http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 54, 2019

Pages 305–322

http://topology.nipissingu.ca/tp/

On the Notion of Tree-Likeness for Generalized Continua

by

Włodzimierz J. Charatonik, Tomás Fernández-Bayort, and Antonio Quintero

Electronically published on May 13, 2019

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 54 (2019)
Pages 305-322

E-Published on May 13, 2019

ON THE NOTION OF TREE-LIKENESS FOR GENERALIZED CONTINUA

WŁODZIMIERZ J. CHARATONIK, TOMÁS FERNÁNDEZ-BAYORT, AND ANTONIO QUINTERO

Abstract. A variety of equivalent approaches to tree-likeness is available in classical continuum theory. In absence of compactness, some of those equivalences do not hold. In this paper, we compare the class of generalized continua defined as inverse limits of locally finite trees with proper bonding maps with the class of those for which any open cover admits acyclic refinements. We show that the latter is precisely the subclass of the former consisting of those generalized continua with exhausting sequences of tree-like continua. In addition, we show that locally injective proper maps onto tree-like generalized continua are homeomorphisms for the second definition but not for the first one, which, notwithstanding, is still reflected by such maps.

1. Introduction

The proper category is widely accepted as the most convenient framework for the study of the topology of locally compact spaces; in particular, classes of spaces and maps of interest in continuum theory are extended to the proper category. Recall that a map $f : X \to Y$ is said to be *proper* (also termed *perfect* in the literature) if for any compact subset $K \subset Y$, $f^{-1}(K)$ is compact in X. It is well known that proper maps are closed maps [6, Theorem 3.7.18].

2010 *Mathematics Subject Classification.* 54F15, 18A30, 54C10, 54D35.
Key words and phrases. acyclic cover, Freudenthal compactification, generalized continuum, inverse limit, proper map, tree-like.

This paper is focussed on the extension to the proper category of the well-known class of tree-like continua. These spaces are described as inverse limits of sequences of finite trees. Alternatively, a tree-like continuum X is characterized by the property that all open covers of X admit acyclic refinements. See [10, Theorem 2.1].

Passing to the proper category via the inverse limit approach, the authors define in [5] a locally compact metric space to be a *tree-like space* if it can be expressed as the limit of an inverse sequence of locally finite trees with proper bonding maps. An interesting property of this notion of tree-likeness is proved in [5, Theorem 4.1]: The tree-likeness of a generalized continuum is equivalent to the tree-likeness of its Freudenthal compactification.

In this paper, we compare the definition of tree-likeness in [5] with the definition based on the existence of acyclic refinements for arbitrary open covers. Similar to the compact setting, this second approach to tree-likeness can be characterized by purely homotopical properties (theorems 3.1 through 3.4).

Nevertheless, in contrast to the classical continuum theory, the two notions of tree-likeness are not equivalent in the proper category since the second approach is strictly stronger than the first one (Example 4.3). We prove that both approaches coincide exactly on the class of those generalized continua with an exhausting sequence of tree-like continua (Theorem 4.4).

We finish the paper by showing that local homeomorphisms (and, more generally, locally injective proper surjections) onto tree-like generalized continua are homeomorphisms for the stronger definition (Theorem 6.5), but not for the weaker one (Example 6.2). Nevertheless, the latter is still reflected by such maps (Theorem 6.3). These results are extensions to the proper category of a theorem for continua due to Jo W. Heath [9], improving an older result by T. Maćkowiak [13].

2. Some Preliminaries

Throughout this paper, a *continuum* (*generalized continuum*, respectively) is a connected compact (locally compact, respectively) metric space.

It follows from [6, Theorem 5.1.27] that any generalized continuum is second countable and σ-compact [6, Corollary 4.1.16 and Exercise 3.8.C(b)]. Moreover, local compactness, together with σ-compactness, readily implies the existence of *exhausting sequences* in a generalized continuum X, that is, increasing sequences of compact subsets $X_n \subset X$ with $X = \cup_{n=1}^{\infty} X_n$ and $X_n \subset \mathrm{int} X_{n+1}$.

Given an exhausting sequence $\{X_n\}_{n\geq 1}$ of the generalized continuum X, a *Freudenthal end of* X, $\varepsilon = (Q_n)_{n\geq 1}$, is a decreasing nested sequence of quasicomponents $Q_n \subset X - \mathrm{int}X_n$. Recall that the *quasicomponent* of a point x is defined to be the intersection of all open and closed sets containing x.

Let $\mathcal{F}(X)$ denote the set of all Freudenthal ends of X. The set $\widehat{X} = X \cup \mathcal{F}(X)$ admits a compact metrizable topology, called the *Freudenthal compactification of* X, for which the subspace $\mathcal{F}(X) \subset \widehat{X}$ turns out to be compact and 0-dimensional. Any proper map $f : X \to Y$ between generalized continua extends to a continuous map $\widehat{f} : \widehat{X} \to \widehat{Y}$ with a continuous restriction $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$; see [2] for details.

3. Spaces with Enough Acyclic Refinements in the Proper Category

A cover of a space X is termed *acyclic* if it does not contain cyclic chains of length ≥ 3. Here, a *cyclic chain* is a sequence of distinct subsets of X, $V_1, \ldots, V_n$, such that $V_i \cap V_{i+1} \neq \varnothing$ if $1 \leq i \leq n-1$ and $V_1 \cap V_n \neq \varnothing$. We will say that X *has enough acyclic refinements* if any open cover of X admits an acyclic refinement. Clearly, having enough acyclic refinements implies $\dim X \leq 1$. Hence, by using the one-point compactification, [7, Theorem 1.11.4] shows that X admits an embedding into $\mathbb{R}^3$ as a closed set. Furthermore, the following theorem (collecting previous results in [4] and [8]) characterizes in a purely homotopical manner the existence of enough acyclic covers; compare with [10].

Theorem 3.1. *Let X be a 1-dimensional generalized continuum. Then the following statements are equivalent, where G ranges over the class of graphs.*

(i) *Any open cover of X admits a locally finite countable acyclic refinement.*
(ii) *X has enough acyclic refinements.*
(iii) *Any continuous map $f : X \to G$ is inessential.*
(iv) *Any proper map $f : X \to G$ is inessential.*
(v) *Any proper map $f : X \to G$ factorizes as a composite of proper maps $X \to T \to G$ through a tree T.*

Here, by a *graph*, we mean a 1-dimensional polyhedron, and *trees* are graphs with no cycles. We follow the convention of [14], and so graphs and trees will be assumed to be locally finite throughout the paper. Notice that Theorem 3.1(iii) is an extension of the well-known characterization of tree-like continua due to J. H. Case and R. E. Chamberlin [4, Theorem 1]. Recall that a map $f : X \to Y$ is termed *inessential* if it is homotopic to a constant map.

Proof of Theorem 3.1. Both (i)⇒(ii) and (iii)⇒(iv) are immediate. Moreover, (ii)⇒(iii) appears as part of the proof of [4, Theorem 1], whereas the equivalence (iv)⇔(v) is proved in [8, Lemma 5].

It remains to check (v)⇒(i). Let $\mathcal{U}$ be any open cover of X. By using the paracompactness, the local compactness, the 1-dimensionality, and the Lindelöff property of X, we can refine $\mathcal{U}$ by a countable locally finite open cover $\mathcal{V} = \{V_i\}_{i\geq 1}$ of order 2 whose elements have compact closure. In particular, the nerve of $\mathcal{V}$, $N = N(\mathcal{V})$, is a graph and the canonical barycentric map $\alpha : X \to N$ is proper. Therefore, by (v), α factorizes up to proper homotopy as a composite of proper maps $\alpha = h \circ g : X \to T \to N$ where T is a tree.

Let $\mathcal{S} = \{S_n\}_{i\geq 1}$ be the canonical cover of N by the open stars $S_i = \overset{\circ}{st}\,(V_i; N)$ and $\mathcal{W} = \{W_i\}_{i\geq 1}$ be the open cover of T with $W_i = h^{-1}(S_i)$. By [15, p. 126], there exists a subdivision $\mathfrak{T}$ of T such that the cover $\mathcal{U}$ consisting of the open stars of the vertices of $\mathfrak{T}$ refines $\mathcal{W}$. Notice that $\mathcal{U}$ is acyclic and so is the cover $g^{-1}\mathcal{U}$ defined by the counterimages by g of the elements of $\mathcal{U}$. Moreover, $g^{-1}\mathcal{U}$ refines $g^{-1}\mathcal{W} = \alpha^{-1}\mathcal{S} = \mathcal{V}$. The last equality holds since the definition of α yields $\alpha^{-1}(\overset{\circ}{st}\,(V_i; N)) = V_i$ for all $i \geq 1$. □

In [11], it is proved that for ordinary continua, the arbitrary graph G in Theorem 3.1(iii) (i.e., the Case–Chamberlin theorem) can be replaced by the wedge of two circles $S^1 \vee S^1$. Next, we improve Theorem 3.1 by the following addendum, which extends the main result in [11] to generalized continua.

Theorem 3.2. *Under the assumptions of Theorem 3.1, each of the conditions listed there is equivalent to the following one:*
(iii′) *Any continuous map $f : X \to S^1 \vee S^1$ is inessential.*

In order to prove Theorem 3.2, we simply modify the proof in [11] to surmount the fact that non-compact sets of $\mathbb{R}^n$ do not have a countable base of neighbourhoods. For this we use that graphs are ANR.

Proof of Theorem 3.2. Clearly, (ii)⇒(iii′) follows from Theorem 3.1. Conversely, assume that there exists an essential map $f : X \to G$ for some graph G. As done in [11], let $g : G \to S^1 \vee S^1$ be a continuous map such that $g_* : \pi_1(G) \to \pi_1(S^1 \vee S^1)$ is an injective homomorphism. Here we use [15, Theorem 8.1.11] and the well-known fact that a free group with a countable base is isomorphic to a subgroup of the free group with two generators.

We claim that the composite $h = g \circ f : X \to S^1 \vee S^1$ is essential. For this, embed X as a closed set in $\mathbb{R}^3$ and let $f' : U \to G$ be an extension of f to a connected open neighbourhood of X. Here, we use that G is an ANR.

Assume for a moment that h is inessential. Then by the homotopy extension property, h extends to an inessential map $h' : W \to S^1 \vee S^1$ where W is an open neighbourhood of X. Let Ω' be the connected component of $W \cap U$ containing X. As both $g \circ f'$ and h' extend h, we use that $S^1 \vee S^1$ is an ANR to find a connected open neighbourhood of X, $\Omega'' \subset \Omega'$, and a homotopy $g \circ f'' \simeq h''$ for the restrictions $f'' = f'|\Omega''$ and $h'' = h'|\Omega''$. Then $g_* \circ f''_* = h''_* : \pi_1(\Omega'') \to \pi_1(S^1 \vee S^1)$ is the trivial homomorphism. As g_* is injective, f''_* is trivial as well. Hence, by [15, Theorem 2.4.5], f'' lifts to the universal covering space of G, which is a tree and therefore contractible. Here, we use that Ω'' is locally path connected. Thus, f'' is inessential, and so is f, which is a contradiction. $\square$

After Theorem 3.2, one may ask for a "universal" infinite graph to replace in the proper setting the arbitrary graph in Theorem 3.1(iv) and (v). This is done in the following theorem where Σ^1 denotes the string of circles in Figure 1.

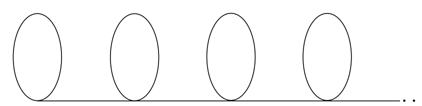
$\Sigma^1 \equiv$ 

FIGURE 1

Theorem 3.3. *Under the assumptions of Theorem 3.1, each of the conditions listed there is equivalent to each of the following two:*

(iv′) *Any proper map $f : X \to \Sigma^1$ factorizes as a composite of proper maps $X \to T \to \Sigma^1$ through a locally finite tree T.*

(v′) *Any proper map $f : X \to \Sigma^1$ is inessential.*

For the proof of Theorem 3.3, we recall that the choice of a root vertex $v_0 \in T$ yields a canonical ordering on the vertex set of a tree T. Namely, we write $w \le v$ if w lies in the unique arc $\Gamma_v \subset T$ from v_0 to v. The number of edges in Γ_v is termed the *height* of v in T.

Proof of Theorem 3.3. Clearly, (ii)⇒(iv′) follows from Theorem 3.1. Since trees are contractible, (iv′)⇒(v′) is immediate. We will prove (v′)⇒(ii) by contradiction. By Theorem 3.1, let us assume that $g : X \to G$ is an essential map into a graph G. We will reach a contradiction by constructing an essential proper map $f : X \to \Sigma^1$ as follows.

The classification of the proper homotopy types of graphs in [1] provides us with a proper homotopy equivalence $\psi_G : G \to S(G)$ where $S(G)$ is a "tree of circles," that is, a tree $T_{S(G)}$ with finitely many (possibly none) circles attached at each vertex. Then the composite $g' = \psi_G \circ g : X \to S(G)$ is also essential.

Let $r : T_{S(G)} \to \mathbb{R}_{\geq 0}$ be the proper map which carries each vertex $v \in T_{S(G)}$ to its height and the edges of $T_{S(G)}$ are mapped linearly onto the corresponding interval. Let $S'(G)$ be the "string of circles" obtained by attaching at $n \in \mathbb{R}_{\geq 0} = T_{S'(G)}$ as many circles as circles of $S(G)$ are attached at vertices of height n. The obvious proper extension of r, $r' : S(G) \to S'(G)$, induces a homeomorphism $S(G)/T_{S(G)} \cong S'(G)/\mathbb{R}_{\geq 0}$. Hence, r' is an ordinary homotopy equivalence since the quotient maps $S(G) \to S(G)/T_{S(G)}$ and $S'(G) \to S'(G)/\mathbb{R}_{\geq 0}$ are as well.

Finally, by [1], there is a proper homotopy equivalence $h_1 : S'(G) \to \Sigma^1$ if $S(G)$ contains infinitely many circles (or, equivalently, $\pi_1(G)$ has infinite rank), or a proper homotopy equivalence $h_2 : S'(G) \to S_n \subset \Sigma^1$ where S_n is the union of $\mathbb{R}_{\geq 0}$ with the n first circles of Σ^1, if $\pi_1(G)$ has rank n. In each case, the composite $f = h_i \circ r' \circ g' : X \to \Sigma^1$ is the required essential proper map. □

Theorem 3.1 can also be improved with a further equivalent condition involving the proper shape in the sense of [2], as in the theorem below.

Theorem 3.4. *Under the assumptions of Theorem 3.1, each of the conditions listed there is equivalent to each of the following two.*

(vi) *X has the proper shape of a tree.*

(vii) *Any proper map into a locally compact ANR, $f : X \to Y$, factorizes up to proper homotopy through a tree.*

Recall that, in [14, Theorem 3.1], having the proper shape of a tree is characterized by the so-called SUV$^\infty$ property. Namely, a space X has the proper shape of a tree if and only if given a closed embedding $X \subset Y$ into some (equivalently, any) locally compact ANR Y and any closed neighbourhood of X in Y and V, there is another closed neighbourhood, $U \subset V$, such that the inclusion $i : U \to V$ factorizes up to proper homotopy as a composite $h \circ g : U \to T \to V$ where T is a tree.

Proof of Theorem 3.4. (vi)⇒(vii). We start by embedding X in the Euclidean space $\mathbb{R}^3$ as a closed set. As Y is a locally compact ANR, there is a proper extension of f to a closed neighbourhood of X, $\widetilde{f} : V \to Y$ ([2, Lemma 3.2]). Moreover, as X has the proper shape of a tree, there is a closed neighbourhood $X \subset U \subset V$ such that the inclusion $i : U \to V$ is properly homotopic to a composite $h \circ g : U \to T \to V$ for some tree T. Then $f = \widetilde{f}|X$ is properly homotopic to the composite $\widetilde{f} \circ h \circ g|X$, which is the required factorization of f.

(vii)$\Rightarrow$(ii) Since graphs are ANR, condition (vii) yields that any proper map $f : X \to G$ into a graph is inessential and so (ii) follows from Theorem 3.1.

Finally, as X is 1-dimensional, it can be embedded as a closed set into $\mathbb{R}^3$ and so in order to prove (ii)$\Rightarrow$(vi), it will suffice to check the SUV$^\infty$ property for a closed embedding of X into $\mathbb{R}^3$. Let V be any closed neighbourhood of X in $\mathbb{R}^3$. By using [16, Theorem 35], we find a triangulated (infinite) polyhedron $Z \subset V$ which is also a closed neighbourhood of X. Next, we consider the locally finite open cover of Z, $\mathcal{S} = \{S_v\}_{v\in Z}$, consisting of the open stars $S_v = \overset{\circ}{st}\,(v; Z)$ of the vertices of Z, and let $i^{-1}\mathcal{S}$ be the locally finite open cover of X formed with the counterimages $i^{-1}(S_v)$ for the inclusion $i : X \to Z$. By hypothesis, there is a locally finite acyclic refinement $\mathcal{O} \prec i^{-1}\mathcal{S}$ whose nerve $N(\mathcal{O})$ is a tree and the canonical barycentric map $\alpha : X \to N(\mathcal{O})$ is proper. Here, we use that $\mathcal{O}$ is locally finite and the open sets in $\mathcal{S}$, and so the open sets in $\mathcal{O}$, have compact closure. Let us consider the following diagram

$$\begin{tikzcd}
W \arrow[dr, "\widetilde{\alpha}"'] & X \arrow[l, "j"'] \arrow[r, "i"] \arrow[d, "\alpha"] & Z \arrow[r, "k"] & V \subset \mathbb{R}^3 \\
& N(\mathcal{O}) \arrow[r, "\pi"] & N(i^{-1}\mathcal{S}) \arrow[u, "\psi", "\cong"'] &
\end{tikzcd}$$

where π is the simplicial map defined by fixing for each $O \in \mathcal{O}$ an open set $\pi(O) = i^{-1}(S_v)$ with $O \subset i^{-1}(S_v)$, and ψ is the simplicial isomorphism which sends $i^{-1}(S_v)$ to v. Moreover, $\widetilde{\alpha}$ is an extension of α to a closed neighbourhood $X \subset W \subset V$ given by [2, Lemma 3.2]. Notice that the square is commutative up to proper homotopy since for each $x \in X$, $\psi \circ \pi \circ \alpha(x)$ lies in an edge of any simplex of Z containing x. By [2, Lemma 3.4], one finds a closed neighbourhood of X, $U \subset W$, such that the inclusion $U \subset V$ is properly homotopic to the composite $\psi \circ \pi \circ \widetilde{\alpha}|U$. $\square$

4. Tree-Likeness and the Existence of Enough Acyclic Refinements

This section is devoted to the comparison of tree-likeness and the existence of enough acyclic refinements in the proper category. In order to fix the terminology, we introduce the following definition.

Definition 4.1. A 1-dimensional generalized continuum X is termed *strongly tree-like* if it satisfies the equivalent conditions listed in §3. Namely,

(i) any open cover of X admits a locally finite countable acyclic refinement.
(ii) X has enough acyclic refinements;

(iii) any continuous map $f : X \to G$ is inessential;
(iv) any proper map $f : X \to G$ is inessential;
(v) any proper map $f : X \to G$ factorizes as a composite of proper maps $X \to T \to G$ through a tree T;
(vi) X has the proper shape of a tree;
(vii) any proper map into a locally compact ANR, $f : X \to Y$, factorizes up to proper homotopy through a tree.

See also the equivalent variations (iii′) in Theorem 3.2 and (iv′) and (v′) in Theorem 3.3.

The following result is an immediate consequence of Theorem 3.1 and [8, Theorem 11].

Proposition 4.2. *Any strongly tree-like generalized continuum X is tree-like.*

However, in contrast to the compact case (see [10, Theorem 2.1]), the converse of Proposition 4.2 does not hold in the non-compact setting as the following example shows.

Example 4.3. Let $X \subset [0, 1] \times \mathbb{R}_+$ be the plane subspace depicted in thick lines in Figure 2 below.

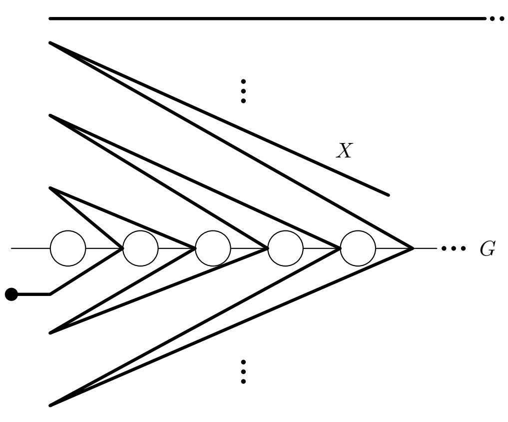


FIGURE 2

Notice that X is homeomorphic to the space obtained by removing the origin from the $\sin \frac{1}{x}$-curve. Then $\widehat{X}$ is homeomorphic to the $\sin \frac{1}{x}$-curve and, hence, X is a one-ended tree-like generalized continuum by [5, Theorem 4.1] (see the Introduction). However, in [8, Example 12], it is shown that the obvious projection of X onto the graph G depicted in thin lines in

Figure 2 (proper homotopy equivalent to Σ^1 in Figure 1) is a proper surjection which does not factorize through a tree. Thus, X is not strongly tree-like.

The following theorem characterizes the class of generalized continua for which both notions of tree-likeness coincide. The proof is rather lengthy and it will be postponed to the next section.

Theorem 4.4. *For any generalized continuum X, the following statements are equivalent.*

(i) *X is strongly tree-like.*
(ii) *X is tree-like and admits an exhausting sequence of subcontinua.*
(iii) *X admits an exhausting sequence consisting of tree-like subcontinua.*

A theorem due to T. B. McLean (see [8]) shows that tree-like continua are preserved by confluent maps. As an immediate consequence of Theorem 4.4, we get the extension to the proper category of McLean's theorem for the strong version of tree-likeness.

Proposition 4.5. *Let X be a strongly tree-like generalized continuum. If $f : X \to Y$ is a confluent proper surjection, then Y is also strongly tree-like.*

Recall that a continuous surjection $f : X \to Y$ is *confluent* if for every subcontinuum $B \subset Y$ we have that $f(A) = B$ for each connected component $A \subset f^{-1}(B)$. In that case, the restriction $f : A \to f(A) = B$ is also confluent. See [8].

In the proof of Proposition 4.5, we use the following lemma; see [5, Theorem 5.6] for a more general result.

Lemma 4.6. *Any subcontinuum $Z \subset X$ of a tree-like generalized continuum X is a tree-like continuum.*

Proof of Proposition 4.5. By Theorem 4.4, there is an exhausting sequence $X = \cup_{n\geq 1} X_n$ consisting of tree-like subcontinua. The properness of f yields that the images $\{f(X_n)\}_{n\geq 1}$ form an exhausting sequence of Y. We claim that each $f(X_n)$ is tree-like as well, and so Y is strongly tree-like by again applying Theorem 4.4.

To check the claim, take any component $C \subset f^{-1}(f(X_n))$. As f is confluent, $f(C) = f(X_n)$, and the restriction $f|C : C \to f(X_n)$ is also confluent. As C is tree-like (Lemma 4.6), so is $f(X_n)$ by McLean's theorem. $\square$

Remark 4.7. For arbitrary confluent proper maps, McLean's theorem does not hold in the non-compact setting for the weaker notion of tree-likeness; see [5, Remark 5.5] for an example. However, it still holds for end-preserving (i.e., the map induced on ends is a homeomorphism) confluent maps; see [5, Theorem 5.4].

5. Proof of Theorem 4.4

The proof will be given in two parts. For the first part we will use Lemma 4.6 above.

Proof of (ii)⇒(iii)⇒(i) in Theorem 4.4. We get (ii)⇒(iii) as an immediate consequence of Lemma 4.6. Moreover, if (iii) holds, let $\{X_n\}_{n\geq 1}$ be an exhausting sequence of X consisting of tree-like subcontinua and let $f : X \to G$ be any continuous map into a graph G. Consider the universal covering space $p : \widetilde{G} \to G$ and choose points $x_0 \in X_1$ and $\widetilde{x}_0 \in \widetilde{G}$ with $p(\widetilde{x}_0) = f(x_0)$. Since each X_n is a tree-like continuum, each restriction $f_n = f|X_n : X_n \to G$ is inessential, and there exists a lifting $\widetilde{f}_n : X_n \to \widetilde{G}$ with $\widetilde{f}_n(x_0) = \widetilde{x}_0$. Moreover, by the uniqueness of liftings on connected spaces it follows that $\widetilde{f}_{n+1}$ agrees with $\widetilde{f}_n$ on X_n and, therefore, we have a well-defined lifting of f by $\widetilde{f} = \cup_{n\geq 1}\widetilde{f}_n : X = \cup_{n\geq 1}X_n \to \widetilde{G}$. Here, we use that $\{X_n\}_{n\geq 1}$ is an exhausting sequence. Finally, as $\widetilde{G}$ is contractible, the composite $f = p \circ \widetilde{f}$ is inessential and, hence, X is strongly tree-like. This shows (iii)⇒(i). □

For the remaining part of the proof of Theorem 4.4, we recall from [12, Definition V.47.VIII.1] that the *constituent* of $x \in X$ is the union of all subcontinua of X containing x. Notice that a generalized continuum X is continuumwise connected if and only if X has just one constituent. Notice also that the intersection of any constituent with a compact set $A \subset X$ consists of a union of components of A.

The following lemmas concerning constituents will be crucial for the rest of the proof of Theorem 4.4.

Lemma 5.1. *Assume that the generalized continuum X has an exhausting sequence $\{X_n\}_{n\geq 1}$ with the following property:*

(P) *For each $n \geq 1$, there exists some $m_n > n$ such that for any constituent $A \subset X$, the intersection $A \cap X_n$ is contained in a component of X_{m_n}.*

Then X admits an exhausting sequence $\{X'_s\}_{s\geq 1}$ for which the following property holds:

(Q) *Each component $C \subset X'_s$ $(s \geq 1)$ can be written as an intersection $C = A_C \cap X'_s$ for some constituent A_C of X.*

Proof. We next construct inductively an increasing sequence of integers $1 = m_0 < m_1 < \ldots < m_s < \ldots$ and an increasing sequence of compact sets $\{X'_s\}_{s\geq 1}$ as follows. Given m_{s-1} (with $m_0 = 1$), let $\mathcal{A}_s$ be the family of all constituents of X which meet $X_{m_{s-1}}$. Then $m_s > m_{s-1}$ is chosen according to property (P); that is, for each $A \in \mathcal{A}_s$, there exists a component $C_A \subset X_{m_s}$ with $A \cap X_{m_{s-1}} \subset C_A$. We now define $X'_s = \cup_{A\in\mathcal{A}_s} C_A$. We claim that X'_s is a closed set of X_{m_s} and, hence, compact. Indeed, let $\{x_n\}_{n\geq 1} \subset X'_s$ be a sequence converging to some $x \in X_{m_s}$. As each C_A is compact, we can assume without loss of generality that $x_n \in C_{A_n}$ for a sequence of components $C_{A_n} \subset X_{m_s}$. Let $p_n \in A_n \cap X_{m_{s-1}}$. By compactness, there is no loss of generality in assuming, in addition, that $\{p_n\}_{n\geq 1}$ converges to some $p \in X_{m_{s-1}}$. Then both p and x belong to the inferior limit $\mathrm{Li}C_{A_n} \subset X_{m_s}$.

Recall that $\mathrm{Li}C_{A_n}$ consists of all points $x \in X$ such that each open neighbourhood of x meets eventually all C_{A_n}'s. Moreover, $L = \mathrm{Li}C_{A_n}$ is a continuum by [12, Theorem V.47.II.6]. Let A be the constituent of X containing L. As $p \in X_{m_{s-1}} \cap L \subset X_{m_{s-1}} \cap A$, we have $A \in \mathcal{A}_s$, and so $X_{m_{s-1}} \cap A \subset C_A$. Hence, $x \in L \subset C_A \subset X'_s$.

Furthermore, one readily checks $X_{m_{s-1}} \subset X'_s \subset X_{m_s}$, and so $X'_s \subset X_{m_s} \subset \mathrm{int}X_{m_{s+1}} \subset \mathrm{int}X'_{s+1}$. Hence, $\{X'_s\}_{s\geq 1}$ is an exhausting sequence of X. It remains to check that this sequence satisfies property (Q). In fact, if C is a component of X'_s, then C is a component of X_{m_s} for which there exists a constituent A with $\varnothing \neq A \cap X_{m_{s-1}} \subset C$. Then, obviously, $C \subset A \cap X'_s$. Moreover, by construction, any other component $Z \subset X_{m_s}$, $Z \neq C$, with $Z \subset A$ misses $X_{m_{s-1}}$ (otherwise, $Z \cap C \neq \varnothing$), and so it lies outside X'_s. Thus, $C = A \cap X'_s$. $\square$

Remark 5.2. Notice that property (Q) of the sequence $\{X'_s\}$ in Lemma 5.1 is equivalent to

(Q′) for any constituent $A \subset X$, the intersection $A \cap X'_s$ $(s \geq 1)$ is either empty or a whole component of X'_s.

One readily derives from property (Q′) that for any component $C \subset X'_{s+1}$, the intersection $C \cap X'_s$ is either empty or a whole component of X'_s.

Lemma 5.3. *Any generalized continuum X satisfying property (P) in Lemma 5.1 admits an exhausting sequence consisting of subcontinua.*

Proof. By Lemma 5.1, there exists an exhausting sequence of X, $\{X_s\}_{s\geq 1}$, satisfying property (Q′) in Remark 5.2. Thus, it will suffice to check that X is continuumwise connected; that is, X reduces to a unique constituent, and so each X_s is a continuum.

To prove that X is continuumwise connected, assume to the contrary that X has at least two constituents A and B. So, by property (Q$'$), whenever the intersections $A \cap X_s$ and $B \cap X_s$ are not empty, they are (distinct) components of X_s. We can assume without loss of generality that both A and B meet X_1.

Now we observe that, for each $s \geq 1$, the components of X_s define a u.s.c. decomposition with a 0-dimensional compact quotient $\pi_s : X_s \to \mathcal{D}_s$ [12, theorems V.46.V.3 and V.47.VI.1] such that the inclusion $i : X_s \to X_{s+1}$ induces a continuous map $\widetilde{i} : \mathcal{D}_s \to \mathcal{D}_{s+1}$. Moreover, by property (Q$'$) of the sequence $\{X_s\}_{s\geq 1}$, if a component in $\mathcal{D}_{s+1}$ meets X_s, it does it in a whole component; see Remark 5.2. Hence, the map $\widetilde{i}$ is injective and so $\mathcal{D}_s$ can be identified with a closed set of $\mathcal{D}_{s+1}$.

Let $\alpha_s, \beta_s \in \mathcal{D}_s$ be the classes of the components $A \cap X_s$ and $B \cap X_s$ ($s \geq 1$), respectively. The separation theorem for dimension 0 in [7, Theorem 1.2.6] gives us a decomposition $\mathcal{D}_1 = \mathcal{D}_1^A \cup \mathcal{D}_1^B$, into two non-empty disjoint closed sets of $\mathcal{D}_1$ with $\alpha_1 \in \mathcal{D}_1^A$ and $\beta_1 \in \mathcal{D}_1^B$. This way X_1 is decomposed as the disjoint union of the closed sets $P_1 = \pi_1^{-1}(\mathcal{D}_1^A)$ and $Q_1 = \pi_1^{-1}(\mathcal{D}_1^B)$.

As $\mathcal{D}_1^A$ and $\mathcal{D}_1^B$ can be regarded as closed sets of $\mathcal{D}_2$, we apply again the aforementioned separation theorem to get a decomposition of $\mathcal{D}_2$ into two disjoint closed sets $\mathcal{D}_2^A$ and $\mathcal{D}_2^B$, with $\mathcal{D}_1^A \subset \mathcal{D}_2^A$ and $\mathcal{D}_1^B \subset \mathcal{D}_2^B$. Notice also that $\alpha_2 \in \mathcal{D}_2^A$ and $\beta_2 \in \mathcal{D}_2^B$. This way we have $X_2 = P_2 \cup Q_2$ decomposed as the disjoint union of the closed sets $P_2 = \pi_2^{-1}(\mathcal{D}_2^A)$ and $Q_2 = \pi_2^{-1}(\mathcal{D}_2^B)$. Moreover, $P_1 \subset P_2$ and $Q_1 \subset Q_2$.

By proceeding inductively we can find increasing sequences $\{P_s\}_{s\geq 1}$ and $\{Q_s\}_{s\geq 1}$ where P_s and Q_s are disjoint closed sets of X_s with $X_s = P_s \cup Q_s$, and $A \cap X_s \subset P_s$ and $B \cap X_s \subset Q_s$. As $\{X_s\}_{s\geq 1}$ is an exhausting sequence, the unions $P = \cup_{s=1}^{\infty} P_s$ and $Q = \cup_{s=1}^{\infty} Q_s$ are disjoint nonempty closed sets of X with $X = P \cup Q$. This contradicts the connectedness of X. Thus, X is necessarily continuumwise connected. $\square$

We are ready to complete the proof of Theorem 4.4. For this we use the following easy lemma.

Lemma 5.4. *Let X be any generalized continuum and $K, L \subset X$ be two compact sets with $K \subset \mathrm{int}L$. Assume that $C \subset L$ is an irreducible continuum between $x, y \in K$ with $C - K \neq \varnothing$. Then, for any $p \in C \cap (\mathrm{int}L - K)$ and any open neighbourhood of p, $U \subset \mathrm{int}L - K$; the difference $D = C - (U \cup \mathrm{int}K)$ is a non-connected compact set.*

Proof. Obviously, D is compact. It cannot be connected since, otherwise, the union $C' = D \cup (C \cap K)$ would be a continuum containing x and y

and smaller than C. Here, we use that all components of $C \cap K$ meet $\mathrm{Fr}K$ [12, Theorem V.47.III.2]. □

Proof of (i)⇒ (ii) in Theorem 4.4. We proceed by contradiction. Assume that X does not admit an exhausting sequence consisting of subcontinua. Thus, by Lemma 5.3, all exhausting sequences of X fail to have property (P) in Lemma 5.1. Therefore, given any exhausting sequence $\{X_n\}_{n\geq 0}$, there exists $n_0 \geq 0$ such that, for every $n \geq n_0$, one can find a constituent $Z_n \subset X$ such that the intersection $Z_n \cap X_{n_0}$ meets at least two components of X_n, say A_n and B_n.

Moreover, as both A_n and B_n lie in the constituent Z_n, there exists $m_n > n$ such that $A_n \cup B_n$ is contained in a connected component of X_{m_n} $(n \geq n_0)$. After choosing appropriate subsequences and reindexing, if necessary, we can assume $n_0 = 0$ and $m_n = n+1$.

Choose points $a_n \in A_n$ and $b_n \in B_n$, and let $C_n \subset X_{n+1}$ be an irreducible continuum between a_n and b_n. Note that $C_{2n-1} \subset \mathrm{int}X_{2n+1}$, so that, by reindexing these subsequences, we can assume $C_n \subset \mathrm{int}X_{n+1}$.

As a_n and b_n belong to distinct components of X_n, then $C_n - X_n \neq \varnothing$. Take $p_n \in C_n - X_n$. Let $U_n \subset V_{n+1,n}$ be an open neighbourhood of p_n, where for all $m < s$, $V_{s,m} = \mathrm{int}X_s - X_m$ for $m \geq 1$ and $V_{s,0} = \mathrm{int}X_s$. Notice that the $V_{n+1,n}$'s form a locally finite family of pairwise disjoint sets and so do the U_n's.

By Lemma 5.4, we can decompose each compact set $(n \geq 1)$ $D_n = C_n - (U_n \cup \mathrm{int}X_n)$ into two disjoint non-empty compact sets $D_n = D_n^1 \cup D_n^2$. Let $U_n^i \subset V_{n+1,n-1}$ $(i = 1,2)$ be an open neighbourhood of D_n^i missing p_n and C_{n-1} with $U_n^1 \cap U_n^2 = \varnothing$. In particular, $C_n - \mathrm{int}X_n \subset U_n \cup U_n^1 \cup U_n^2$.

Next, we choose an open neighbourhood of X_0, $U_0 \subset \mathrm{int}X_1$, and enlarge the family of open sets $\mathcal{U}_0 = \{U_0, U_n, U_n^i : i = 1,2 \text{ and } n \geq 1\}$ to an open cover of X, $\mathcal{U} = \mathcal{U}_0 \cup (\cup_{n=1}^{\infty}\mathcal{U}_n)$, where $\mathcal{U}_n = \{U_\alpha^n\}_{\alpha\in\Lambda_n}$ is an open cover of $X_n - (X_{n-1} \cup C_{n-1} \cup (C_n \cap \mathrm{Fr}X_n))$ $(n \geq 1$ with $C_0 = \varnothing)$ consisting of open sets $U_\alpha^n \subset V_{n+1,n-1}$ which miss $C_n - \mathrm{int}X_n$.

Let $\mathcal{W} \prec \mathcal{U}$ be any refinement of $\mathcal{U}$. From the construction of the cover $\mathcal{U}$, we derive that for any $W \in \mathcal{W}$ with $W \cap X_0 \neq \varnothing$, either $W \subset \mathrm{int}X_1$ or $W \subset U_\alpha^1$ for some α; in particular, W misses U_n for $n \geq 1$ and U_n^i for $n \geq 2$ and $i = 1,2$. Moreover, for any $W \in \mathcal{W}$ with $W \cap D_n^i \neq \varnothing$, we have $W \subset U_n^i$. Similarly, if $p_n \in W$, then $W \subset U_n$.

The compactness of X_0 allows us to assume without loss of generality that the sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ converge to points $a, b \in X_0$, respectively (possibly, $a = b$). Let $W_0, W_0' \in \mathcal{W}$ with $a \in W_0$ and $b \in W_0'$.

Suppose $W_0 \cap W_0' \neq \varnothing$ (possibly, $W_0 = W_0'$) and let n be large enough to have $a_n \in W_0$ and $b_n \in W_0'$. The connectedness of C_n yields that C_n is covered with a finite chain in $\mathcal{W}$ between a_n and b_n, that is, a sequence

$\mathcal{C} = \{W_1, \ldots, W_s\} \subset \mathcal{W}$ with $a_n \in W_1$, $b_n \in W_s$, and $W_i \cap W_{i-1} \neq \varnothing$. We also assume that the elements of $\mathcal{C}$ are distinct, except possibly W_1 and W_s. As observed above, any two points $d_1 \in D_n^1$ and $d_2 \in D_n^2$ must lie in disjoint sets of the chain $\mathcal{C}$ distinct from W_1, W_s, and any element of $\mathcal{C}$ containing p_n. Therefore, the length of $\mathcal{C}$ is ≥ 4. Moreover, as W_0 and W_0' meet X_0, it follows that $\mathcal{C} \cup \{W_0, W_0'\}$ contains a cyclic chain of length ≥ 3. This is a contradiction to the strong tree-likeness of X.

Otherwise, if $W_0 \cap W_0' = \varnothing$, we also consider the continuum C_{n+2} and a chain $\mathcal{C}' = \{W_1', \ldots, W_{s'}'\} \subset \mathcal{W}$ between a_{n+2} and b_{n+2} given by the connectedness of C_{n+2}. As above, the length of $\mathcal{C}'$ is ≥ 4. Furthermore, as the sets U_{n+2}^i and U_{n+2} are disjoint with U_n^i and U_n, $\mathcal{C}' - \mathcal{C}$ contains at least three different elements, and so a cyclic chain of length ≥ 3 can be found in $\mathcal{C} \cup \mathcal{C}' \cup \{W_0, W_0'\}$. This leads again to a contradiction, and the proof of Theorem 4.4 is finished. □

6. Local Homeomorphisms and Tree-Likeness in the Proper Category

A theorem due to Maćkowiak [13] states that any local homeomorphism of a continuum onto a tree-like continuum is a homeomorphism. Later, Maćkowiak's theorem was extended to locally injective surjections onto tree-like continua by Heath [9]; see also [3, Proposition 1]. Recall that a continuous map $f : X \to Y$ is said to be *locally injective* if, for each $x \in X$, there is an open neighbourhood $\Omega(x)$ of x such that the restriction $f|\Omega(x)$ is injective.

Example 6.2 below shows that the proper analogue of Heath's theorem does not hold for tree-like generalized continua. It does under the assumption of continuumwise connectedness, as below.

Proposition 6.1. *Let $f : X \to Y$ be a locally injective proper surjection from a continuumwise connected generalized continuum X onto a tree-like generalized continuum Y. Then f is a homeomorphism.*

Proof. Let $x \neq x'$ be two distinct points of X and take a continuum $\Gamma \subset X$ containing both points. Then the restriction $f|\Gamma : \Gamma \to f(\Gamma)$ is locally injective and $f(\Gamma)$ is a tree-like continuum by Lemma 4.6. Hence, $f|\Gamma$ is a homeomorphism by [9] or [3, Proposition 1], and so $f(x) \neq f(x')$. This shows that $f : X \to Y$ is injective and, hence, a homeomorphism since proper maps are closed maps [6, Theorem 3.7.18]. □

The next example shows that the continuumwise connectedness of Y cannot be dropped from Proposition 6.1. We actually give an example of a proper local homeomorphism between tree-like generalized continua which is not a homeomorphism.

Example 6.2. Let $X \subset [0,1] \times \mathbb{R}_+$ be the generalized continuum in Example 4.3. For $0 \leq k \leq 3$, let $X_k \subset S^1 \times \mathbb{R}_+$ denote the image of the generalized continuum X by the canonical embedding $\varphi_k : [0,1] \times \mathbb{R}_+ \to S^1 \times \mathbb{R}_+$ given by $\varphi_k(t,y) = (e^{i\frac{(k+t)\pi}{2}}, y)$. Then, if $Z = \cup_{k=0}^{3} X_k$, the local homeomorphism $f : S^1 \times \mathbb{R}_+ \to S^1 \times \mathbb{R}_+$, given by $f(e^{i\theta}, y) = (e^{i2\theta}, y)$, restricts to a local homeomorphism $f : Z \to f(Z)$ where Z and $f(Z)$ are non-homeomorphic tree-like generalized continua.

Although locally injective proper surjections onto tree-like generalized continua may fail to be homeomorphisms, they always reflect tree-likeness. Namely, by the use of [5, Theorem 4.1] (see the Introduction), we modify the proof of the main theorem in [9] to get the following result.

Theorem 6.3. *Let $f : X \to Y$ be a locally injective proper surjection between generalized continua and assume that Y is tree-like. Then X is also tree-like.*

In the proof of Theorem 6.3, we will need the following lemma whose proof is a straightforward application of the properness of f.

Lemma 6.4. *Any locally injective proper surjection $f : X \to Y$ between generalized continua is finite-to-one. Moreover, given any compact set $L \subset Y$, there exists $\epsilon > 0$ such that $d(x,x') > \epsilon$ whenever $x \neq x'$ and $f(x) = f(x') \in L$. Here, d is any metric on X.*

Proof of Theorem 6.3. According to [5, Theorem 4.1], $\widehat{Y}$ is tree-like and X will be tree-like if $\widehat{X}$ is too. We next use the induced map $\widehat{f} : \widehat{X} \to \widehat{Y}$ (not necessarily locally injective, see Example 6.2) to derive the tree-likeness of $\widehat{X}$ from the tree-likeness of $\widehat{Y}$.

As $\widehat{X}$ is a continuum, it will suffice to show that an arbitrary open cover of $\widehat{X}$, say $\mathcal{U}$, admits an acyclic refinement (see [10, Theorem 2.1]).

Let us start by choosing a finite cover of the 0-dimensional space of ends $\mathcal{F}(Y)$ by pairwise disjoint open sets in $\widehat{Y}$, $\{E_j\}_{j=1}^{n}$. Then we find pairwise disjoint open sets W_i in $\widehat{X}$ $(1 \leq i \leq m)$ with $\mathcal{F}(X) \subset \cup_{i=1}^{m} W_i$ and $\widehat{f}(W_i) \subset E_{j(i)}$ for some $j(i)$. Here, we use the continuity of $\widehat{f}$ and the 0-dimensionality of $\mathcal{F}(X)$. Furthermore, by choosing the diameters of the W_i's smaller than the Lebesgue number of $\mathcal{U}$, we have that each W_i is contained in some element of $\mathcal{U}$.

Notice that for the compact complement $K = \widehat{X} - \cup_{i=1}^{m} W_i$, we have $\widehat{Y} - f(K) \subset \cup_{j=1}^{n} E_j$. For each $1 \leq j \leq n$, we form the open set $\Theta_j = \cup_{j(i)=j} W_i$.

Let $d = \widehat{d}|X$ be the restriction of a metric $\widehat{d}$ on $\widehat{X}$. Then Lemma 6.4 yields that, for each $y \in Y$, the fibre $f^{-1}(y)$ is finite and, furthermore, there is $\epsilon > 0$ such that $d(x,x') > \epsilon$ for any two distinct points

$x, x' \in f^{-1}(y)$ with $y \in f(K)$. In particular, for each $y \in f(K)$, the open balls $B_x^y \subset X$ of centre $x \in f^{-1}(y)$ and radius $\frac{\epsilon}{2}$ are pairwise disjoint. Moreover, by choosing ϵ smaller than the Lebesgue number of $\mathcal{U}$, we have that each B_x^y is contained in some element of $\mathcal{U}$. Clearly, the balls B_x^y cover $f^{-1}(f(K))$, and so the family $\mathcal{V} = \{W_i; 1 \leq i \leq m\} \cup \{B_x^y; y \in f(K), x \in f^{-1}(y)\}$ is a refinement of $\mathcal{U}$.

For each $y \in f(K)$, $U_y = \cup\{B_x^y; x \in f^{-1}(y)\}$ is an open neighbourhood of $f^{-1}(y)$ and the difference $\Omega_y = \widehat{Y} - \widehat{f}(\widehat{X} - U_y)$ is an open set for which it is easily checked that $\widehat{f}^{-1}(\Omega_y) \subset U_y$. Here, we use that $\widehat{f}$ is a closed map. Similarly, we consider the open set $\Omega_j = \widehat{Y} - \widehat{f}(\widehat{X} - \Theta_j)$ $(1 \leq j \leq n)$ for which $\widehat{f}^{-1}(\Omega_j) \subset \Theta_j$.

We claim that the family $\mathcal{O} = \{\Omega_y\}_{y \in f(K)} \cup \{\Omega_j\}_{1 \leq j \leq n}$ is an open cover of $\widehat{Y}$. Indeed, if $y \in \widehat{Y} - f(K) \subset \cup_{j=1}^{n} E_j$, then there is a unique j_0 with $y \in E_{j_0}$. Moreover, as $f^{-1}(y)$ does not meet K, we get $f^{-1}(y) \subset \Theta_{j_0}$, whence $y \in \Omega_{j_0}$.

Let $\mathcal{G} = \{G_\alpha\}_{\alpha \in \Lambda}$ $(G_\alpha \neq G_{\alpha'}$ for $\alpha \neq \alpha')$ be an acyclic refinement of $\mathcal{O}$ provided by the tree-likeness of $\widehat{Y}$. We construct from $\mathcal{G}$ an acyclic refinement $\mathcal{L}$ of $\mathcal{U}$ as follows.

Let Λ_0 be the set of indexes $\alpha \in \Lambda$ with $G_\alpha \subset \Omega_j$ for some j, and let $\Lambda_1 = \Lambda - \Lambda_0$. For each $\alpha \in \Lambda_0$, we take $j(\alpha)$ with $G_\alpha \subset \Omega_{j(\alpha)}$ and for each pair (α, i) with $j(i) = j(\alpha)$, we consider the open set $L(\alpha, i) = f^{-1}(G_\alpha) \cap W_i$ if this set is not empty. Furthermore, for each $\alpha \in \Lambda_1$, we choose $y(\alpha) \in f(K)$ with $G_\alpha \subset \Omega_{y(\alpha)}$ and for each pair (α, x) with $x \in f^{-1}(y(\alpha))$, we consider the open set $L(\alpha, x) = f^{-1}(G_\alpha) \cap B_x^{y(\alpha)}$ if this set is not empty.

We will finish the proof by checking that

$$\mathcal{L} = \{L(\alpha, i); \alpha \in \Lambda_0, j(i) = j(\alpha)\} \cup \{L(\alpha, x); \alpha \in \Lambda_1, x \in f^{-1}(y(\alpha))\}$$

is an acyclic refinement of $\mathcal{U}$. Notice that $L(\alpha, i) \subset W_i$ for $\alpha \in \Lambda_0$, whereas $L(\alpha, x) \subset B_x^{y(\alpha)}$ for $\alpha \in \Lambda_1$. Thus, $\mathcal{L}$ refines $\mathcal{V}$ and, hence, $\mathcal{U}$.

In order to see that $\mathcal{L}$ covers $\widehat{X}$, let $z \in \widehat{X}$ and take an element $G_\alpha \in \mathcal{G}$ with $f(z) \in G_\alpha$. If $G_\alpha \subset \Omega_{j(\alpha)}$, then $z \in f^{-1}(G_\alpha) \subset \Theta_{j(\alpha)}$, and so $z \in L(\alpha, i)$ for some i with $j(i) = j(\alpha)$. Similarly, if $G_\alpha \subset \Omega_{y(\alpha)}$ with $y(\alpha) \in f(K)$, then $z \in f^{-1}(\Omega_{y(\alpha)})$ lies in some ball $B_x^{y(\alpha)}$ and, hence, $z \in L(\alpha, x)$.

It remains to check the acyclicity of $\mathcal{L}$. For this, assume that $\mathcal{L}$ contains a cyclic sequence of distinct elements $L_1, \ldots, L_n$ $(n \geq 3)$; that is, $L_k \cap L_{k+1} \neq \varnothing$ for $1 \leq k \leq n-1$ and $L_n \cap L_1 \neq \varnothing$.

For each $k \leq n$, let α_k denote the first component of the pair indexing L_k in $\mathcal{L}$ above. Necessarily, the corresponding sets G_{α_k} meet each other cyclically, and the acyclicity of $\mathcal{G}$ yields that at least two of the indices coincide. Assume $\alpha_1 = \alpha_2 = \alpha$. If $\alpha \in \Lambda_0$, and $L_1 = L(\alpha, i_1)$ and $L_2 = L(\alpha, i_2)$ with $j(i_1) = j(i_2) = j(\alpha)$, we have $W_{i_1} \cap W_{i_2} \neq \varnothing$, and so $i_1 = i_2$ since the W_i's are pairwise disjoint. Similarly, if $\alpha \in \Lambda_1$, and $L_1 = (\alpha, x_1)$ and $L_2 = L(\alpha, x_2)$ with $x_1, x_2 \in f^{-1}(y(\alpha))$, we have $x_1 = x_2$ since the balls $B_x^{y(\alpha)}$ are pairwise disjoint. Thus, $L_1 = L_2$. This contradiction shows that the cover $\mathcal{L}$ is an acyclic refinement of the arbitrary cover $\mathcal{U}$, and we are done. $\square$

For strongly tree-like spaces, the proper analogue of Heath's theorem in [9] (see also [3, Proposition 1]) holds in full generality. Namely, we have the following theorem.

Theorem 6.5. *Let $f : X \to Y$ be a locally injective proper surjection from a generalized continuum X onto a strongly tree-like generalized continuum Y. Then f is a homeomorphism.*

The proof mimics the proof of [3, Proposition 1] for ordinary continua, which is more easily adapted to the non-compact setting than the proof in [9]. Indeed, the crucial arguments of the proof in [3] are the existence of star refinements for arbitrary open covers of a continuum and the fact that continuous maps between continua are closed. In the proper setting, one uses the paracompactness of X to obtain the suitable star refinements [6, Theorem 5.1.12] and the properness of f to guarantee that f is closed.

Acknowledgments. The authors wish to thank the referee for his/her helpful comments.

References

[1] R. Ayala, E. Dominguez, A. Márquez, and A. Quintero, *Proper homotopy classification of graphs*, Bull. London Math. Soc. **22** (1990), no. 5, 417–421.

[2] B. J. Ball and R. B. Sher, *A theory of proper shape for locally compact metric spaces*, Fund. Math. **86** (1974), 163–192.

[3] Taras Banakh, Zdzisław Kosztołowicz, and Sławomir Turek, *Characterizing chainable, tree-like, and circle-like continua*, Colloq. Math. **124** (2011), no. 1, 1–13.

[4] J. H. Case and R. E. Chamberlin, *Characterizations of tree-like continua*, Pacific J. Math. **10** (1960), 73–84.

[5] Włodzimierz J. Charatonik, Tomás Fernández-Bayort, and Antonio Quintero, *The Freudenthal compactification of tree-like generalized continua*, Topology Proc. **42** (2013), 173–193.

[6] Ryszard Engelking, *General Topology.* Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[7] Ryszard Engelking, *Theory of Dimensions: Finite and Infinite.* Sigma Series in Pure Mathematics, 10. Lemgo: Heldermann Verlag, 1995.

[8] T. Fernández-Bayort and A. Quintero, *Homotopic properties of confluent maps in the proper category*, Topology Appl. **156** (2009), no. 18, 2960–2970.

[9] Jo W. Heath, *Each locally one-to-one map from a continuum onto a tree-like continuum is a homeomorphism*, Proc. Amer. Math. Soc. **124** (1996), no. 8, 2571–2573.

[10] J. Krasinkiewicz, *Curves which are continuous images of tree-like continua are movable*, Fund. Math. **89** (1975), no. 3, 233–260.

[11] J. Krasinkiewicz, *On one-point union of two circles*, Houston J. Math. **2** (1976), no. 1, 91–95.

[12] K. Kuratowski, *Topology. Vol. II.* New edition, revised and augmented. Translated from the French by A. Kirkor. New York-London: Academic Press and Warsaw: PWN, 1968.

[13] T. Maćkowiak, *Local homeomorphisms onto tree-like continua*, Colloq. Math. **38** (1977), no. 1, 63–68.

[14] R. B. Sher, *Property SUV^∞ and proper shape theory*, Trans. Amer. Math. Soc. **190** (1974), 345–356.

[15] Edwin H. Spanier, *Algebraic Topology.* Corrected reprint. New York-Berlin: Springer-Verlag, 1982.

[16] J. H. C. Whitehead, *Simplicial spaces, nuclei and m-groups*, Proc. London Math. Soc. (2) **45** (1939), no. 4, 243–327.

(Charatonik) Department of Mathematics and Statistics; Missouri University of Science and Technology; Rolla, MO 65409 USA

Email address: wjcharat@mst.edu

(Fernández-Bayort) Departamento de Matemáticas; Instituto Pablo Neruda; C/ Manuel Rodríguez Navarro 4; 41950 Castilleja de la Cuesta; Sevilla, Spain

Email address: tfernandez@andaluciajunta.es

(Quintero) Departamento de Geometría y Topología; Facultad de Matemáticas; Universidad de Sevilla, Apartado 1160; 41080 Sevilla, Spain

Email address: quintero@us.es