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HOMOLOGICALLY EQUIVALENT DISCRETE MORSE FUNCTIONS

MICHAEL AGIORGOUSIS, BRIAN GREEN, ALEX ONDERDONK, KIM RICH, AND NICHOLAS A. SCOVILLE

Abstract. A theory of homological equivalence of discrete Morse functions is developed in this paper, extending the work of R. Ayala, L. M. Fernández, D. Fernández-Ternero, and J. A. Vilches [*Discrete Morse theory on graphs*, Topology Appl. **156** (2009), no. 18, 3091–3100] and Ayala, Fernández, and J. A. Vilches [*Characterizing equivalent discrete Morse functions*, Bull. Braz. Math. Soc. (N.S.) **40** (2009), no. 2, 225–235]. This is accomplished by defining the homological sequence associated with a discrete Morse function on any finite simplicial complex and developing its basic properties. These properties allow us to show that certain homological sequences may be viewed as lattice walks satisfying parameters. We count the number of discrete Morse functions up to homological equivalence on all collapsible 2-dimensional complexes by constructing discrete Morse functions inducing the desired sequence. The paper concludes with an example to illustrate our construction.

1. Introduction

Discrete Morse theory was invented by Robin Forman [6] as an analogue of "smooth" Morse theory popularized by J. Milnor [10]. Many classical results in Morse theory, such as the Morse inequalities, carry over into the discrete setting [8]. Applications of discrete Morse theory are vast,

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ranging from applications in configuration spaces [11] to computer science search problems [7].

Let f and g be two discrete Morse functions defined on a 1-dimensional simplicial complex, i.e., a graph. Inspired by Liviu I. Nicolaescu [12], R. Ayala et al. [1] study the homological sequence of a discrete Morse function by introducing the notion of f and g being homologically equivalent, and they count the number of excellent discrete Morse functions on all graphs [5]. The authors continue their study of homological sequences in [3] and [4], where in the latter paper they define the homological sequence on 2-dimensional simplicial complexes. We continue their work in this paper by defining the homological sequence induced by an excellent discrete Morse function for all finite simplicial complexes. We are then able to prove in Theorem 3.4 that the homological sequence of any excellent discrete Morse function exhibits the same kind of behavior as Ayala et al. prove in the 1- and 2-dimensional case. From the properties we prove in Theorem 3.4, it is then immediate that an upper bound for the number of excellent discrete Morse functions, with $m = 2k+1$ critical values, on a given collapsible complex of dimension n is the number of lattice walks on $\mathbb{Z}^n$ of length $2k$ that start and end at $(1,0,0,\ldots,0)$ with each value $(a_1, a_2, \ldots, a_n)$ in the walk satisfying $a_i \geq 0$ for all $2 \leq i \leq n$ and $a_1 \geq 1$. In [12], Nicolaescu proves that for $n = 2$, the number of such walks is given by $C_k C_{k+1}$ where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k^{th} Catalan number. In fact, Nicolaescu derives this computation while counting the number of *smooth* Morse functions up to homological equivalence on S^2. We develop an alternative formula for this value in Proposition 4.1. We give a construction in Theorem 4.3 to show that when Δ is a collapsible 2-dimensional simplicial complex, we may construct $C_k C_{k+1}$ such discrete Morse functions. Our paper concludes with an example of the construction in Example 4.4.

2. Preliminaries

Let $[n] = \{1, 2, 3, \ldots, n\}$. An *abstract simplicial complex* Δ on $[n]$ is a collection of nonempty subsets of $[n]$ such that

(1) if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$;
(2) $\{i\} \in \Delta$ for every $i \in [n]$.

An element $\sigma \in \Delta$ of cardinality $i+1$ is called an *i-dimensional face* or an *i-face* of Δ. A 0-face is sometimes called a *vertex*. If $\sigma, \tau \in \Delta$ with $\tau \subseteq \sigma$, then σ is a *face* of τ and τ is a *coface* of σ. The *dimension* of Δ, denoted $\dim(\Delta)$, is the maximum of the dimensions of all its faces. We use $\sigma^{(i)}$ to denote a simplicial complex of dimension i, and we write $\tau < \sigma^{(i)}$ to denote any subcomplex of σ of dimension strictly less than i.

Definition 2.1. A *discrete Morse function* f on Δ is a function $f\colon \Delta \to \mathbb{R}$ such that for every p-simplex $\sigma \in \Delta$, we have

$$|\{\tau^{(p-1)} < \sigma : f(\tau) \geq f(\sigma)\}| \leq 1$$

and

$$|\{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\}| \leq 1.$$

A p-simplex $\sigma \in \Delta$ is said to be *critical* with respect to a discrete Morse function f if

$$|\{\tau^{(p-1)} < \sigma : f(\tau) \geq f(\sigma)\}| = 0$$

and

$$|\{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\}| = 0.$$

Example 2.2. The 2-dimensional simplicial complex is labeled with discrete Morse function f. The critical vertices are $f^{-1}(0)$, $f^{-1}(2)$, and $f^{-1}(3)$, while the critical edges are $f^{-1}(5)$, $f^{-1}(6)$, $f^{-1}(7)$, and $f^{-1}(10)$. The 2-simplex with value 8 is not critical, while the 2-simplex with value 11 is critical.

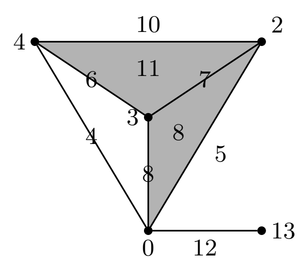

FIGURE 1. A discrete Morse function on a 2-dimensional complex.

We say that a discrete Morse function f is *excellent* if the critical values of f satisfy $c_0 < c_1 < ... < c_{m-1}$, i.e., all critical values of f are distinct.

Let $c \in \mathbb{R}$. The *level subcomplex* $\Delta(c)$ is the subcomplex of Δ consisting of all simplices τ with $f(\tau) \leq c$ as well as their faces, i.e.,

$$\Delta(c) = \bigcup_{f(\tau)\leq c} \bigcup_{\sigma \leq \tau} \sigma.$$

For each critical value $c_0, c_1, \ldots, c_{m-1}$ of f, we are interested in studying the behavior of the Betti numbers of the level subcomplexes $\Delta(c_0) \subset \Delta(c_1) \subset \ldots \subset \Delta(c_{m-1})$. We review simplicial homology and Betti numbers below.

2.1. Homological Sequences.

We briefly recall the theory of simplicial homology. Since we are only interested in the Betti numbers, we use coefficients in $\mathbb{R}$. Let Δ be a simplicial complex on $[n]$. Denote by $F_i(\Delta)$ the set of i-dimensional faces of Δ. Let $\sigma \in F_i(\Delta)$. Then to each σ, we associate the symbol e_σ to represent a basis element in the vector space $k^{|F_i(\Delta)|}$ generated by all the elements of $F_i(\Delta)$. The boundary operators $\partial_i\colon k^{|F_i(\Delta)|} \to k^{|F_{i-1}(\Delta)|}$ are defined as follows: Let $\sigma \in F_i(\Delta)$ and define $\partial_i(e_\sigma) = \sum\limits_{j\in\sigma} \operatorname{sgn}(j,\sigma)e_{\sigma-j}$ where $\operatorname{sgn}(j,\sigma) = (-1)^{i-1}$ if j is the i^{th} element of σ when the elements of σ are listed in increasing order. Then $\operatorname{im}(\partial_{i+1}) \subseteq \ker(\partial_{i+1})$, and we define the i^{th} *(unreduced) homology* of Δ to be the vector space $H_i(\Delta) = \ker(\partial_i)/\operatorname{im}(\partial_{i+1}) = k^{\operatorname{nul}(\partial_i)-\operatorname{rank}(\partial_{i+1})}$. The i^{th} *Betti number* of Δ is defined to be $b_i(\Delta) = \operatorname{nul}(\partial_i) - \operatorname{rank}(\partial_{i+1})$. Clearly, $b_j(\Delta) = 0$ for $j > n$.

Now let $f\colon \Delta \to \mathbb{R}$ be an excellent discrete Morse function on Δ. To each level subcomplex $\Delta(c_i)$, we consider the Betti numbers $b_i(\Delta(c_i))$. The *homological sequence of* f is given by the $n+1$ maps $B_0^f, B_1^f, \ldots, B_n^f\colon \{0,1,\ldots,m-1\} \to \mathbb{N}\cup\{0\}$ defined by $B_k^f(i) = b_k(\Delta(c_i))$ for all $0 \le k \le n$ and $0 \le i \le m-1$. We usually write $B_k(i)$ for $B_k^f(i)$ when the discrete Morse function f is clear from the context.

Example 2.3. Consider the discrete Morse function f in Example 2.2. This is an excellent discrete Morse function with critical values $0, 2, 3, 5, 6, 7, 10$, and 11. To find the homological sequence of f, we list the Betti numbers of $\Delta(0), \Delta(2), \Delta(3), \Delta(5), \Delta(6), \Delta(7), \Delta(10)$, and $\Delta(11)$. The homological sequence is given in the following table.

$$\begin{array}{l}
B_0 : 1 \quad 2 \quad 3 \quad 2 \quad 1 \quad 1 \quad 1 \quad 1\\
B_1 : 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 1\\
B_2 : 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0
\end{array}$$

Notice that only one value changes when moving from column to column and that the last column is the homology of the original simplex Δ even though $\Delta \neq \Delta(11)$. These observations and others are true of the homological sequence of any excellent discrete Morse function. We prove this in Theorem 3.4.

Two excellent discrete Morse functions $f, g\colon \Delta \to \mathbb{R}$ with m critical values are *homologically equivalent* if $B_k^f(i) = B_k^g(i)$ for all $0 \le k \le m-1$ and $0 \le i$. Homologically equivalent discrete Morse functions are first introduced and studied in [1]. When Δ is a 1-dimensional simplicial complex, the authors show the following.

Proposition 2.4 ([1]). *If f is an excellent discrete Morse function on a 1-dimensional simplicial complex, then the homological sequence of f satisfies $|B_0(i+1) - B_0(i)| = 0, 1$ and $B_1(i+1) - B_1(i) = 0, 1$. In addition, for all $i = 0, 1, \ldots, m-2$, exactly one of the following holds:*

(1) $B_0(i) = B_0(i+1)$;
(2) $B_1(i) = B_1(i+1)$.

In Theorem 3.4, we generalize this result to the homological sequence of an excellent discrete Morse function on any finite simplicial complex.

3. Homological Sequences

In order to generalize Proposition 2.4, the following lemmas are required, the first a classical result in discrete Morse theory [6] and the second a well-known fact about homology.

Lemma 3.1. [6, Theorem 3.3] *If $a < b$ are real numbers such that $[a, b]$ contains no critical values of f, then $b_i(\Delta(a)) = b_i(\Delta(b))$ for all integers $i \geq 0$.*

Lemma 3.2. *Let σ^p be a p-dimensional simplex such that $\sigma^p \notin \Delta$ and $\Delta \cup \sigma^p$ is a simplicial complex. Write $\overline{\Delta} = \Delta \cup \sigma^p$. For every integer $i \geq 0$, exactly one of the following holds:*

(1) $b_p(\overline{\Delta}) - b_p(\Delta) = 1$ *and* $b_{p-1}(\overline{\Delta}) - b_{p-1}(\Delta) = 0$;
(2) $b_{p-1}(\overline{\Delta}) - b_{p-1}(\Delta) = -1$ *and* $b_p(\overline{\Delta}) - b_p(\Delta) = 0$.

Furthermore, $b_d(\overline{\Delta}) = b_d(\Delta)$ for all $d \neq p, p-1$.

Lemma 3.3. *Let Δ be a simplicial complex with excellent discrete Morse function f and suppose that f has global minimum a. Then there is a unique 0-critical simplex σ such that $f(\sigma) = a$.*

Proof. By Lemma 3.1, $\Delta(x) = \emptyset$ for all $x < a$. Since f is excellent, there exists a unique simplex σ such that $f(\sigma) = a$. Thus, $\Delta(a) = \{\sigma\}$ and $|\Delta(a)| - |\Delta(x)| = 1$ so that σ must be a 0-dimensional critical simplex. $\square$

The following result can be interpreted as saying that the homological sequence of any excellent discrete Morse function is "well behaved" in the sense that only one Betti number can change for each subsequent level subcomplex by a value of ± 1.

Theorem 3.4. *Let f be an excellent discrete Morse function on a connected n-dimensional simplicial complex Δ with m critical values $c_0, c_1, \ldots, c_{m-1}$. Then each of the following holds:*

(1) $B_0(0) = B_0(m-1) = 1$ *and* $B_d(0) = 0$ *for all* $d \in \mathbb{Z}^{\geq 1}$.
(2) *For all* $0 \leq i < m-1$, $|B_d(i+1) - B_d(i)| = 0$ *or* 1 *whenever* $0 \leq d \leq n$ *and* $B_d(i) = 0$ *whenever* $d \geq n+1$.

(3) $B_d(m-1) = b_d(\Delta)$.
(4) *For each* $i = 0, 1, \ldots, m-2$, *either*
(a) $B_{p-1}(i) = B_{p-1}(i+1)$
or
(b) $B_p(i) = B_p(i+1)$
where $p = \dim(f^{-1}(c_i))$.
Furthermore, $B_d(i) = B_d(i+1)$ *for any* $d \neq p, p-1$ *and* $1 \leq d \leq n$.

Proof. We proceed in order. For (1), choose $y \in \mathbb{N}$ such that $\Delta(c_{m-1} + y) = \Delta$. By Lemma 3.1, $b_0(\Delta_{c_{m-1}}) = b_0(\Delta(c_{m-1}+y)) = b_0(\Delta)$. Since Δ is connected, $b_0(\Delta(c_{m-1})) = B_0(m-1) = 1$. By Lemma 3.3, $\Delta(0) = \sigma_0$. Thus, $B_d(0) = 0$ for all $d \in \mathbb{Z}^{\geq 1}$. This proves the first assertion.

For (2), we note that by Lemma 3.1, $b_d(\Delta(c_i)) = b_d(\Delta(x))$ for any $x \in [c_i, c_{i+1})$. Since f is excellent, there exists $\epsilon > 0$ such that $\Delta(c_{i+1}) = \Delta(c_{i+1} - \epsilon) \cup \sigma^p$ where σ^p is a critical p-simplex such that $f(\sigma^p) = c_{i+1}$. We now apply Lemma 3.2 for each of the following cases: if $p = d$, then $B_d(i+1) - B_d(i) = 0$ or 1. If $p = d+1$, then $B_d(i+1) - B(i) = -1$ or 0. Otherwise, $B_d(i+1) - B_d(i) = 0$. This proves (2).

For (3), observe that $m-1$ is the maximum critical value. By Lemma 3.1, B_d is constant for all values $x > c_{m-1}$. Since there is a $y \in \mathbb{N}$ such that $\Delta(c_{m-1}+y) = \Delta$, we see that $B_d(m-1) = b_d(\Delta)$.

Finally, we apply Lemma 3.1 to see that $b_d(\Delta(c_i)) = b_d(\Delta(x))$ for all $x \in [c_i, c_{i+1})$. Since f is excellent, there exists $\epsilon > 0$ such that $\Delta(c_{i+1}) = \Delta(c_{i+1} - \epsilon) \cup \sigma_p$ as in the proof of (2). Observe that, by Lemma 3.2, the addition of a p-dimensional simplex will change either B_p or B_{p-1}, leaving all other values fixed. □

4. Counting Discrete Morse Functions

Let f be an excellent discrete Morse function on Δ. If Δ is 1-dimensional and collapsible (i.e., a tree), then [1, Theorem 6.1] shows that the number of homological sequences with $m = 2k+1$ critical values is given by the k^{th} Catalan number. To see this, observe that by Theorem 3.4, we have that $B_0(i+1) - B_0(i) = \pm 1$ and that $B_j = 0$ for all $j \geq 2$. Furthermore, $B_0(0) = B_0(m-1) = 1$ so that B_0 is a walk in $\mathbb{Z}^+$ starting and ending at 1 with length $m - 1 = 2k$ with step size ± 1. As pointed out in [1, p. 3096], this value is known [9] to be precisely the k^{th} Catalan number.

In this section, we wish to extend these results by investigating the case where Δ is a 2-dimensional collapsible, connected simplicial complex. Indeed, suppose Δ is a collapsible, connected 2-dimensional simplicial complex, and let $f \colon \Delta \to \mathbb{R}$ be an excellent discrete Morse function with $m = 1 + 2k$ critical values. We associate a lattice walk with the induced homological sequence of f by considering the walk $\{(B_d(0), B_d(1))\}_{d=0}^{m-1}$.

By Theorem 3.4(1), $B_0(0) = 1$ and $B_1(0) = 0$, so the walk begins at $(1,0)$. By (1) and (3), $B_0(m-1) = 1$ and $B_1(m-1) = b_1(\Delta) = 0$ since Δ is collapsible; hence, the walk ends at $(1,0)$. By (2) and (4), each subsequent value can change by ± 1 in exactly one coordinate, so that this is a lattice walk of step size ± 1. Since $B_0(i) > 0$ and $B_1(i) \geq 1$ for all i, we obtain a lattice walk of length $2k$ in $\mathbb{Z}^2$ starting and ending at $(1,0)$ with first coordinate positive and second coordinate nonnegative. The number of such walks has been computed explicitly by Nicolaescu [12] to be C_kC_{k+1}, the product of consecutive Catalan numbers. We obtain an alternative formula for this value.

Proposition 4.1. *Let f be an excellent discrete Morse function on a 2-dimensional collapsible complex with $m = 2k+1$ critical values. An upper bound for the number of homology equivalence classes of excellent discrete Morse functions is*

$$\sum_{\ell=0}^{k} \binom{m-1}{2\ell} C_{k-\ell}C_\ell.$$

Proof. By Theorem 3.4, $B_1(0) = B_1(m-1) = 0$ so that each time a value in the B_1 increases by 1, it must also decrease by 1. Hence, the number of times B_1 changes (increases or decreases) is even, say B_1 changes 2ℓ times. Since $B_1(0) = 0$, there are $m-1$ positions to place 2ℓ changes in B_1, giving us $\binom{m-1}{2\ell}$. For any fixed choice, the B_1 sequence exhibits a walk in $\mathbb{Z}^{\geq 0}$ starting and ending at 0 with ℓ steps of size ± 1. Since there are exactly C_ℓ such walks, there are $\binom{m-1}{2\ell}C_\ell$ choices for position and value of the B_1 sequence.

Now recall that when B_1 changes, B_0 remains constant. Thus, there are $m - 2\ell = 2(k-l)+1$ available inputs in the B_0 sequence for a total of $C_{k-\ell}$ arrangements for B_0.

Thus, the total number of possible sequences on m critical values is

$$\sum_{\ell=0}^{k} \binom{m-1}{2\ell} C_{k-\ell}C_\ell,$$

which is what we desired to show. $\square$

Remark 4.2. Note that Proposition 4.1 then implies the well-known fact that if $m = 2k+1$, then $\sum\limits_{\ell=0}^{k} \binom{m-1}{2\ell} C_{k-\ell}C_\ell = C_kC_{k+1}$.

We now show that the above upper bound is the actual number of excellent discrete Morse functions on a collapsible 2-dimensional simplicial complex up to homological equivalence. As noted, for $m = 2k+1$ critical values, this yields $C_k C_{k+1}$ classes. In [12], Nicolaescu computes this value to count the number of smooth Morse functions on the 2-sphere S^2 up to homological equivalence. Hence, the theorem below provides another nice symmetry between smooth and discrete Morse theory. In [4, Theorem 5.1], Ayala et al. give a sketch of a proof which counts the number of excellent discrete Morse functions on all compact orientable surfaces. Below we give the full details of an alternative proof using the construction of excellent discrete Morse functions on any 1-dimensional simplicial complex found in [5, Theorem 4.3]. Let $\Delta^1 := \{\sigma \in \Delta : \dim(\sigma) \leq 1\}$ denote the 1-skeleton of Δ. The technical hypothesis about Δ^1 is to ensure that we can apply this construction to obtain any homological sequence. (See introductory remarks of the proof of Theorem 4.3 [5].)

Theorem 4.3. *Let Δ be a collapsible 2-dimensional simplicial complex such that Δ^1 contains at least one vertex of degree 1 or that Δ^1 is a non-trivial bridgeless graph. Then the number of excellent discrete Morse functions up to homological equivalence with $m = 2k+1$ critical elements on Δ is $C_k C_{k+1}$.*

Proof. By Remark 4.2, we know that the number of excellent discrete Morse functions on Δ with $m = 2k+1$ critical values is bounded above by $C_k C_{k+1}$. Hence, given a homological sequence, we construct an excellent discrete Morse function with $m = 2k+1$ critical values which realizes this sequence. As observed in the proof of Proposition 4.1, for a fixed $0 \leq \ell \leq k$, there are 2ℓ nonzero entries in the row B_1. By Theorem 3.4, any such homological sequence on Δ is of the form

$$
\begin{array}{llllllll}
B_0\colon & n_0 & n_1 & \dots & n_{t_1} & n_{t_1} & n_{t_1+1} & \dots \\
 & n_{t_i} & n_{t_i} & \dots & n_{t_{2\ell}} & n_{t_{2\ell}} & n_{t_{2\ell+1}} & \dots & 1 \\
B_1\colon & 0 & 0 & \dots & 0 & 1 & 1 & \dots \\
 & s \pm 1 & s & \dots & 1 & 0 & 0 & \dots & 0.
\end{array}
$$

Homological sequence A

We will construct an excellent discrete Morse function g on Δ with homological sequence A. This is accomplished by first constructing an excellent discrete Morse function on Δ^1, the 1-skeleton of Δ.

Begin by subdividing Δ as necessary to obtain enough simplices. Remove the 2-simplices of Δ to obtain Δ^1. The resulting skeleton is a graph with $b := b_1(\Delta^1)$ independent cycles. By [4, Theorem 5.1] there is an excellent discrete Morse function f on Δ^1 with the following homological sequence:

B_0:	n_0	n_0	n_0	n_0	$\dots$	n_0	n_1	n_2	$\dots$
	n_{t_1}	n_{t_1},	$\dots$	1					
B_1:	0	1	2	3	$\dots$	$b-\ell$	$b-\ell$	$b-\ell$	$\dots$
	$b-\ell$	$b-\ell+1$	$\dots$	b.					

Homological sequence B

Homological sequence B is obtained by starting with the homological sequence A and inserting $b-\ell$ cycles after the initial critical value. This has the effect of shifting the B_0 sequence of homological sequence A by $b-\ell$ entries to the right, and the corresponding B_1 value is $b-\ell$. Since the first time B_1 increases in homological sequence A is at n_{t_1}, homological sequence B also has B_1 increase after n_{t_1} except that it increases from $b-\ell$ to $b-\ell+1$. Continue in this manner until the first time B_1 decreases in homological sequence A.

We define $g = f$ on Δ^1. We will next insert the 2-simplices back into Δ^1 and label them so that the sequence on Δ^1 is transformed into homological sequence A on Δ. Since Δ is collapsible, each 2-simplex $\tau \in \Delta$ may be associated with a critical edge bounding τ. Hence, there is a one-to-one correspondence between the first $b-\ell$ critical edges and 2-simplices bounding that edge. Call these critical edges $e_1, e_2, \dots, e_{b-\ell}$ and their corresponding 2-simplices $d_1, d_2, \dots, d_{b-\ell}$. Define $g(d_i) = f(e_i)$ where $1 \leq i \leq b-\ell$. Since f is excellent, the critical edge e_i is the only simplex with value $f(e_i)$. Hence, defining $g(f_i) = f(e_i)$ will still yield a discrete Morse function, but now each of the e_i is not critical under the function g. In addition, each d_i is non-critical.

Now let t_j be the first index in homological sequence A such that $B_1(t_j) - B_1(t_{j+1}) = 1$. Choose any 2-simplex that has not yet been labeled and whose boundary is in the current level subcomplex. Label this simplex so that it has a value greater than the maximum of all values of the current level subcomplex, but less than the values on $f(\Delta^1)$ that are not yet in the current level subcomplex. This will ensure that the 2-simplex is critical, so that $B_1(t_j) - B_1(t_{j+1}) = 1$. Repeat this step as necessary.

The resulting discrete Morse function g on Δ will have the given homological sequence. □

We give an example of the construction in Theorem 4.3.

Example 4.4. Let Δ be the collapsible 2-dimensional complex given in Figure 2.

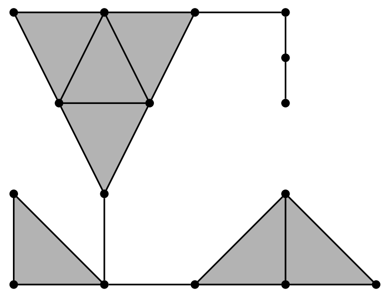

FIGURE 2. Collapsible 2-dimensional complex.

We will construct an excellent discrete Morse function on Δ with the following homological sequence:

$$\begin{array}{cccccccccccc} B_0 & 1 & 2 & 1 & 2 & 2 & 3 & 3 & 2 & 1 & 1 & 1 \\ B_1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0. \end{array}$$

Use the result of [5, Theorem 4.3] to construct an excellent discrete Morse function f on Δ^1 with the following homological sequence:

$$\begin{array}{ccccccccccccccc} B_0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 3 & 2 & 1 & 1 \\ B_1 & 0 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 7. \end{array}$$

Such a discrete Morse function is given below.

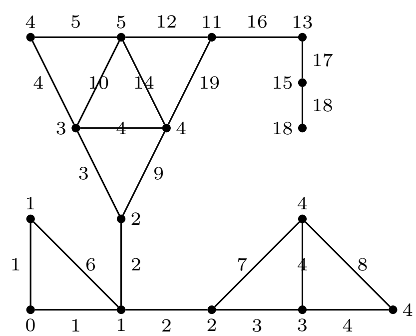

FIGURE 3. Discrete Morse function on Δ^1.

Now pick five 2-simplices and label them with the maximum value of their boundary edge value.

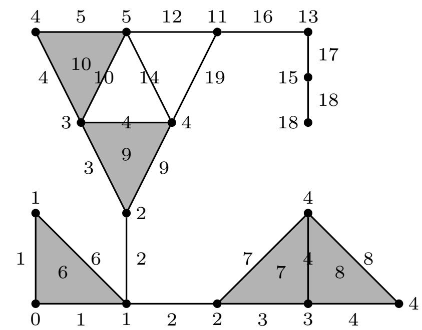

FIGURE 4. Discrete Morse function with desired homological sequence.

The remaining two 2-simplices are labeled slightly greater than the maximum of all the simplices in the current level subcomplex, where the current level subcomplex is determined by when B_1 decreases in our original homological sequence. Hence, the discrete Morse function g is given in Figure 5.

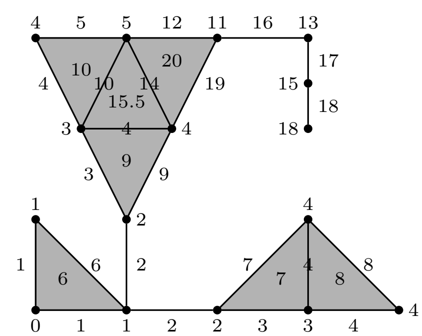

FIGURE 5. Discrete Morse function g.

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