\varprojlim G_s$.

Let A and B be two arcs in $\varprojlim G_s$ with endpoint $(0,0,0,\ldots)$. Suppose $\mathbf{x}$ is in A such that $x_0 \neq 0$ and $\mathbf{y}$ is in B such that $y_0 \neq 0$. Since $x_0 \neq 0$, $y_0 \neq 0$, and both A and B contain $(0,0,0,\ldots)$, there exists $t_0 \in (0, \min(x_0, y_0))$ such that the arc with endpoints $(0,0,0,\ldots)$ and $(t_0, f(t_0), f^2(t_0), \ldots)$ is in both A and B. Thus, $A \cap B$ contains a nondegenerate arc with endpoint $(0,0,0,\ldots)$, and as A and B were arbitrarily chosen with endpoint $(0,0,0,\ldots)$, it follows that $(0,0,0,\ldots)$ must be an endpoint of every arc containing it. Therefore, $(0,0,0,\ldots)$ is an endpoint of $\varprojlim G_s$. Similar arguments will show that every other point $\mathbf{p} \in P$ is an endpoint of $\varprojlim G_s$. Therefore, the collection of endpoints in this case is a Cantor set. Further, all such inverse limits in this case will be homeomorphic, and the inverse limit is drawn in Figure 23. Note that each of the drawn branches will have a Cantor set of branches, and each of those branches will have a Cantor set of branches, etc.

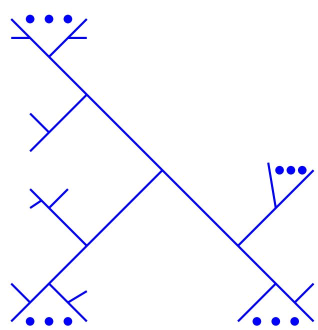

FIGURE 23. Inverse limit of G_s for $s < 1$

6. Functions H_s Obtained via Attaching onto the Graph of T_s the Straight Line from the Critical Point to $\left(\frac{1}{2}, 1\right)$

In this section, we consider set-valued functions obtained from the symmetric tent map T_s by attaching an arc from the critical point to $(\frac{1}{2}, 1)$; we refer to the family of such functions by H_s where s indicates the slope of the associated tent map T_s. We observe how the collection of endpoints varies as the parameter s is changed.

6.1. H_s WITH $1 < s < 2$.

Now consider the function H_s where $1 < s < 2$; the graph of H_s is included in Figure 24. We show that the set of endpoints of $\varprojlim H_s$ is equal to

$$\{\mathbf{p} \in \varprojlim H_s : p_i = 1 \text{ for some } i \geq 0\} \cup \{(0, 0, 0, \ldots)\}.$$

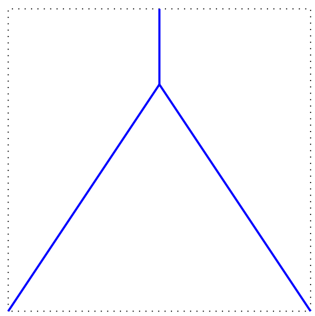

FIGURE 24. The set-valued function $H_s : [0, 1] \to 2^{[0,1]}$

As in past cases, figures 25 and 26 show the decomposition of H_s^{-1} into the union of two mappings f and g.

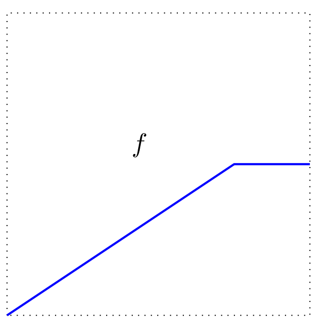

FIGURE 25.
$f : [0, 1] \to [0, 1]$

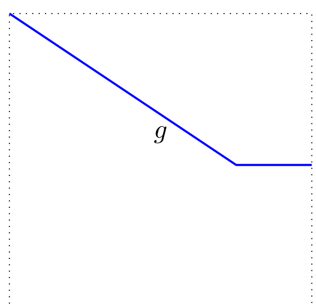

FIGURE 26.
$g : [0, 1] \to [0, 1]$

In this family, we have that

$$f(x) = \begin{cases} \frac{x}{s} & x < \frac{s}{2} \\ \frac{1}{2} & x \geq \frac{s}{2} \end{cases}$$

and

$$g(x) = \begin{cases} 1 - \frac{x}{s} & x < \frac{s}{2} \\ \frac{1}{2} & x \geq \frac{s}{2} \end{cases}.$$

Let

$$P = \{\mathbf{p} \in \varprojlim H_s : p_i = 1 \text{ for some } i \in \mathbb{N}\} \cup \{(0,0,0,\ldots)\}.$$

Observe that $(p_0, p_1, \ldots, p_{n-1})$ will be an endpoint of Γ_n for all $n \in \mathbb{N}$, and every endpoint of Γ_n will be of the form $(p_0, p_1, \ldots, p_{n-1})$ for some $\mathbf{p} \in P$; therefore, P is the set of potential endpoints of $\varprojlim H_s$. We claim that the collection of points $\mathbf{p} \in P$ is precisely the collection of endpoints of $\varprojlim H_s$.

Note that $\varinjlim f$ is an arc with endpoints

$$(0,0,0,\ldots) \text{ and } \left(1, \frac{1}{2}, f\left(\frac{1}{2}\right), f^2\left(\frac{1}{2}\right), \ldots\right),$$

whereas $\varinjlim(g, f, f, f, \ldots)$ is an arc with endpoints

$$\left(0, 1, \frac{1}{2}, f\left(\frac{1}{2}\right), f^2\left(\frac{1}{2}\right), \ldots\right) \text{ and } \left(1, \frac{1}{2}, f\left(\frac{1}{2}\right), f^2\left(\frac{1}{2}\right), \ldots\right).$$

If $\mathbf{x} \in \varprojlim H_s$ is a point with $x_0 \in [\frac{s}{2}, 1]$ and $x_i = f(x_{i-1})$ for all $i \in \mathbb{N}$, then $\mathbf{x}$ is in both $\varinjlim f$ and $\varinjlim(g, f, f, f, \ldots)$. In general, let A and B be two arcs in $\varprojlim H_s$ with endpoint $(1, \frac{1}{2}, f(\frac{1}{2}), f^2(\frac{1}{2}), \ldots)$. Suppose $\mathbf{x} \in A$ with $x_0 \neq 1$ and $\mathbf{y} \in B$ with $y_0 \neq 1$. Let $t_0 = \max(x_0, y_0, \frac{s}{2})$. Then there exists a nondegenerate arc with endpoints $(1, \frac{1}{2}, f(\frac{1}{2}), f^2(\frac{1}{2}), \ldots)$ and $(t_0, \frac{1}{2}, f(\frac{1}{2}), f^2(\frac{1}{2}), \ldots)$ such that the arc is in both A and B. Thus, $A \cap B$ contains a nondegenerate arc with endpoint $(1, \frac{1}{2}, f(\frac{1}{2}), f^2(\frac{1}{2}), \ldots)$, and as A and B were arbitrarily chosen with endpoint $(1, \frac{1}{2}, f(\frac{1}{2}), f^2(\frac{1}{2}), \ldots)$, it follows that $(1, \frac{1}{2}, f(\frac{1}{2}), f^2(\frac{1}{2}), \ldots)$ must be an endpoint of every arc containing it. Therefore, $(1, \frac{1}{2}, f(\frac{1}{2}), f^2(\frac{1}{2}), \ldots)$ is an endpoint of $\varprojlim H_s$. A similar argument will show that every point in P is an endpoint of $\varprojlim H_s$.

6.2. H_s WITH $s = 1$.

Now consider the function H_s where $s = 1$; the graph of H_1 is included in Figure 27. We show that $\varprojlim H_1$ is homeomorphic to $\varprojlim F_1$ in subsection 4.2. In particular, it has countably many endpoints.

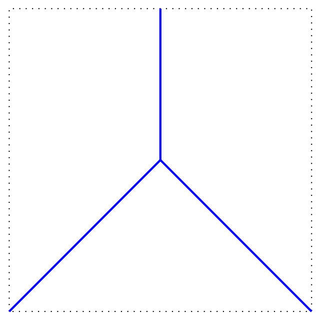

FIGURE 27. The set-valued function $H_1 : [0,1] \to 2^{[0,1]}$

Once again, we decompose H_1^{-1} into the union of two mappings, as seen in figures 28 and 29.

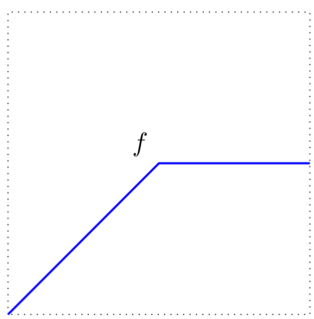

FIGURE 28.
$f : [0,1] \to [0,1]$

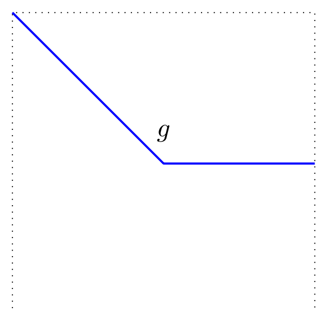

FIGURE 29.
$g : [0,1] \to [0,1]$

Note that $f(x) = g(x) = \frac{1}{2}$ for all $x \in [\frac{1}{2}, 1]$, $f(x) = x$ on $[0, \frac{1}{2}]$, and $g(x) = 1 - x$ on $[0, \frac{1}{2}]$. This implies that for $\varprojlim H_1 = \varinjlim(f \cup g)$, every point $x_n \in [\frac{1}{2}, 1]$ is such that $x_i = \frac{1}{2}$ for all $i \geq n+1$, regardless of whether f or g is applied to obtain x_i. Additionally, if $x_n \in [0, \frac{1}{2}]$, then whenever f is applied to obtain x_{n+1}, we will have $x_{n+1} = x_n \leq \frac{1}{2}$, and whenever g is applied to obtain x_{n+1}, we will have $x_{n+1} = 1 - x_n \geq \frac{1}{2}$. Hence, every point in $\varprojlim H_1$ is of the form $\mathbf{p} = (p_1, p_2, p_3, \ldots)$ where every $p_i \in \{a, 1-a, \frac{1}{2}\}$ where $a \in [0, \frac{1}{2}]$, and if $p_n \in \{1-a, \frac{1}{2}\}$, then $p_{n+1} = \frac{1}{2}$, and $p_n = a$ may be followed by either $p_{n+1} = a$ or $p_{n+1} = 1 - a$.

Therefore, we may think of Γ_2 as the union of three arcs:

(1) the arc of the form (a, a), where $a \in [0, \frac{1}{2}]$;
(2) the arc of the form $(a, 1-a)$, where $a \in [0, \frac{1}{2}]$; and

(3) the arc of the form $(1-a, \frac{1}{2})$, where $a \in [0, \frac{1}{2}]$.

Similarly, we may think of Γ_3 as the union of four arcs:

(1) the arc of the form (a, a, a), where $a \in [0, \frac{1}{2}]$;
(2) the arc of the form $(a, a, 1-a)$, where $a \in [0, \frac{1}{2}]$;
(3) the arc of the form $(a, 1-a, \frac{1}{2})$, where $a \in [0, \frac{1}{2}]$; and
(4) the arc of the form $(1-a, \frac{1}{2}, \frac{1}{2})$, where $a \in [0, \frac{1}{2}]$.

Inductively, we may think of Γ_n as the union of $n+1$ arcs, all joined at the point $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. We conclude that, in general, the endpoints of $\varprojlim H_1$ will be all points of the form $(0, 0, 0, \ldots)$, $(0, 0, \ldots, 0, 1, \frac{1}{2}, \frac{1}{2}, \ldots)$, or $(1, \frac{1}{2}, \frac{1}{2}, \ldots)$. That is, $\varprojlim H_1$ will have a countable collection of endpoints. Additionally, $\varprojlim H_1$ is homeomorphic to $\varprojlim F_1$; that is, $\varprojlim H_1$ is a countable fan with vertex $(\frac{1}{2}, \frac{1}{2}, \ldots)$.

6.3. H_s WITH $s < 1$.

We finally consider the function H_s where $s < 1$; the graph of an example of a function H_s with $s < 1$ is included in Figure 30. We show that $\varprojlim H_s$ is homeomorphic to $\varprojlim F_s$ in subsection 4.3. In particular, for each $0 < s < t < 1$, the inverse limits $\varprojlim H_s$ and $\varprojlim H_t$ are homeomorphic with countably many endpoints.

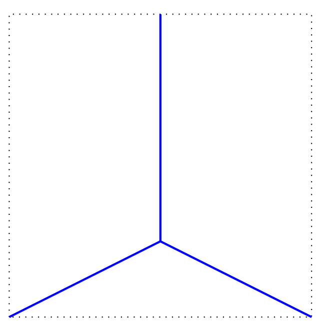

FIGURE 30. The set-valued function $H_s : [0, 1] \to 2^{[0,1]}$

Figures 31 and 32 show the decomposition of H_s^{-1} into the union of two mappings.

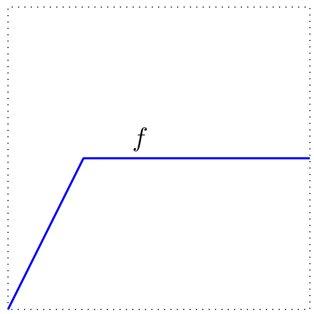

FIGURE 31.
$f : [0,1] \to [0,1]$

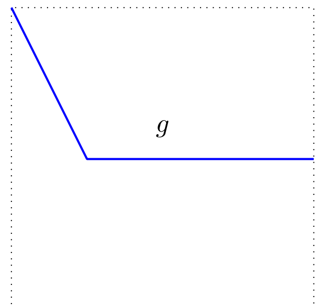

FIGURE 32.
$g : [0,1] \to [0,1]$

Referring to the critical point of T_s as $(\frac{1}{2}, \frac{s}{2})$, the graphs of f and g are identical on the domain $[\frac{s}{2}, 1]$, where $\frac{s}{2} < \frac{1}{2}$. Note that $\varinjlim f$ is the arc with endpoints $(0,0,0,\ldots)$ and $(1, \frac{1}{2}, \frac{1}{2}, \ldots)$, whereas $\varinjlim g$ is the arc with endpoints $(0, 1, \frac{1}{2}, \frac{1}{2}, \ldots)$ and $(1, \frac{1}{2}, \frac{1}{2}, \ldots)$. Further, observe that if $x_n \in [\frac{s}{2}, 1]$, then $x_i = \frac{1}{2} \in [\frac{s}{2}, 1]$ for all $i \geq n$ and it will not matter whether we apply f or g to obtain x_{i+1}. Therefore, without loss of generality, we can assume that once g has been applied to a point x_i, it will be applied to all subsequent iterates. The point $(\frac{s}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$ is a branch point in $\varprojlim H_s$. That is, the arc connecting $(\frac{s}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$ and $(1, \frac{1}{2}, \frac{1}{2}, \ldots)$ is exactly the intersection of $\varinjlim f$ and $\varinjlim g$, and it is exactly at this point where the two arcs diverge. Let A and B be arcs with endpoint $(1, \frac{1}{2}, \frac{1}{2}, \ldots)$. Let $\mathbf{x} \in A$ be such that $x_0 \neq 1$ and $\mathbf{y} \in B$ be such that $y_0 \neq 1$; let $t_0 = \max\{x_0, y_0, \frac{s}{2}\}$. Then the non-degenerate arc with endpoint $(t_0, \frac{1}{2}, \frac{1}{2}, \ldots)$ and $(1, \frac{1}{2}, \frac{1}{2}, \ldots)$ is contained in $A \cap B$ (moreover, this arc is contained in the arc connecting the branch point $(\frac{s}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$ and $(1, \frac{1}{2}, \frac{1}{2}, \ldots)$). As A and B were arbitrarily chosen arcs with the endpoint $(1, \frac{1}{2}, \frac{1}{2}, \ldots)$, it follows that $(1, \frac{1}{2}, \frac{1}{2}, \ldots)$ is an endpoint of $\varprojlim H_s$. Due to the location of the branch points in $\varprojlim H_s$, this argument can be generalized to show that every point of the form $(0,0,0,\ldots)$, $(0, \ldots, 0, 1, \frac{1}{2}, \frac{1}{2}, \ldots)$, or $(1, \frac{1}{2}, \frac{1}{2}, \ldots)$ is an endpoint of $\varprojlim H_s$. Further, by observing the endpoints of Γ_n, there are no other possible endpoints of $\varprojlim H_s$. Thus, there is a countable collection of endpoints for $\varprojlim H_s$ when $s < 1$. Additionally, $\varprojlim H_s$ is homeomorphic to $\varprojlim H_t$ whenever $s, t \in (0,1)$, and $\varprojlim H_s$ is homeomorphic to $\varprojlim F_s$ for all $0 < s < 1$.

7. Observations about Endpoints in This Family of Set-Valued Functions

We would like to conclude this paper by making some observations about the collections of endpoints of $\varprojlim T_s$, $\varprojlim F_s$, $\varprojlim G_s$, and $\varprojlim H_s$ for various parameters $s \in (0,2)$. First of all, for any symmetric tent map T_s $(0 < s < 2)$, the point $(0,0,0,\ldots)$ is always an endpoint of $\varprojlim T_s$ and will remain an endpoint of $\varprojlim F_s$, $\varprojlim G_s$, and $\varprojlim H_s$. No other points of $\varprojlim T_s$ will ever be an endpoint of $\varprojlim F_s$, $\varprojlim G_s$, or $\varprojlim H_s$; however, in some cases, $\varprojlim F_s$, $\varprojlim G_s$, and $\varprojlim H_s$ may contain other endpoints.

Recall that in the case where $s > 1$, the collection of endpoints of $\varprojlim T_s$ is always contained in the set $\mathcal{F} = \{\mathbf{x} \in \varprojlim T_s : x_i \in \omega(c, T_s) \text{ for all } i \in \mathbb{N}\}$, and for many parameters s, the collection of endpoints is exactly the set $\mathcal{F}$; see [1], [5], [3], [10]. Hence, it may seem surprising that the only endpoint of $\varprojlim F_s$ is $(0,0,0,\ldots)$, regardless of which slope $s > 1$ is selected. Although the collection of endpoints of $\varprojlim F_s$ and $\varprojlim F_t$ is the same for $s, t > 1$, we make no claims about the overall structures of $\varprojlim F_s$ and $\varprojlim F_t$ and how they compare to each other. The one commonality is that the arc connecting the critical point and the point $(1,1)$ somehow makes all the non-zero endpoints of $\varprojlim T_s$ non-endpoints, while at the same time not introducing any new endpoints. On the other, $\varprojlim G_s$ will always contain infinitely many endpoints, and $\varprojlim H_s$ will always contain an uncountable collection of endpoints; again, none of the original endpoints of $\varprojlim T_s$ will be endpoints of $\varprojlim G_s$ or $\varprojlim H_s$, but in these cases, additional endpoints are introduced due to the attached arc.

In the case where $s = 1$, $\varprojlim T_1$ is precisely an arc with endpoints $(0,0,0,\ldots)$ and $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots)$. Although $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots)$ is an endpoint of $\varprojlim T_1$, this point is no longer an endpoint of $\varprojlim F_1$, $\varprojlim H_1$, or $\varprojlim G_1$; however, each newly obtained inverse limit contains infinitely many copies of $\varprojlim T_1$ that are all attached at $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots)$. It is rather interesting that $\varprojlim F_1$ is homeomorphic to $\varprojlim H_1$ and each is a (countable) fan. It appears that attaching an arc from the critical point to $(1,1)$ or from the critical point to $(\frac{1}{2},1)$ will result in similar behavior because the domain of each arc is contained in the image of the arc. However, when attaching an arc from the critical point to $(0,1)$, the domain and range of the arc only coincide on the value $\frac{1}{2}$. This will still result in a fan, but in this case, $\varprojlim G_1$ is Cantor fan. That is, somehow the arc from the critical point to $(0,1)$ introduces more possible preimage paths than when the other two arcs are attached.

In some ways, the most interesting differences between $\varprojlim T_s$, $\varprojlim F_s$, $\varprojlim G_s$, and $\varprojlim H_s$ occur when $s < 1$. For these parameters $\varprojlim T_s$ is a singleton point $(0,0,0,\ldots)$, but the other associated inverse limits are more interesting. As was the case when $s = 1$, $\varprojlim H_s$ and $\varprojlim F_s$ are homeomorphic and $\varprojlim H_s$ is homeomorphic to $\varprojlim H_t$ when $0 < s, t < 1$. However, $\varprojlim G_s$ is much more complicated than $\varprojlim F_s$ and $\varprojlim H_s$ and as in the case when $s = 1$, there will be uncountably many endpoints for $\varprojlim G_s$.

We were able to precisely draw the inverse limits $\varprojlim F_s$, $\varprojlim G_s$, and $\varprojlim H_s$ when $0 < s \leq 1$ because it is well known what $\varprojlim T_s$ is in each of those cases. Because $\varprojlim T_s$ is much more complicated when $s > 1$ and $\varprojlim T_s$ is embedded in $\varprojlim F_s$, $\varprojlim G_s$, and $\varprojlim H_s$, we do not have a complete understanding of the structure of those inverse limits. It is clear, based on studying the collection of endpoints for the inverse limits for these families of set-valued functions, that adding even a small arc to a function can have a significant effect on the topological structure of the associated inverse limit.

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