http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 54, 2019

Pages 323–334

http://topology.nipissingu.ca/tp/

Motion Planning in Connected Sums of Real Projective Spaces

by

Daniel C. Cohen and Lucile Vandembroucq

Electronically published on May 16, 2019

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: (Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 54 (2019)
Pages 323-334

E-Published on May 16, 2019

MOTION PLANNING IN CONNECTED SUMS OF REAL PROJECTIVE SPACES

DANIEL C. COHEN AND LUCILE VANDEMBROUCQ

Abstract. The topological complexity $\mathsf{TC}(X)$ is a homotopy invariant of a topological space X, motivated by robotics, and providing a measure of the navigational complexity of X. The topological complexity of a connected sum of real projective planes, that is, a high genus nonorientable surface, is known to be maximal. We use algebraic tools to show that the analogous result holds for connected sums of higher dimensional real projective spaces.

1. Introduction

Let X be a finite, path-connected CW-complex. Viewing X as the space of configurations of a mechanical system, the motion planning problem consists of constructing an algorithm which takes as input pairs of configurations $(x_0, x_1) \in X \times X$, and produces a continuous path $\gamma\colon [0,1] \to X$ from the initial configuration $x_0 = \gamma(0)$ to the terminal configuration $x_1 = \gamma(1)$. The motion planning problem is of significant interest in robotics; see, for example, Jean-Claude Latombe [15] and Micha Sharir [17].

2010 *Mathematics Subject Classification.* Primary 55S40, 55M30; Secondary 55N25, 70Q05.

Key words and phrases. connected sum, real projective space, topological complexity.

The first author was partially supported by the Simons Foundation and by the Mathematisches Forschungsinstitut Oberwolfach.

The research of the second author was partially supported by Portuguese funds through FCT (Fundação para a Ciência e Tecnologia) within the project UID/MAT/00013/2013 and by the Polish National Science Centre grant 2016/21/P/ST1/03460 within the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 665778.

Michael Farber develops a topological approach to the motion planning problem in [9], [10], and [11]. Let $I = [0,1]$ be the unit interval, and let X^I be the space of continuous paths $\gamma\colon I \to X$ (with the compact-open topology). The map $\mathrm{ev}\colon X^I \to X \times X$, defined by sending a path to its endpoints, $\mathrm{ev}(\gamma) = (\gamma(0), \gamma(1))$, is a fibration, with fiber $\Omega(X)$, the based loop space of X. The motion planning problem requests a section of this fibration, a map $s\colon X \times X \to X^I$ satisfying $\mathrm{ev} \circ s = \mathrm{id}_{X\times X}$. It would be desirable for the motion planning algorithm to depend continuously on the input. However, there exists a globally continuous section $s\colon X \times X \to PX$ if and only if X is contractible; see [9, Theorem 1]. This prompts the study of the discontinuities of such algorithms and leads to the following definition from [10].

Definition 1.1. A *motion planner* for X is a collection of subsets $F_0, F_1, \ldots, F_m$ of $X \times X$ and continuous maps $s_i\colon F_i \to PX$ such that

(1) the sets F_i are pairwise disjoint, $F_i \cap F_j = \emptyset$ if $i \neq j$, and cover $X \times X$,
$$X \times X = F_0 \cup F_1 \cup \cdots \cup F_m;$$
(2) $\mathrm{ev} \circ s_i = \mathrm{id}_{F_i}$ for each i; and
(3) each F_i is a Euclidean neighborhood retract.

Refer to the sets F_i as local domains of the motion planner and the maps s_i as local rules. Call a motion planner optimal if it requires a minimal number of local domains (rules, respectively).

Definition 1.2. For a finite, path-connected CW-complex X, the (reduced) *topological complexity of* X, $\mathsf{TC}(X)$, is one less than the number of local domains in an optimal motion planner for X, $\mathsf{TC}(X) = m$ if there exists an optimal motion planner $F_0, F_1, \ldots, F_m$ for X.

1.1. Motion planning in cell complexes.

We briefly recall from [10, §3] a construction of a motion planner for a finite cell complex. Recall that X is a finite, path-connected CW-complex, and let X^k be the k-dimensional skeleton of X. Assume that $\dim(X) = n$, and for $k = 0, 1, \ldots, n$, let $V^k = X^k \setminus X^{k-1}$ be the union of the open k-cells of X. For $i = 0, 1, \ldots, 2n$, the sets $F_i = \bigcup_{k+l=i} V^k \times V^l \subset X \times X$ are homeomorphic to disjoint unions of balls, so are Euclidean neighborhood retracts. Note that $F_0 \cup F_1 \cup \cdots \cup F_{2n} = X \times X$.

To define a local rule $s_i\colon F_i \to X^I$, since F_i is the union of disjoint sets $V^k \times V^l$ (which are both open and closed in F_i), it suffices to construct a continuous map $s_{k,l}\colon V^k \times V^l \to X^I$ satisfying $\mathrm{ev} \circ s_{k,l} = \mathrm{id}_{V^k \times V^l}$. Pick a point $v_k \in V^k$ for each k, and fix a path $\gamma_{k,l}$ in X from v_k to v_l for each k and l. Then, for any $(x,y) \in V^k \times V^l$, one can construct a path $s_{k,l}(x,y)$

from x to y by first moving from x to v_k in the cell V^k, then traversing the fixed path $\gamma_{k,l}$, and finally moving from v_l to y in V^l.

This construction exhibits a motion planner for X with $2\dim(X)+1$ local domains. Consequently, we have the upper bound $\mathsf{TC}(X) \leq 2\dim(X)$ (for a finite, path-connected CW-complex X). This upper bound is achieved by many spaces of interest in topology and applications. For instance, it is well known that $\mathsf{TC}(\Sigma_g) = 4$ for an orientable surface Σ_g of genus $g \geq 2$; see [9]. More recent work of Alexander Dranishnikov [6], [7] and the authors [3] shows that the same holds for nonorientable surfaces of high genus. Observe that the construction above provides an optimal motion planner in these instances.

1.2. Main result.

The objective of this note is to establish a higher dimensional analog of these last results. Let $\mathcal{P}^n_g = \mathbb{RP}^n \# \cdots \# \mathbb{RP}^n$ be the connected sum of g copies of the real projective space $\mathbb{RP}^n$.

Theorem 1.3. *For $n \geq 2$ and $g \geq 2$, we have $\mathsf{TC}(\mathcal{P}^n_g) = 2n$.*

Thus, applying the construction in §1.1 above to a standard CW decomposition of the space $\mathcal{P}^n_g$ yields an optimal motion planner for this space.

When $n = 2$, $\mathcal{P}^2_g = N_g$ is the nonorientable surface of genus g, and it has been established in [3] that $\mathsf{TC}(N_g) = 4$ for $g \geq 2$, completing results obtained by Dranishnikov [6], [7] in the case $g \geq 4$. So we focus on the case $n \geq 3$ below. As we will see, the methods developed in [3] admit extensions to this higher dimensional case.

Remark 1.4. The case $g = 1$, with $\mathcal{P}^n_1 = \mathbb{RP}^n$, is significantly more subtle. As shown by Farber, Serge Tabachnikov, and Sergey Yuzvinsky [12], for $n \neq 1, 3, 7$, the topological complexity and immersion dimension of $\mathbb{RP}^n$ are equal, $\mathsf{TC}(\mathbb{RP}^n) = \mathrm{imm}(\mathbb{RP}^n)$.

Remark 1.5. For closed n-dimensional manifolds M and N, techniques analogous to those presented here provide conditions under which $\mathsf{TC}(M \# N) = \mathsf{TC}(M) = 2n$ is maximal; see Remark 3.2.

2. Preliminaries

Let $p\colon E \to B$ be a fibration. The (reduced) sectional category, or Schwarz genus, of p, denoted by $\mathrm{secat}(p)$, is the smallest integer m such that B can be covered by $m+1$ open subsets, over each of which p has a continuous section. Classical references include A. S. Schwarz [16] and I. M. James [14]. The following result makes clear the topological nature of the motion planning problem.

Theorem 2.1 (cf. [11, §4.2])**.** *If X is a finite CW-complex, then the topological complexity of X is equal to the sectional category of the path-space fibration* $\mathrm{ev}\colon X^I \to X \times X$, $\mathsf{TC}(X) = \mathrm{secat}(\mathrm{ev})$.

The equality $\mathsf{TC}(X) = \mathrm{secat}(\mathrm{ev}\colon X^I \to X \times X)$ yields the following estimates:

$$\max\{\mathrm{cat}(X), \mathrm{zcl}_{\Bbbk}(X)\} \le \mathsf{TC}(X) \le 2\,\mathrm{cat}(X) \le 2\dim(X); \text{ see [9]}.$$

Here, $\mathrm{cat}(X)$ is the reduced Lusternik–Schnirelmann (LS) category of X and $\mathrm{zcl}_{\Bbbk}(X)$ is the zero-divisors cup-length of the cohomology of X with coefficients in a field $\Bbbk$. More precisely, $\mathrm{zcl}_{\Bbbk}(X)$ is the nilpotency of the kernel of the cup product $H^*(X;\Bbbk)\otimes H^*(X;\Bbbk) \to H^*(X;\Bbbk)$, the smallest nonnegative integer n such that any $(n+1)$-fold cup product in this kernel is trivial.

As noted in §1.1, the upper bound $\mathsf{TC}(X) \le 2\dim(X)$ may also be obtained from an explicit motion planner construction. We will not make further use of the lower bounds $\mathrm{cat}(X)$ and $\mathrm{zcl}_{\Bbbk}(X)$, which are included here primarily for context and are both insufficient for our purposes. Indeed for $g \ge 2$, one can show that $\mathrm{cat}(\mathcal{P}_g^n) = n$ and $\mathrm{zcl}_{\mathbb{Z}_2}(\mathcal{P}_g^n) = 2n-1$. Following [3], we will instead utilize the topological complexity analog of the classical Berstein–Schwarz cohomology class, which informs on the LS category; see [4, Theorem 2.51].

Let X be a space and $\pi = \pi_1(X)$ its fundamental group. Let $\mathbb{Z}[\pi]$ be the group ring of π, $\epsilon\colon \mathbb{Z}[\pi] \to \mathbb{Z}$ the augmentation map, and $I(\pi) = \ker(\varepsilon\colon \mathbb{Z}[\pi] \to \mathbb{Z})$ the augmentation ideal. Recall that $\mathbb{Z}[\pi]$ and $I(\pi)$ are both (left) $\mathbb{Z}[\pi\times\pi]$-modules through the action given by

$$(a,b)\cdot \sum n_i a_i = \sum n_i(a a_i \bar{b}).$$

Here, $n_i \in \mathbb{Z}$, $a, b, a_i \in \pi$, and $\bar{b}$ is the inverse of b. In general, (see [19, §6]), left $\mathbb{Z}[\pi\times\pi]$-modules correspond to local coefficient systems on $X \times X$, which we denote by the same symbols.

Let $\mathfrak{v} = \mathfrak{v}_X \in H^1(X\times X; I(\pi))$ be the Costa–Farber canonical class of X introduced in [5], corresponding to the crossed homomorphism $\pi\times\pi \to I(\pi)$, $(a,b) \mapsto a\bar{b} - 1$. The significance of this cohomology class in the context of topological complexity is given by the following result.

Theorem 2.2 ([5, Theorem 7])**.** *Suppose that X is a CW-complex of dimension $n \ge 2$. Then $\mathsf{TC}(X) = 2n$ if and only if the $2n^{th}$ power of $\mathfrak{v}$ does not vanish:*

$$\mathsf{TC}(X) = 2n \iff \mathfrak{v}^{2n} \neq 0 \textit{ in } H^{2n}(X\times X; I(\pi)^{\otimes 2n}).$$

Here $I(\pi)^{\otimes 2n} = I(\pi)\otimes_{\mathbb{Z}} I(\pi) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} I(\pi)$ is the tensor product of $2n$ copies of $I(\pi)$, with the diagonal action of $\pi\times\pi$.

3. Reduction to the Case $g = 2$

Let π_g denote the fundamental group of the space $\mathcal{P}_g^n$. Since $n \geq 3$, we have $\pi_g = \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ (g copies). As in [3], we will prove that $\mathsf{TC}(\mathcal{P}_g^n) = 2n$ by proving that the evaluation of $\mathfrak{v}^{2n} \in H^{2n}(\mathcal{P}_g^n \times \mathcal{P}_g^n; I(\pi_g)^{\otimes 2n})$ on the $\mathbb{Z}_2$ top class $[\mathcal{P}_g^n \times \mathcal{P}_g^n]_{\mathbb{Z}_2} \in H_{2n}(\mathcal{P}_g^n \times \mathcal{P}_g^n; \mathbb{Z}_2)$ does not vanish and use the bar resolution to carry out the calculation. As noted in [5, Corollary 8], if $f : X \to Y = K(\pi, 1)$ induces an isomorphism of fundamental groups, we have $(f \times f)^* \mathfrak{v}_Y = \mathfrak{v}_X$. In general, for $f : X \to Y$ and $\rho = \pi_1(f) : \pi_1(X) \to \pi_1(Y)$, we have

$$(f \times f)^* \mathfrak{v}_Y = I(\rho)\mathfrak{v}_X \in H^1(X \times X; I(\pi_1(Y))) \tag{3.1}$$

Let $f_g \colon \mathcal{P}_g^n \to K(\pi_g, 1)$ denote the canonical map, the unique (up to homotopy) map such that $\pi_1(f_g) = \mathrm{id}$. We then analyze the cohomology class $\mathfrak{v}^{2n}$, its evaluation on the homology class $[\mathcal{P}_g^n \times \mathcal{P}_g^n]_{\mathbb{Z}_2}$ in particular, using the cap product diagram (cf. [1, Ch. V, §10])

$$\begin{tikzcd}
H_{2n}(\mathcal{P}_g^n \times \mathcal{P}_g^n; \mathbb{Z}_2) \otimes H^{2n}(\mathcal{P}_g^n \times \mathcal{P}_g^n; I(\pi_g)^{\otimes 2n}) \arrow[r, "\cap"] \arrow[d, shift right=12ex, "(f_g \times f_g)_*"] & I(\pi_g; \mathbb{Z}_2)^{\otimes 2n}_{\pi_g \times \pi_g} \arrow[d, "="] \\
H_{2n}(\pi_g \times \pi_g; \mathbb{Z}_2) \otimes H^{2n}(\pi_g \times \pi_g; I(\pi_g)^{\otimes 2n}) \arrow[r, "\cap"] \arrow[u, shift left=4ex, "(f_g \times f_g)^*"'] & I(\pi_g; \mathbb{Z}_2)^{\otimes 2n}_{\pi_g \times \pi_g}.
\end{tikzcd}$$

Here $I(\pi_g; \mathbb{Z}_2) = I(\pi_g) \otimes \mathbb{Z}_2$, and $I(\pi_g; \mathbb{Z}_2)^{\otimes 2n}_{\pi_g \times \pi_g}$ denotes the coinvariants of $I(\pi_g; \mathbb{Z}_2)^{\otimes 2n}$ with respect to the diagonal action of $\pi_g \times \pi_g$, which coincides with $H_0(\mathcal{P}_g^n \times \mathcal{P}_g^n; I(\pi_g)^{\otimes 2n} \otimes \mathbb{Z}_2) = H_0(\pi_g \times \pi_g; I(\pi_g)^{\otimes 2n} \otimes \mathbb{Z}_2)$.

As in [3, Theorem 14], the study of the general case $g \geq 2$ can be reduced to the case $g = 2$. Consider the projection $\mathcal{P}_g^n \to \mathcal{P}_{g-1}^n$ that collapses the last $\mathbb{RP}^n$ connected summand of $\mathcal{P}_g^n$ and induces the projection $\pi_g \to \pi_{g-1}$ which sends the last $\mathbb{Z}_2$ to 1. We have a (homotopy) commutative diagram

$$\begin{tikzcd}
\mathcal{P}_g^n \arrow[r, "f_g"] \arrow[d] & K(\pi_g, 1) \arrow[d] \\
\mathcal{P}_{g-1}^n \arrow[r, "f_{g-1}"] & K(\pi_{g-1}, 1)
\end{tikzcd} \tag{3.2}$$

The space $\mathcal{P}_g^n$ admits CW-complex structure, based on the standard CW decomposition of $\mathbb{RP}^n$ with a single cell in each dimension. Identify the $(n-1)$-skeleton of the last $\mathbb{RP}^n$ connected summand of $\mathcal{P}_g^n$ with $\mathbb{RP}^{n-1}$, and note that $(\mathcal{P}_g^n, \mathbb{RP}^{n-1})$ is an NDR-pair. Identifying $H_*(\mathcal{P}_g^n, \mathbb{RP}^{n-1}; \mathbb{Z}_2)$ with the reduced homology of $\mathcal{P}_g^n / \mathbb{RP}^{n-1} \simeq \mathcal{P}_{g-1}^n$ in the long exact homology sequence of this pair (cf. [13, Theorem 2.13]), we conclude that the projection $\mathcal{P}_g^n \to \mathcal{P}_{g-1}^n$ induces an isomorphism $\mathbb{Z}_2 \cong H_{2n}(\mathcal{P}_g^n;\mathbb{Z}_2) \longrightarrow H_{2n}(\mathcal{P}_{g-1}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

Write $F_k = (f_k \times f_k)_*$ and $\mathfrak{v}_k = \mathfrak{v}_{\pi_k}$. Considering the morphism $I(\pi_g;\mathbb{Z}_2) \to I(\pi_{g-1};\mathbb{Z}_2)$ induced by the projection $\pi_g \to \pi_{g-1}$, the naturality condition (3.1), and the diagram (3.2), we obtain the following commutative diagram

$$\begin{tikzcd}
H_{2n}(\mathcal{P}_g^n \times \mathcal{P}_g^n) \arrow[r, "F_g"] \arrow[d, "\cong"'] & H_{2n}(\pi_g \times \pi_g) \arrow[r, "\cap \mathfrak{v}_g^{2n}"] \arrow[d] & I(\pi_g;\mathbb{Z}_2)^{\otimes 2n}_{\pi_g \times \pi_g} \arrow[d] \\
H_{2n}(\mathcal{P}_{g-1}^n \times \mathcal{P}_{g-1}^n) \arrow[r, "F_{g-1}"] & H_{2n}(\pi_{g-1} \times \pi_{g-1}) \arrow[r, "\cap \mathfrak{v}_{g-1}^{2n}"] & I(\pi_{g-1};\mathbb{Z}_2)^{\otimes 2n}_{\pi_{g-1} \times \pi_{g-1}}
\end{tikzcd}$$

where the $\mathbb{Z}_2$ coefficients in homology are suppressed. Therefore, if the bottom horizontal map does not annihilate the generator, then neither does the top horizontal map. In other words, as in [3], the calculation can be reduced to the "genus" $g = 2$ case. Thus, for $n \geq 3$, Theorem 1.3 will follow from the following proposition which will be proved in the next section.

Proposition 3.1. *For $n \geq 3$, $\mathfrak{v}^{2n}([\mathcal{P}_2^n \times \mathcal{P}_2^n]_{\mathbb{Z}_2}) \neq 0$.*

Remark 3.2. We note that, for M and N closed n-manifolds, a similar argument to the one above permits one to conclude that $\mathsf{TC}(M\#N) = \mathsf{TC}(M) = 2n$ as soon as $\mathfrak{v}^{2n}([M \times M]_{\mathbb{Z}_2})$ is nonzero. Actually, using [8, Lemma 7] (and $\mathbb{Z}$-fundamental classes instead of $\mathbb{Z}_2$ top classes), we can see that $\mathsf{TC}(M\#N)$ is maximal as soon as $\mathsf{TC}(M)$ is maximal whenever N is orientable. Note also that, for simply-connected orientable manifolds, Dranishnikov and Rustam Sadykov [8] establish the more general result that $\mathsf{TC}(M\#N) \geq \mathsf{TC}(M)$.

4. The Case $g = 2$

In this section, we prove Proposition 3.1.

4.1. Algebraic preliminaries.

Refer to Kenneth S. Brown [2] and Charles A. Weibel [18] as standard references for cohomology of groups and homological algebra. We will use the normalized bar resolution $\bar{B}_*(\pi)$ of $\mathbb{Z}$ as a trivial $\mathbb{Z}[\pi]$-module:

$$\cdots \longrightarrow \bar{B}_n(\pi) \xrightarrow{\partial_n} \cdots \longrightarrow \bar{B}_1(\pi) \xrightarrow{\partial_1} \bar{B}_0(\pi) = \mathbb{Z}[\pi] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

Here $\bar{B}_n(\pi)$ is the free $\mathbb{Z}[\pi]$-module with basis

$$\{[g_1|\cdots|g_n], (g_1,\ldots,g_n) \in \bar{\pi}^n\},$$

where $\bar{\pi} = \{g \in \pi \mid g \neq 1\}$ and ∂_n is the $\mathbb{Z}[\pi]$ morphism given by

$$\begin{aligned}\partial_n([g_1|\cdots|g_n]) = g_1 \cdot [g_2|\cdots|g_n] \\ &+ \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\cdots|g_n] \\ &+ (-1)^n[g_1|\cdots|g_{n-1}]\end{aligned}$$

(with $[h_1|\cdots|h_k] = 0$ if $h_i = 1$ for some i). The homology of the space $K(\pi, 1)$ (or of the group π) with coefficients in $\mathbb{Z}_2$ is then the homology of the chain complex $\bar{B}_*(\pi; \mathbb{Z}_2) := \bar{B}_*(\pi) \otimes_\pi \mathbb{Z}_2 = (\bar{B}_*(\pi))_\pi \otimes \mathbb{Z}_2$ (with differential $\partial \otimes \mathrm{id}$).

We now describe a cycle representing the image of the $\mathbb{Z}_2$ top class of $\mathcal{P}_2^n = \mathbb{RP}^n \# \mathbb{RP}^n$ under the map induced by $f_2 : \mathcal{P}_2^n \to K(\pi_2, 1)$. We have $H_i(\pi_2; \mathbb{Z}_2) = H_i(\mathbb{RP}^\infty \vee \mathbb{RP}^\infty; \mathbb{Z}_2)$. Let $\mathbf{a}_i$ and $\mathbf{b}_i$ be the homology classes (with $\mathbf{a}_0 = \mathbf{b}_0$) corresponding to the two branches of the wedge. As the two projections $\mathcal{P}_2^n = \mathbb{RP}^n \# \mathbb{RP}^n \to \mathbb{RP}^n$ each induce an isomorphism $H_n(\mathcal{P}_2^n; \mathbb{Z}_2) \to H_n(\mathbb{RP}^n; \mathbb{Z}_2)$, the image of the $\mathbb{Z}_2$ top cell of $\mathbb{RP}^n \# \mathbb{RP}^n$ under the map $f_2 : \mathcal{P}_2^n \to K(\pi_2, 1)$ can be identified with the element $\mathbf{c}_n = \mathbf{a}_n + \mathbf{b}_n$ of $H_n(\pi_2; \mathbb{Z}_2)$ and we are reduced to describe cycles representing the classes $\mathbf{a}_n$ and $\mathbf{b}_n$.

Writing $\pi_2 = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \,|\, a^2 = 1, b^2 = 1\rangle$, the classes $\mathbf{a}_i$ and $\mathbf{b}_i$ are represented by the following cycles of $\bar{B}_i(\pi_2; \mathbb{Z}_2)$:

$$\alpha_i = [a|a|\cdots|a], \qquad \beta_i = [b|b|\cdots|b].$$

As our calculation will use portions of the calculation carried out in [3], we will use the isomorphism from $\pi_2 = \langle a, b \mid a^2 = 1, b^2 = 1\rangle$ to the infinite dihedral group $D = \langle x, y \mid yxy = x, x^2 = 1\rangle$ given by $a \mapsto x$ and $b \mapsto yx$. We will then work with the following cycles of $\bar{B}_i(D; \mathbb{Z}_2)$ as representatives of the classes $\mathbf{a}_i$ and $\mathbf{b}_i$:

$$\alpha_i' = [x|x|\cdots|x], \qquad \beta_i' = [yx|yx|\cdots|yx].$$

For $X = K(\pi, 1)$, the Costa–Farber TC canonical cohomology class $\mathfrak{v} \in H^1(X \times X; I(\pi))$ can be described as the class of the canonical degree 1 cocycle, $\nu\colon \bar{B}_1(\pi \times \pi) \to I(\pi)$, which is well defined on the normalized bar resolution and given by

$$\nu([(g, h)]) = g\bar{h} - 1$$

for $[(g, h)] \in \bar{B}_1(\pi \times \pi)$, and $\bar{h} = h^{-1}$ as above. As in [3], we have the following explicit expression of the n^{th} power of $\mathfrak{v} \in H^1(X \times X; I(\pi))$.

Lemma 4.1. *The n^{th} power of the canonical* TC *cohomology class* $\mathfrak{v}$ *is the class of the cocycle* ν^n *of degree* n *given by*

$$\begin{aligned} \nu^n\colon \bar{B}_n(\pi\times\pi) &\to I(\pi)^{\otimes n} \\ [(g_1,h_1)|\cdots|(g_n,h_n)] &\mapsto (-1)^{n(n-1)/2}\cdot u_1\otimes u_2\otimes\cdots\otimes u_n \end{aligned}$$

where $u_1 = g_1\bar{h}_1 - 1$ *and* $u_i = (g_1\cdots g_{i-1})(g_i\bar{h}_i-1)(\bar{h}_{i-1}\cdots\bar{h}_1)$ *for each* i, $2\le i\le n$.

We will also use the Eilenberg–Zilber chain equivalence (well defined on normalized bar resolutions)

$$EZ\colon \bar{B}_*(\pi)\otimes\bar{B}_*(\pi)\longrightarrow \bar{B}_*(\pi\times\pi), \tag{4.1}$$

which is the $\mathbb{Z}[\pi\times\pi]\cong\mathbb{Z}[\pi]\otimes\mathbb{Z}[\pi]$ morphism given by

$$\begin{aligned} EZ_n\colon \bigoplus_{i=0}^{n}\bar{B}_i(\pi)\otimes\bar{B}_{n-i}(\pi) &\to \bar{B}_n(\pi\times\pi) \\ [g_1|\cdots|g_i]\otimes[h_{i+1}|\cdots|h_n] &\mapsto \sum_{\sigma\in\mathcal{S}_{i,n-i}}\operatorname{sgn}(\sigma)[q_{\sigma^{-1}(1)}|\cdots|q_{\sigma^{-1}(n)}] \end{aligned}$$

where $\mathcal{S}_{i,n-i}$ denotes the set of $(i,n-i)$ shuffles, $\operatorname{sgn}(\sigma)$ is the signature of the shuffle σ (which can be omitted over $\mathbb{Z}_2$), and

$$q_k = \begin{cases}(g_k,1) & \text{if } 1\le k\le i,\\ (1,h_k) & \text{if } i+1\le k\le n.\end{cases}$$

Example 4.2. We find an explicit expression of $\nu^4(EZ(\alpha_2'\otimes\beta_2'))$ in $I(D;\mathbb{Z}_2)^{\otimes 4}$, which will be useful in the proof of Proposition 3.1. Since $\alpha_2' = [x|x]$ and $\beta_2' = [yx|yx]$, we have

$$\begin{aligned} EZ(\alpha_2'\otimes\beta_2') = {} & [x_1|x_1|y_2x_2|y_2x_2] + [x_1|y_2x_2|x_1|y_2x_2] \\ & + [x_1|y_2x_2|y_2x_2|x_1] + [y_2x_2|x_1|x_1|y_2x_2] \\ & + [y_2x_2|x_1|y_2x_2|x_1] + [y_2x_2|y_2x_2|x_1|x_1] \end{aligned}$$

where $x_1=(x,1)$, $x_2=(1,x)$, $y_1=(y,1)$, and $y_2=(1,y)$. Using Lemma 4.1 together with the fact that $x^2=1$ and $(yx)^2=1$, we obtain

(4.2)
$$\begin{aligned} \nu^4(EZ(\alpha_2'\otimes\beta_2')) = {} & (x-1)\otimes(1-x)\otimes(yx-1)\otimes(1-yx) \\ & + (x-1)\otimes x(yx-1)\otimes(1-x)yx\otimes(1-yx) \\ & + (x-1)\otimes x(yx-1)\otimes x(1-yx)\otimes(1-x) \\ & + (yx-1)\otimes(x-1)yx\otimes(1-x)yx\otimes(1-yx) \\ & + (yx-1)\otimes(x-1)yx\otimes x(1-yx)\otimes(1-x) \\ & + (yx-1)\otimes(1-yx)\otimes(x-1)\otimes(1-x) \end{aligned}$$

The image of this expression in the coinvariants $I(D;\mathbb{Z}_2)^{\otimes 4}_{D\times D}$ corresponds to the element $\mathfrak{v}^4(\mathbf{a}_2 \times \mathbf{b}_2) \in H_0(D\times D; I(D;\mathbb{Z}_2)^{\otimes 4}) \cong I(D;\mathbb{Z}_2)^{\otimes 4}_{D\times D}$.

Let $Y = \langle y \,|\, y^2 = 1\rangle$ and $Z = \langle z \,|\, z^2 = 1\rangle$. We have $I(Y;\mathbb{Z}_2) \cong \mathbb{Z}_2(y-1)$ and $I(Z;\mathbb{Z}_2) \cong \mathbb{Z}_2(z-1)$. Consider the projection $I(D;\mathbb{Z}_2) \to I(Y;\mathbb{Z}_2)$ sending x to 1 and the projection $I(D;\mathbb{Z}_2) \to I(Z;\mathbb{Z}_2)$ sending both x and y to z (and hence $yx \mapsto 1$). One can check that, after projection onto $I(Y;\mathbb{Z}_2)^{\otimes 2} \otimes I(Z;\mathbb{Z}_2)^{\otimes 2} \cong \mathbb{Z}_2$, (4.2) yields a unique non-zero term $(y-1)\otimes(y-1)\otimes(z-1)\otimes(z-1)$ which corresponds to the element $[y_2x_2|y_2x_2|x_1|x_1] \in \bar{B}_4(D\times D)$.

4.2. Proof of Proposition 3.1.

The statement will follow from the fact that the image of the $\mathbb{Z}_2$ top class of $\mathcal{P}_2^n$,

$$\mathbf{c}_n = (f_2)_*([\mathcal{P}_2^n]_{\mathbb{Z}_2}) \in H_n(\pi_2;\mathbb{Z}_2) = H_n(D;\mathbb{Z}_2),$$

satisfies $\mathfrak{v}^{2n}(\mathbf{c}_n \times \mathbf{c}_n) = \mathfrak{v}^{2n} \cap (\mathbf{c}_n \times \mathbf{c}_n) \neq 0$.

Let $G = D\times D$ and $X = K(G,1)$, and let $\Delta\colon X \to X\times X$ be the diagonal map. For the G-modules $M = I(D)^{\otimes 2n}$, $M' = I(D)^{\otimes 4}$, and $M'' = I(D)^{\otimes 2n-4}$, and the homology and cohomology classes $\zeta \in H_{2n}(G;\mathbb{Z}_2)$ and $\omega \in H^{2n}(G\times G; M) = H^{2n}(G\times G; M'\otimes M'')$, a cap product diagram as in [1, Ch. V, §10] yields

$$\Delta_*(\zeta \cap \Delta^*(\omega)) = \Delta_*(\zeta) \cap \omega$$

in $H_0(G\times G; M\otimes\mathbb{Z}_2)$. Fixing $\omega = \mathfrak{v}^4 \times \mathfrak{v}^{2n-4} \in H^{2n}(G\times G; M'\otimes M'')$, so that $\Delta^*(\omega) = \mathfrak{v}^4 \cup \mathfrak{v}^{2n-4} = \mathfrak{v}^{2n}$, we obtain the commuting diagram

$$\begin{tikzcd}
H_{2n}(G;\mathbb{Z}_2) \arrow[r, "\Delta_*"] \arrow[d, "\mathfrak{v}^{2n}"'] & H_{2n}(G\times G;\mathbb{Z}_2) \arrow[d, "\mathfrak{v}^4\times\mathfrak{v}^{2n-4}"] \\
H_0(G; M\otimes\mathbb{Z}_2) \arrow[r, "\Delta_*"] & H_0(G\times G; M'\otimes M''\otimes\mathbb{Z}_2)
\end{tikzcd}$$

where the vertical maps are cap products with the indicated cohomology classes.

Let $\kappa_{4,2n-4}$ denote the composition of the Künneth isomorphism and the projection indicated below

$$H_{2n}(G\times G;\mathbb{Z}_2) \to \bigoplus_{i+j=2n} H_i(G;\mathbb{Z}_2)\otimes H_j(G;\mathbb{Z}_2) \twoheadrightarrow H_4(G;\mathbb{Z}_2)\otimes H_{2n-4}(G;\mathbb{Z}_2),$$

and let $\Delta_{4,2n-4} = \kappa_{4,2n-4} \circ \Delta_*$. As $G = D \times D$ and $M = M' \otimes M'' = I(D)^{\otimes 2n}$, identifying zero-dimensional homology groups in the above commuting diagram yields
(4.3)

$$\begin{tikzcd} H_{2n}(D \times D;\mathbb{Z}_2) \arrow[r, "\Delta_{4,2n-4}"] \arrow[d, "\mathfrak{v}^{2n}"'] & H_4(D \times D;\mathbb{Z}_2) \otimes H_{2n-4}(D \times D;\mathbb{Z}_2) \arrow[d, "\mathfrak{v}^4 \otimes \mathfrak{v}^{2n-4}"] \\ I(D;\mathbb{Z}_2)^{\otimes 2n}_{D\times D} \arrow[r] & I(D;\mathbb{Z}_2)^{\otimes 4}_{D\times D} \otimes I(D;\mathbb{Z}_2)^{\otimes 2n-4}_{D\times D} \end{tikzcd}$$

As above, we consider $Y = \langle y \,|\, y^2 = 1\rangle$ and $Z = \langle z \,|\, z^2 = 1\rangle$ and the projections $I(D;\mathbb{Z}_2) \to I(Y;\mathbb{Z}_2)$ and $I(D;\mathbb{Z}_2) \to I(Z;\mathbb{Z}_2)$. We then compose the $\mathfrak{v}^{2n-4}$ portion of the right-hand vertical map in the diagram (4.3) with the projection

$$I(D;\mathbb{Z}_2)^{\otimes 2n-4}_{D\times D} \longrightarrow I(Y;\mathbb{Z}_2)^{\otimes n-2} \otimes I(Z;\mathbb{Z}_2)^{\otimes n-2}.$$

There is no need to pass to coinvariants since the $D\times D$ action on $I(Y;\mathbb{Z}_2)$ and $I(Z;\mathbb{Z}_2)$ is trivial. Observe that

$$I(Y;\mathbb{Z}_2)^{\otimes n-2} \otimes I(Z;\mathbb{Z}_2)^{\otimes n-2} \cong \mathbb{Z}_2(y-1)^{\otimes n-2} \otimes \mathbb{Z}_2(z-1)^{\otimes n-2} \cong \mathbb{Z}_2.$$

Since $\mathbf{c}_n = \mathbf{a}_n + \mathbf{b}_n$, the expression $\Delta_{4,2n-4}(\mathbf{c}_n \times \mathbf{c}_n)$ decomposes as

$$\sum_{i=0}^{4}(\mathbf{a}_i \times \mathbf{a}_{4-i}) \otimes (\mathbf{a}_{n-i} \times \mathbf{a}_{n-4+i}) + \sum_{i=0}^{4}(\mathbf{a}_i \times \mathbf{b}_{4-i}) \otimes (\mathbf{a}_{n-i} \times \mathbf{b}_{n-4+i})$$
$$+ \sum_{i=0}^{4}(\mathbf{b}_i \times \mathbf{a}_{4-i}) \otimes (\mathbf{b}_{n-i} \times \mathbf{a}_{n-4+i}) + \sum_{i=0}^{4}(\mathbf{b}_i \times \mathbf{b}_{4-i}) \otimes (\mathbf{b}_{n-i} \times \mathbf{b}_{n-4+i})$$

Now, we can check that, among the right-hand components, the only terms on which the projection of $\mathfrak{v}^{2n-4}$ on $I(Y;\mathbb{Z}_2)^{\otimes n-2} \otimes I(Z;\mathbb{Z}_2)^{\otimes n-2}$ does not vanish are $\mathbf{a}_{n-2}\times\mathbf{b}_{n-2}$ and $\mathbf{b}_{n-2}\times\mathbf{a}_{n-2}$, represented respectively by $EZ(\alpha'_{n-2}\otimes\beta'_{n-2})$ and $EZ(\beta'_{n-2}\otimes\alpha'_{n-2})$. Furthermore, calculating in $I(Y;\mathbb{Z}_2)^{\otimes n-2} \otimes I(Z;\mathbb{Z}_2)^{\otimes n-2}$, we have

$$\mathfrak{v}^{2n-4}(\mathbf{a}_{n-2} \times \mathbf{b}_{n-2}) = \mathfrak{v}^{2n-4}(\mathbf{b}_{n-2} \times \mathbf{a}_{n-2}) = (y-1)^{\otimes n-2} \otimes (z-1)^{\otimes n-2}.$$

Consequently, in $I(D;\mathbb{Z}_2)^{\otimes 4}_{D\times D} \otimes I(Y;\mathbb{Z}_2)^{\otimes n-2} \otimes I(Z;\mathbb{Z}_2)^{\otimes n-2}$, we have

$$\mathfrak{v}^{2n}(\mathbf{c}_n \times \mathbf{c}_n) = \mathfrak{v}^4(\mathbf{a}_2 \times \mathbf{b}_2 + \mathbf{b}_2 \times \mathbf{a}_2) \otimes (y-1)^{\otimes n-2} \otimes (z-1)^{\otimes n-2}.$$

We now check that $\mathfrak{v}^4(\mathbf{a}_2\times\mathbf{b}_2+\mathbf{b}_2\times\mathbf{a}_2)$ does not vanish in $I(D;\mathbb{Z}_2)^{\otimes 4}_{D\times D}$. For this, recall the expression (in $I(D;\mathbb{Z}_2)^{\otimes 4}$) of $\nu^4(EZ(\alpha'_2\otimes\beta'_2))$ which has been obtained in (4.2).

By considering, as in [3], the projection

$$I(D;\mathbb{Z}_2)^{\otimes 4} \to I(D;\mathbb{Z}_2) \otimes \bigwedge^3(I(D;\mathbb{Z}_2)),$$

together with the relation $xyx = \bar{y}$, we see that the expression (4.2) reduces to

$$(yx - x) \otimes (1 - yx) \wedge (1 - \bar{y}) \wedge (1 - x).$$

Calculating the image of $\nu^4(EZ(\beta_2' \otimes \alpha_2'))$ in $I(D;\mathbb{Z}_2) \otimes \bigwedge^3(I(D;\mathbb{Z}_2))$ in an analogous manner yields

$$(yx - x) \otimes (1 - yx) \wedge (1 - y) \wedge (1 - x).$$

As in [3], we then send the first component to $I(Y;\mathbb{Z}_2) \cong \mathbb{Z}_2$ (through $x \mapsto 1$) and the statement follows from the fact that the sum of the two elements above is the element

$$s = (x - 1) \wedge (yx - 1) \wedge (y - \bar{y}) \in \bigwedge^3 I(D;\mathbb{Z}_2),$$

which is shown to be nonzero in [3, §3.3.2]. □

Acknowledgment. We thank the referee for pertinent suggestions. The first author also thanks S. Carter, A. Dranishnikov, and S. Ferry for inviting him to participate in the Geometric Topology session at the 52nd Spring Topology and Dynamical Systems Conference.

References

[1] Glen E. Bredon, *Sheaf Theory.* 2nd ed. Graduate Texts in Mathematics, 170. New York: Springer-Verlag, 1997. MR1481706

[2] Kenneth S. Brown, *Cohomology of Groups.* Graduate Texts in Mathematics, 87. New York-Berlin: Springer-Verlag, 1982. MR0672956

[3] Daniel C. Cohen and Lucile Vandembroucq, *Topological complexity of the Klein bottle,* J. Appl. and Comput. Topology **1** (2017), no. 2, 199–213.

[4] Octav Cornea, Gregory Lupton, John Oprea, and Daniel Tanré, *Lusternik-Schnirelmann Category.* Mathematical Surveys and Monographs, 103. Providence, RI: American Mathematical Society, 2003. MR1990857

[5] Armindo Costa and Michael Farber, *Motion planning in spaces with small fundamental groups*, Commun. Contemp. Math. **12** (2010), no. 1, 107–119. MR2649230

[6] Alexander Dranishnikov, *The topological complexity and the homotopy cofiber of the diagonal map for non-orientable surfaces*, Proc. Amer. Math. Soc. **144** (2016), no. 11, 4999–5014. MR3544546

[7] Alexander Dranishnikov, *On topological complexity of non-orientable surfaces*, Topology Appl. **232** (2017), 61–69. MR3720882

[8] Alexander Dranishnikov and Rustam Sadykov, *On the LS-category and topological complexity of a connected sum*, Proc. Amer. Math. Soc. **147** (2019), no. 5, 2235–2244. MR3937697

[9] Michael Farber, *Topological complexity of motion planning*, Discrete Comput. Geom. **29** (2003), no. 2, 211–221. MR1957228

[10] Michael Farber, *Instabilities of robot motion*, Topology Appl. **140** (2004), no. 2-3, 245–266. MR2074919

[11] Michael Farber, *Invitation to Topological Robotics.* Zurich Lectures in Advanced Mathematics. Zürich: European Mathematical Society (EMS), 2008. MR2455573

[12] Michael Farber, Serge Tabachnikov, and Sergey Yuzvinsky, *Topological robotics: Motion planning in projective spaces*, Int. Math. Res. Not. **2003** (2003), no. 34, 1853–1870. MR1988783

[13] Allen Hatcher, *Algebraic Topology.* Cambridge: Cambridge University Press, 2002. MR1867354

[14] I. M. James, *On category, in the sense of Lusternik-Schnirelmann*, Topology **17** (1978), no. 4, 331–348. MR0516214

[15] Jean-Claude Latombe, *Robot Motion Planning.* The Kluwer International Series in Engineering and Computer Science. Dordrecht: Kluwer Academic, 1991. MR1137544

[16] A. S. Schwarz, *The genus of a fiber space*, Amer. Math. Soc. Transl. **55** (1966), 49–140. MR0154284

[17] Micha Sharir, *Algorithmic motion planning* in Handbook of Discrete and Computational Geometry. Ed. Edited by Jacob E. Goodman and Joseph O'Rourke. CRC Press Ser. Discrete Math. Appl., Boca Raton, FL: CRC, 1997. 733–754. MR1730196

[18] Charles A. Weibel, *An Introduction to Homological Algebra.* Cambridge Studies in Advanced Mathematics, 38. Cambridge: Cambridge University Press, 1994. MR1269324

[19] George W. Whitehead, *Elements of Homotopy Theory.* Graduate Texts in Mathematics, 61. New York-Berlin: Springer-Verlag, 1978. MR0516508

(Cohen) Department of Mathematics; Louisiana State University; Baton Rouge, LA 70803 USA

Email address: `cohen@math.lsu.edu`

(Vandembroucq) Centro de Matemática; Universidade do Minho; Campus de Gualtar; 4710-057 Braga, Portugal

Email address: `lucile@math.uminho.pt`