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by

Paloma Hernández and Héctor Méndez

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ENTROPY OF INDUCED DENDRITE HOMEOMORPHISM $C(f) : C(D) \to C(D)$

PALOMA HERNÁNDEZ AND HÉCTOR MÉNDEZ

Abstract. Let $f : D \to D$ be a dendrite homeomorphism. Let $C(D)$ denote the hyperspace of all subcontinua of D endowed with the Hausdorff metric. Let $C(f) : C(D) \to C(D)$ be the induced homeomorphism in hyperspace $C(D)$. We show in this paper that the topological entropy of $C(f)$ has only two possible values: 0 or ∞. Also we show that the entropy of $C(f)$ is ∞ if and only if there exists a point $x \in D$ such that x is not an element of the minimal subdendrite of D that contains the union $\alpha(x, f) \cup \omega(x, f)$.

1. Introduction and Some Definitions

A *continuum* is a nonempty compact and connected metric space.

Let $X = (X, d)$ be a continuum. Let 2^X be the collection of all nonempty compact subsets of X endowed with the Hausdorff metric H_d induced by metric d. Each nonempty subset of 2^X, with the corresponding restriction of H_d, is a *hyperspace*.

If Y is a continuum and $Y \subset X$, then Y is a *subcontinuum* of X. Let $C(X)$ denote the hyperspace of all subcontinua of X. Hyperspaces 2^X and $C(X)$ are continua as well; see [15].

A continuum X is an *arc* if it is homeomorphic to the unit interval $[0, 1] \subset \mathbb{R}$, a *simple closed curve* provided that it is homeomorphic to the circle $S^1 = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$, a *graph* if it can be written as the union of finitely many arcs any two of which either are disjoint or intersect only in one or both of their end points, a *tree* if it is a graph which contains no

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simple closed curve, and a *dendrite* provided that it is locally connected and contains no simple closed curve.

Let $\mathbb{N}$ denote the set of all positive integers. A *mapping* is a continuous function. In section 5, we recall the definition and some properties of the topological entropy of a mapping $f : X \to X$.

Let $2^f : 2^X \to 2^X$ be the mapping induced in 2^X by $f : X \to X$. For each $n \in \mathbb{N}$ and for each $A \in 2^X$, $\left(2^f\right)^n (A) = f^n(A)$. Let $C(f) : C(X) \to C(X)$ be the restriction of 2^f to hyperspace $C(X)$. If $f : X \to X$ is a homeomorphism, then both 2^f and $C(f)$ are homeomorphisms; see [15]. It is known that if D is a dendrite and $f : D \to D$ is a homeomorphism, then entropy of f is 0; see [2].

In 2010, Merek Lampart and Peter Raith [12] proved that if X is an arc and $f : X \to X$ is a homeomorphism, then the topological entropy of $C(f)$ is 0. In 2013, Mykola Matviichuk [14] proved that for any tree homeomorphism $f : X \to X$, the topological entropy of $C(f)$ is 0 as well. In 2009, Gerardo Acosta, Alejandro Illanes, and Héctor Méndez-Lango [3] produced an example of a dendrite D and a homeomorphism, $f : D \to D$, where the topological entropy of the induced map $C(f)$ is ∞. Recently, in 2016, Haithem Abouda and Issam Naghmouchi [1] introduced another dendrite homeomorphism $f : D \to D$, where the mapping $C(f)$ has infinite topological entropy.

Let $f : X \to X$ be a homeomorphism. The limit sets $\alpha(x, f)$ and $\omega(x, f)$ are defined in section 2.

Our main result in this note is the following: Let D be a dendrite and $f : D \to D$ be a homeomorphism. Then

- the topological entropy of $C(f) : C(D) \to C(D)$ has only two possible values: 0 or ∞;
- the topological entropy of $C(f) : C(D) \to C(D)$ is ∞ if and only if there exists a point $x \in D$ such that x is not an element of the minimal subcontinuum of D that contains the union $\alpha(x, f) \cup \omega(x, f)$.

2. Preliminary Results

Let $X = (X, d)$ be a compact metric space that contains more than one point. Let $f : X \to X$ be a mapping. Given a point x in X, the *orbit of x under f* is the sequence

$$o(x, f) = \{f^n(x) : n \geq 0\},$$

where f^0 denotes the identity map in X, $f^1 = f$, and for each $n \in \mathbb{N}$, $f^{n+1} = f \circ f^n$. If there exists $n \in \mathbb{N}$ with $f^n(x) = x$, then x is a *periodic point* of f. If $f(x) = x$, then x is a *fixed point* of f. Let $Per(f)$ and $Fix(f)$ denote the set of all periodic points and of all fixed points of f, respectively. If $x \in Per(f)$, then $n_0 = \min\{n \in \mathbb{N} : f^n(x) = x\}$ is the *period* of x.

The *omega limit set of* x *under* f is the set

$$\omega(x, f) = \left\{ y \in X : \exists \{n_1 < n_2 < \cdots\} \subset \mathbb{N} \text{ with } \lim_{i \to \infty} f^{n_i}(x) = y \right\}.$$

If $x \in \omega(x, f)$, then it is said that x is a *recurrent point* of f. Let $R(f)$ denote the set of all recurrent points of f. Let $\Lambda(f) = \cup \{\omega(x, f) : x \in X\}$. Note that

$$Fix(f) \subset Per(f) \subset R(f) \subset \Lambda(f).$$

If $f : X \to X$ is a homeomorphism, the *alpha limit set of* x *under* f is the set

$$\alpha(x, f) = \omega(x, f^{-1}).$$

A nonempty subset $A \subset X$ is *invariant* under $f : X \to X$ if $f(A) \subset A$; it is *strongly invariant* provided that $f(A) = A$. An invariant subset A is a *minimal set* of f provided that $A = \omega(a, f)$ for every point $a \in A$.

Proposition 2.1 contains some basic properties of $\omega(x, f)$. See [5].

Proposition 2.1. *Let $x, y \in X$, then*

- *$\omega(x, f)$ is closed and nonempty;*
- *$\omega(x, f)$ is strongly invariant under f;*
- *for each open set $U \subset X$ with $\omega(x, f) \subset U$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $f^n(x) \in U$;*
- *for each $m \in \mathbb{N}$, $f(\omega(x, f^m)) = \omega(f(x), f^m)$;*
- *for each $m \in \mathbb{N}$,*

  $$\omega(x, f) = \omega(x, f^m) \cup \omega(f(x), f^m) \cup \cdots \cup \omega(f^{m-1}(x), f^m).$$

  Thus, $\omega(x, f)$ is finite if and only if for some $m \in \mathbb{N}$, $\omega(x, f^m)$ is finite;
- *if for some $N \in \mathbb{N}$, $\omega(x, f^N) = \omega(y, f^N)$, then $\omega(x, f) = \omega(y, f)$;*
- *if $\lim_{n \to \infty} d(f^n(x), f^n(y)) = 0$, then $\omega(x, f) = \omega(y, f)$;*
- *if $card(\omega(x, f))$ is finite, say N, then there exists $y \in Per(f)$ of period N with $\omega(x, f) = \{y, f(y), f^2(y), \ldots, f^{N-1}(y)\}$.*

Let $\varepsilon > 0$. Then $B(x, \varepsilon)$ denotes the open ball around $x \in X$ with radius ε. If $A \subset X$, then the symbols $cl(A)$, $int(A)$, and $bd(A)$ stand for the closure, the interior, and the boundary of A in X.

Furthermore, if $A \neq \emptyset$,

$$N(A, \varepsilon) = \{y \in X : \text{ there is } x \in A,\ d(y, x) < \varepsilon\} = \cup \{B(x, \varepsilon) : x \in A\},$$

and $diam(A) = \sup\{d(x, y) : x, y \in A\}$. The symbol $card(A)$ stands for the cardinality of A.

Let A and B be two elements of 2^X. Then

$$H_d(A,B) = \inf\{\varepsilon > 0 : A \subset N(B,\varepsilon) \quad \text{and} \quad B \subset N(A,\varepsilon)\}$$

defines a metric in 2^X, the *Hausdorff metric*. See [10] and [15].

Let $\{A_n : n \in \mathbb{N}\}$ be a sequence in 2^X and $A \in 2^X$. If

$$\lim_{n\to\infty} H_d(A_n, A) = 0,$$

then we write $\lim A_n = A$.

3. Dendrites

In this section we recall some basic properties of dendrites and of maps defined on dendrites. Let $D = (D,d)$ denote a nondegenerate dendrite.

Theorem 3.1. *The following conditions hold:*

- *Every connected subset of D is arcwise connected.*
- *Each subcontinuum of D is a dendrite.*
- *For each pair $A, B \in C(D)$, $A \cap B = \emptyset$, there exist open and connected subsets of D, U, and V, such that $A \subset U$, $B \subset V$, and $cl(U) \cap cl(V) = \emptyset$.*
- *The intersection of any two connected subsets of D is connected.*
- *For every dendrite mapping $f : D \to D$, $Fix(f) \neq \emptyset$.*

Proof of Theorem 3.1 can be found in [15].

Let $x \in D$. It is said that x is an *end point* of D provided that $D \setminus \{x\}$ is connected; x is a *cut point* of D if $D \setminus \{x\}$ is not connected. The *order* of x, $ord(x)$, is the cardinality of the set of all components of $D \setminus \{x\}$. Each point of D is of order $\leq \aleph_0$ (see [15]). If $ord(x) \geq 3$, it is said that x is a *branch point* of D.

Proposition 3.2. *Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of nonempty connected subsets of D such that for each pair $n \neq m$, $A_n \cap A_m = \emptyset$. Then*

$$\lim_{n\to\infty} diam\,(A_n) = 0.$$

Proposition 3.2 is proved in [13].

Given two distinct points a and b in D, there is only one arc from a to b contained in D. We denote such an arc with $[a,b]$. Also, we use the following notation: $(a,b] = [a,b] \setminus \{a\}$, $[a,b) = [a,b] \setminus \{b\}$, and $(a,b) = [a,b] \setminus \{a,b\}$.

Let $x \in End(D)$. Then for each pair of distinct points $a, b \in D$, $x \in [a,b]$ implies $x = a$ or $x = b$.

For each $A \in 2^D$ there exists a unique subcontinuum of D, $D_{\min}(A) \in C(D)$, such that $D_{\min}(A)$ is irreducible about A,

$$D_{\min}(A) = \bigcap \{B \in C(D) : A \subset B\}.$$

Proposition 3.3. *For each* $A \in 2^D$, $D_{\min}(A) = \cup_{a,b \in A}[a, b]$.

It is not difficult to prove Proposition 3.3.

We refer to $D_{\min}(A)$ as the minimal subdendrite of D that contains $A \in 2^D$.

Let $\varphi : 2^D \to C(D)$ defined by $\varphi(A) = D_{\min}(A)$.

Theorem 3.4. *The function* $\varphi : 2^D \to C(D)$ *is continuous.*

Theorem 3.4 is proved in [8].

Other properties of the mapping $\varphi : 2^D \to C(D)$ are the following:

- Let $a, b \in D$, $a \neq b$. Then $\varphi(\{a\}) = \{a\}$ and $\varphi(\{a, b\}) = [a, b]$.
- If $A \in 2^D$ is finite, then $\varphi(A)$ is a tree.
- For each $A \in 2^D$, $End(\varphi(A)) \subset A$.
- For each $A \in C(D)$, $\varphi(A) = A$.

Proposition 3.5 contains some results already known (see [9] and [16]). Most of them are consequences of Theorem 3.4.

Proposition 3.5. *Let* $a, b \in D$, $a \neq b$. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of points in* D *such that* $lim_{n\to\infty} a_n = a$ *and* $lim_{n\to\infty} b_n = b$.

- *For each* $\varepsilon > 0$, *there is* $\delta > 0$ *such that if* $d(a, b) < \delta$, *then* $diam([a, b]) < \varepsilon$.
- $\lim_{n\to\infty} diam\,([a_n, a]) = 0$.
- *For every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any pair of points* u *and* v *in* D, *if* $d(a, u) < \delta$ *and* $d(b, v) < \delta$, *then* $H_d([a, b], [u, v]) < \varepsilon$.
- $\lim[a_n, b_n] = [a, b]$.
- *For each point* $x \in (a, b)$, *there exists* $\delta > 0$ *such that for each pair of points* u *and* v *in* D *with* $d(a, u) < \delta$ *and* $d(b, v) < \delta$, $x \in [u, v]$.
- *For each arc* $[s, t] \subset [a, b]$, $\{s, t\} \cap \{a, b\} = \emptyset$, *there exists* $\delta > 0$ *such that for each pair of points* u *and* v *in* D *with* $d(a, u) < \delta$ *and* $d(b, v) < \delta$, $[s, t] \subset [u, v]$.
- *For each arc* $[s, t] \subset [a, b]$, $\{s, t\} \cap \{a, b\} = \emptyset$, *there exists* $n_0 \in \mathbb{N}$ *such that for each* $n \geq n_0$, $[s, t] \subset [a_n, b_n]$.

4. Dynamics of Dendrite Homeomorphisms

We collect in this section some basic properties of dendrite homeomorphisms. Let $D = (D, d)$ be a nondegenerate dendrite.

Proposition 4.1. *Let* $f : D \to D$ *be a homeomorphism. Then for each arc* $[a, b]$ *contained in* D, $f([a, b]) = [f(a), f(b)]$.

The proof of Proposition 4.1 can be found in [16].

Corollary 4.2. *Let $\varphi : 2^D \to C(D)$ given by $\varphi(A) = D_{\min}(A)$. Let $f : D \to D$ be a homeomorphism.*

- *Then for each $A \in 2^D$, $\varphi(f(A)) = f(\varphi(A))$.*
- *If $A \in 2^D$ is strongly invariant under f, then $D_{\min}(A)$ is strongly invariant under f as well.*

  In particular, for every $x \in D$, $D_{\min}(\alpha(x, f) \cup \omega(x, f))$ is strongly invariant under f.

Proof. The first part follows from Proposition 3.3 and Proposition 4.1.

The second part of the corollary is immediate from the first part. $\square$

Proposition 4.3. *Let $f : D \to D$ be a homeomorphism. Let $a, b \in D$ be two distinct points such that $f(a) = a$ and $f(b) = b$. Then for each point x in the arc $[a, b]$, $card(\omega(x, f)) = 1$ and $card(\alpha(x, f)) = 1$. Furthermore, if $Fix(f) \cap (a, b) = \emptyset$, then one of the following two conditions holds.*

(1) *For every $x \in (a, b)$, $\alpha(x, f) = \{a\}$ and $\omega(x, f) = \{b\}$, or*
(2) *for every $x \in (a, b)$, $\alpha(x, f) = \{b\}$ and $\omega(x, f) = \{a\}$.*

Proposition 4.3 is proved in [9].

Corollary 4.4. *Let $f : D \to D$ be a homeomorphism. Let $a, b \in D$, $a \neq b$, be periodic points of f. Then for each $x \in [a, b]$, $card(\alpha(x, f))$ and $card(\omega(x, f))$ are finite.*

Proof. Let $a, b \in Per(f)$ be two points of periods n and m, respectively.

Let $N = m \cdot n$. Then $f^N(a) = a$ and $f^N(b) = b$.

According to Proposition 4.3, for each $x \in [a, b]$,

$$card(\omega(x, f^N)) = 1 \quad \text{and} \quad card(\alpha(x, f^N)) = 1.$$

The result is an immediate consequence of Proposition 2.1. $\square$

Theorem 4.5. *Let $f : D \to D$ be a homeomorphism and $x \in D$. Then $\omega(x, f)$ is either a periodic orbit or a Cantor set. Moreover, if $\omega(x, f)$ is a Cantor set, then f restricted to $\omega(x, f)$ is an adding machine.*

Theorem 4.5 is proved in [2].

According to Theorem 4.5, for each $x \in D$, $\omega(x, f)$ is a minimal set of f provided that $f : D \to D$ is a dendrite homeomorphism. Therefore, in this context, every limit set $\omega(x, f)$ is contained in the set $R(f)$. Thus,

$$R(f) = \Lambda(f) = \cup \{\omega(x, f) : x \in X\}.$$

Proposition 4.6 and Corollary 4.7 are immediate consequences of Theorem 4.5.

Proposition 4.6. *Let $f : D \to D$ be a homeomorphism and let $x \in D$. Then*

- *$f(\alpha(x, f)) = \alpha(x, f)$;*
- *$\alpha(x, f)$ is a minimal set of f.*

Corollary 4.7. *Let $f : D \to D$ be a homeomorphism and let $x \in D$.*

- *Then $x \in \omega(x, f)$ if and only if $x \in \alpha(x, f)$.*
- *If $x \notin \omega(x, f) \cup \alpha(x, f)$, then for every $y \in D$, $x \notin \omega(y, f) \cup \alpha(y, f)$.*

Proposition 4.8. *Let $f : D \to D$ be a homeomorphism. Then*

$$R(f) = \Lambda(f) = cl(Per(f)).$$

Proposition 4.8 is proved in [16].

Proposition 4.9. *Let $f : D \to D$ be a homeomorphism. Let $x, z \in D$. If*

$$z \in D_{\min}(\alpha(x, f) \cup \omega(x, f)) \setminus (\alpha(x, f) \cup \omega(x, f)),$$

then $card(\omega(z, f))$ and $card(\alpha(z, f))$ are finite.

Proof. Since $z \in D_{\min}(\alpha(x, f) \cup \omega(x, f)) \setminus (\alpha(x, f) \cup \omega(x, f))$, there exist two points a and b in $\alpha(x, f) \cup \omega(x, f)$ such that $z \in (a, b)$. Note that $a, b \in R(f)$. By Proposition 3.5 and Proposition 4.8, there exist $p, q \in Per(f)$ such that $z \in [p, q]$.

Therefore, by Corollary 4.4, $card(\omega(z, f))$ and $card(\alpha(z, f))$ are finite. □

Corollary 4.10. *Let $f : D \to D$ be a homeomorphism. Let $x \in D$ be a point such that cardinality of $\omega(x, f)$ is infinite. Then $x \in D_{\min}(\alpha(x, f) \cup \omega(x, f))$ if and only if $x \in \alpha(x, f) \cup \omega(x, f)$.*

Proof. The result is immediate from Proposition 4.9. □

Proposition 4.11 is proved in [16].

Proposition 4.11. *Let $f : D \to D$ be a homeomorphism. Let $x \in D$ be a point such that $\omega(x, f)$ is infinite. Let $a \in Fix(f)$ such that $[a, x] \cap Fix(f) = \{a\}$. Then there exists $u \in (a, x]$ such that $u \in \omega(x, f)$. Furthermore, if $u \neq x$, then there exists a sequence $\{N_1 < N_2 < N_3 < \cdots\} \subset \mathbb{N}$ such that for each pair $i, j \in \mathbb{N}$, $i \neq j$,*

$$[f^{N_i}(u), f^{N_i}(x)] \cap [f^{N_j}(u), f^{N_j}(x)] = \emptyset.$$

Corollary 4.12. *Let $f : D \to D$ be a homeomorphism. Let $x \in D$ be a point such that cardinality of $\omega(x, f)$ is infinite. Then $\omega(x, f) = \alpha(x, f)$.*

Proof. Let $x \in D$ such that $card(\omega(x, f)) = \infty$.

Limit sets $\omega(x, f)$ and $\alpha(x, f)$ are strongly invariant under f, and they are minimal sets of f. If $x \in \omega(x, f)$, the result is immediate.

Assume $x \notin \omega(x, f)$. Let $a \in Fix(f)$ such that $[a, x] \cap Fix(f) = \{a\}$. According to Proposition 4.11, there exist $u \in (a, x]$ such that $u \in \omega(x, f)$, and a sequence $\{N_1 < N_2 < N_3 < \cdots\} \subset \mathbb{N}$ such that for each pair $i, j \in \mathbb{N}$, $i \neq j$,

$$[f^{N_i}(u), f^{N_i}(x)] \cap [f^{N_j}(u), f^{N_j}(x)] = \emptyset. \tag{4.1}$$

Since $\omega(x, f)$ is a minimal set, it follows that

$$\alpha(u, f) = \omega(u, f) = \omega(x, f).$$

Let $i, j \in \mathbb{N}$, $i \neq j$. By equation (4.1), and considering homeomorphism $f^{-(N_i+N_j)} : D \to D$, we have that

$$[f^{-N_i}(u), f^{-N_i}(x)] \cap [f^{-N_j}(u), f^{-N_j}(x)] = \emptyset \tag{4.2}$$

as well.

Therefore, by Proposition 3.2,

$$\lim_{i\to\infty} diam([f^{-N_i}(u), f^{-N_i}(x)]) = 0.$$

Hence, $\alpha(x, f) \cap \alpha(u, f) \neq \emptyset$ and $\omega(x, f) = \omega(u, f) = \alpha(u, f) = \alpha(x, f)$. □

Corollary 4.13. *Let $f : D \to D$ be a homeomorphism. Let $x \in D$ be a point such that cardinality of $\alpha(x, f)$ is infinite. Then $\omega(x, f) = \alpha(x, f)$.*

Proof. Since $f^{-1} : D \to D$ is a homeomorphism,

$$\alpha(x, f) = \omega(x, f^{-1}) = \alpha(x, f^{-1}) = \omega(x, f).$$

Thus, $\omega(x, f) = \alpha(x, f)$. □

Corollary 4.14. *Let $f : D \to D$ be a homeomorphism and $x \in D$. Then $\omega(x, f)$ is finite if and only if $\alpha(x, f)$ is finite.*

The proof of Corollary 4.14 is immediate from corollaries 4.12 and 4.13.

Proposition 4.15. *Let $f : D \to D$ be a homeomorphism. Let $x_0 \in D$, $a \in Fix(f)$, $x_0 \neq a$.*

- *If $[a, x_0] \subset [a, f(x_0)]$, then there exists a point $b \in Fix(f)$, $a \neq b$, such that $\lim_{n\to\infty} f^n(x_0) = b$ and for each $n \in \mathbb{Z}$, $f^n(x_0) \in [a, b]$.*
- *If $[a, f(x_0)] \subset [a, x_0]$, then there exists a point $b \in Fix(f)$, $a \neq b$, such that $\lim_{n\to\infty} f^{-n}(x_0) = b$ and for each $n \in \mathbb{Z}$, $f^n(x_0) \in [a, b]$.*

Proposition 4.15 is proved in [16].

Note that in either of the cases considered in Proposition 4.15, for each point $x \in [a, b]$, we have that $card(\omega(x, f)) = 1$ and $card(\alpha(x, f)) = 1$. Corollary 4.16 is an immediate consequence of Proposition 4.15. See also [1, Lemma 2.7].

Corollary 4.16. *Let $f : D \to D$ be a homeomorphism. Let $x_0 \in D$, $a \in Fix(f)$, $x_0 \neq a$. If for some $N \in \mathbb{N}$, either*

- $[a, x_0] \subset [a, f^N(x_0)]$, *or*
- $[a, f^N(x_0)] \subset [a, x_0]$,

holds, then for every $x \in [a, x_0]$, $card(\omega(x, f))$ *and* $card(\alpha(x, f))$ *are finite.*

Proposition 4.17. *Let* $f : D \to D$ *be a homeomorphism. Let* $a, b \in Fix(f)$, $a \neq b$. *If* a *and* b *are end points of* D *and* $card(Fix(f)) = 2$, *then one of the following two conditions holds:*

(1) *For every* $x \in D \setminus \{a, b\}$, $\alpha(x, f) = \{a\}$ *and* $\omega(x, f) = \{b\}$.
(2) *For every* $x \in D \setminus \{a, b\}$, $\alpha(x, f) = \{b\}$ *and* $\omega(x, f) = \{a\}$.

Proposition 4.17 is proved in [9].

Proposition 4.18. *Let* $f : D \to D$ *be a homeomorphism. Let* $x \in D$. *Then*

- *any arc in* D *contains at most two points of* $\omega(x, f)$;
- $End(D_{\min}(\omega(x, f))) = \omega(x, f)$.

Proposition 4.18 is proved in [1].

Lemma 4.19. *Let* $f : D \to D$ *be a homeomorphism. Let* $x \in D$ *such that* $\omega(x, f) = \alpha(x, f) = \{a\}$, $a \neq x$. *Then for each* $n \in \mathbb{N}$, $[a, x] \cap [a, f^n(x)] = \{a\}$.

Proof. Assume that there exists $k \in \mathbb{N}$ such that

$$[a, x] \cap [a, f^k(x)] = [a, u], \tag{4.3}$$

with $a \neq u$.

If $u = x$, then $[a, x] \subset [a, f^k(x)]$. By Proposition 4.15, there exists a point $b \in Fix(f^k)$, $a \neq b$, such that $\lim_{n\to\infty} f^{nk}(x_0) = b$. This contradicts the assumption that $\omega(x, f) = \{a\}$.

From now on we consider $u \neq x$.

Since $u \in [a, x]$ and $a \in Fix(f)$, then $f^k(u) \in [a, f^k(x)]$. Therefore, by (4.3), points a, u, and $f^k(u)$ are in the arc $[a, f^k(x)]$. We have two options:

$$[a, u] \subset [a, f^k(u)] \quad \text{or} \quad [a, f^k(u)] \subset [a, u].$$

If $[a, u] \subset [a, f^k(u)]$, then for each $n \in \mathbb{N}$, we have that

$$[a, u] \subset [a, f^k(u)] \subset [a, f^{nk}(u)] \subset [a, f^{nk}(x)].$$

Hence, $\lim_{n\to\infty} f^{nk}(x)$ is not a. A contradiction.

Now consider the case $[a, f^k(u)] \subset [a, u]$. By (4.3), $[a, f^k(u)] \subset [a, u] \subset [a, x]$. It follows that

$$[a, u] \subset [a, f^{-k}(u)] \subset [a, f^{-k}(x)].$$

Hence, for each $n \in \mathbb{N}$,

$$[a, u] \subset [a, f^{-k}(u)] \subset [a, f^{-nk}(u)] \subset [a, f^{-nk}(x)].$$

Thus, $\lim_{n\to\infty} f^{-nk}(x)$ is not a. This contradicts the assumption that $\alpha(x, f) = \{a\}$. Then for each $n \in \mathbb{N}$, $[a, x] \cap [a, f^n(x)] = \{a\}$. $\square$

Corollary 4.20. *Let $f : D \to D$ be a homeomorphism. Let $x \in D$ be a point such that $\omega(x, f) = \alpha(x, f) = \{a\}$, $a \neq x$. Then for each $n, m \in \mathbb{Z}$, $n \neq m$, $[a, f^n(x)] \cap [a, f^m(x)] = \{a\}$. Furthermore, $A = \cup_{n\in\mathbb{Z}}[a, f^n(x)]$ is a subdendrite of D strongly invariant under f.*

The proof of Corollary 4.20 is immediate from Lemma 4.19.

Proposition 4.21. *Let $f : D \to D$ be a dendrite homeomorphism. Let x_0 be a point of D that it is not a recurrent point, $x_0 \in D \setminus R(f)$. Let U be the component of $D \setminus R(f)$ that contains x_0. Then for each $x \in U$,*

$$\alpha(x, f) = \alpha(x_0, f) \quad \textit{and} \quad \omega(x, f) = \omega(x_0, f).$$

Proposition 4.21 is proved in [9].

5. Topological Entropy

In this section we recall the definition of topological entropy and some of its basic properties. Let $X = (X, d)$ denote a nondegenerate compact metric space. Let $f : X \to X$ be a mapping.

Let $\varepsilon > 0$ and $n \in \mathbb{N}$. A subset $A \subset X$ is said to (n, ε)*-span* X if for any $x \in X$ there exists $a \in A$ with

$$d(f^i(x), f^i(a)) < \varepsilon, \quad \text{for} \quad 0 \le i \le n - 1.$$

Let $r(n, \varepsilon)$ denote the smallest cardinality of any (n, ε)-spanning set for X. Let

$$r(\varepsilon, f) = \limsup_{n\to\infty} \left(\frac{1}{n}\right) \log(r(n, \varepsilon)).$$

The *topological entropy* of f is given by

$$ent(f) = \lim_{\varepsilon\to 0} r(\varepsilon, f).$$

Note that for each $\varepsilon > 0$, $r(\varepsilon, f) \le ent(f)$. See [6].

Proposition 5.1. *Let $M_1, M_2, \ldots, M_k$ be k closed non empty subsets of X. If all of them are invariant under f and $X = M_1 \cup M_2 \cup \cdots \cup M_k$, then $ent(f) = \max\{ent(f|_{M_i}) : 1 \le i \le k\}$.*

The proof of Proposition 5.1 can be found in [4], [5], [6], and [17]

According to [4], the next more general claim is true.

Proposition 5.2. *If X is the union of $\{M_t : t \in T\}$, where each M_t is closed nonempty and invariant under f, then $ent(f) = \sup\{ent(f|_{M_t}) : t \in T\}$.*

Corollary 5.3. *Let $A \subset X$ be a closed and invariant set of $f : X \to X$. Then $ent(f) \geq ent(f\,|_A)$.*

The proof of Corollary 5.3 is immediate from Proposition 5.1.

Proposition 5.4. *Let Y be a compact metric space. Let $g : Y \to Y$ be a mapping and $h : X \to Y$ be a surjective mapping. If for every $x \in X$, $h(f(x)) = g(h(x))$, then $ent(f) \geq ent(g)$. If h is a homeomorphism, then $ent(f) = ent(g)$.*

The proof of Proposition 5.4 can be found in [4], [5], [6], and [17].

Lemma 5.5. *Let $M \subset X$ be a closed set invariant under f. Let $x, y \in X$ such that $x \in M$, $y \notin M$, and $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$. Let $P = M \cup \{f^n(y) : n \geq 0\}$. Then $ent(f|_M) = ent(f|_P)$.*

Proof. Notice that P is a closed subset of X invariant under f. Since $M \subset P$, $ent(f|_M) \leq ent(f|_P)$.

Let $\varepsilon > 0$ and $m \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ such that $d(f^n(y), f^n(x)) < \frac{\varepsilon}{2}$ for each $n \geq n_0$. Let $E \subset M$ be an $(m, \frac{\varepsilon}{2})$-spanning set for $f|_M$ of cardinality $r(m, \frac{\varepsilon}{2}, f|_M)$.

Let $F = E \cup \{f^n(y) : 0 \leq n \leq n_0 - 1\}$. It follows that F is an (m, ε)-spanning set for $f|_P$. Then $r(m, \varepsilon, f|_P) \leq r(m, \frac{\varepsilon}{2}, f|_M) + n_0$. Hence, for each $\varepsilon > 0$, $r(\varepsilon, f|_P) \leq r(\frac{\varepsilon}{2}, f|_M) \leq ent(f|_M)$. Therefore, $ent(f|_P) \leq ent(f|_M)$. □

Theorem 5.6. *If $f : D \to D$ is a dendrite homeomorphism, then $ent(f) = 0$.*

Theorem 5.6 is proved in [2].

Proposition 5.7. *Let D and E be two dendrites. Let $f : D \to D$, $g : E \to E$, and $h : D \to E$ be homeomorphisms such that for each $x \in D$, $h(f(x)) = g(h(x))$. Let $C(f) : C(D) \to C(D)$, $C(g) : C(E) \to C(E)$, and $C(h) : C(D) \to C(E)$ be the corresponding induced homeomorphisms. Then*

- *For each $A \in C(D)$, $C(h)(C(f)(A)) = C(g)(C(h)(A))$, and*
- *$ent(C(f)) = ent(C(g))$.*

6. Entropy of the Induced Dendrite Homeomorphism $C(f)$ (First Part)

Let D denote a nondegenerate dendrite. Let $f : D \to D$ be a homeomorphism. Recall $R(f)$ stands for the set of recurrent points of f.

In this section, we prove the following:

- If there exists a point $x \in D \setminus R(f)$, such that x is not an element of the minimal subcontinuum that contains $\alpha(x,f) \cup \omega(x,f)$, i.e., $x \notin D_{\min}(\alpha(x,f) \cup \omega(x,f))$, then $ent(C(f)) = \infty$.

Example 6.1 plays a key role in our argument.

Example 6.1. Let S be a dendrite contained in the complex plane $\mathbb{C}$ defined in the following way:

- Let $I_0 = \{z = t \cdot e^{i(\frac{\pi}{2})} : 0 \le t \le 1\} = \{z = t \cdot i : 0 \le t \le 1\}$.
- For each $n \in \mathbb{N}$, let $I_n = \{z = t \cdot e^{i(\pi - \frac{\pi}{n+2})} : 0 \le t \le \frac{1}{n+1}\}$.
- For each $n \in \mathbb{N}$, let $I_{-n} = \{z = t \cdot e^{i(\frac{\pi}{n+2})} : 0 \le t \le \frac{1}{n+1}\}$.

Let $S = \cup_{n \in \mathbb{Z}} I_n$. The *vertex* of S is 0, the *beans* of S are I_n, $n \in \mathbb{Z}$. Let $g : S \to S$ be the function defined by

- for each $n \ge 0$, $g(I_n) = I_{n+1}$;

$$g(z) = g\left(t \cdot e^{i(\pi - \frac{\pi}{n+2})}\right) = \frac{n+1}{n+2} \cdot t \cdot e^{i(\pi - \frac{\pi}{n+3})}, \quad 0 \le t \le \frac{1}{n+1}.$$

- for each $n \ge 1$, $g(I_{-n}) = I_{-n+1}$,

$$g(z) = g\left(t \cdot e^{i(\frac{\pi}{n+2})}\right) = \frac{n+1}{n} \cdot t \cdot e^{i(\frac{\pi}{n+1})}, \quad 0 \le t \le \frac{1}{n+1}.$$

The function $g : S \to S$ has the following properties:

- $g : S \to S$ is a homeomorphism.
- $Fix(g) = \{0\} = Per(g) = R(g)$.
- For each pair $n, k \in \mathbb{Z}$, $g^k(I_n) = I_{n+k}$. In particular, for each $n \in \mathbb{Z}$, $g(I_n) = I_{n+1}$.
- For every $x \in S$, $\omega(x,g) = \alpha(x,g) = \{0\}$.
- The entropy of the homeomorphism $C(g) : C(S) \to C(S)$ is ∞.

This dendrite homeomorphism was introduced in [1]. Proofs of all properties of it are in [1] as well.

From now on until the end of the section, let $f : D \to D$ be a dendrite homeomorphism and let $x_0 \in D \setminus R(f)$ be a point such that

$$x_0 \notin D_{\min}(\alpha(x_0,f) \cup \omega(x_0,f)).$$

Consider the following equivalence relation in dendrite D:

- $x \sim y$ if and only if $x, y \in D_{\min}(\alpha(x_0,f) \cup \omega(x_0,f))$.
- If $x \notin D_{\min}(\alpha(x_0,f) \cup \omega(x_0,f))$, then $x \sim y$ if and only if $x = y$.

Let $\mathbf{E} = D/\sim$ be the identification space and let $p : D \to \mathbf{E}$ be the natural mapping. Since $p : D \to \mathbf{E}$ is monotone, then $\mathbf{E}$ is a dendrite (see [7]).

Let $\mathbf{a} = p(D_{\min}(\alpha(x_0,f) \cup \omega(x_0,f)))$. Let $F : \mathbf{E} \to \mathbf{E}$ be the function given by

- $F(\mathbf{a}) = \mathbf{a}$.
- For each $\mathbf{y} \in \mathbf{E}$, $\mathbf{y} \neq \mathbf{a}$, let $F(\mathbf{y}) = p(f(x))$ where $p(x) = \mathbf{y}$.

Note that $D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f))$ is strongly invariant under f. It follows that $F : \mathbf{E} \to \mathbf{E}$ is a homeomorphism, and for each $x \in D$, $p(f(x)) = F(p(x))$.

Since $x_0 \notin D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f))$, $p(x_0) \neq \mathbf{a}$ and

$$\lim_{n\to\infty} F^n(p(x_0)) = \mathbf{a} \quad \text{and} \quad \lim_{n\to-\infty} F^n(p(x_0)) = \mathbf{a}.$$

Therefore,

$$\omega(p(x_0), F) = \alpha(p(x_0), F) = \{\mathbf{a}\}.$$

From now on let $\mathbf{E} = D/\sim$ and $F : \mathbf{E} \to \mathbf{E}$ stand for the dendrite and the homeomorphism defined above.

Proposition 6.2. *The homeomorphism $F : \boldsymbol{E} \to \boldsymbol{E}$ enjoys the following properties:*

- *For each pair $n, m \in \mathbb{Z}$, $n \neq m$,*
$$[\boldsymbol{a}, F^n(p(x_0))] \cap [\boldsymbol{a}, F^m(p(x_0))] = \{\boldsymbol{a}\}.$$
- *The union $\boldsymbol{L} = \cup_{n\in\mathbb{Z}}[\boldsymbol{a}, F^n(p(x_0))]$ is a subdendrite of $\boldsymbol{E}$ strongly invariant under F.*

Proof. Both results are immediate consequences of Lemma 4.19 and Corollary 4.20. □

Corollary 6.3. *Let $u \in D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f))$ such that*

$$[x_0, u] \cap D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)) = \{u\}.$$

Then

- *for each pair $n, m \in \mathbb{Z}$, $n \neq m$,*
$$[f^n(x_0), f^n(u)) \cap [f^m(x_0), f^m(u)) = \emptyset;$$
- *$J = D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)) \cup (\cup_{n\in\mathbb{Z}}[f^n(x_0), f^n(u)])$ is a dendrite contained in D strongly invariant under $f : D \to D$, $p(J) = \boldsymbol{L}$, and for each $x \in J$, $p(f(x)) = F(p(x))$.*

Proof. The mapping $p : D \to \mathbf{E}$ is monotone and $J = p^{-1}(\mathbf{L})$. Now the result is an immediate consequence of Proposition 6.2. □

Proposition 6.4. *Let S be the dendrite and let $g : S \to S$ be the homeomorphism, both described in Example 6.1. Then there exists a homeomorphism $h : \boldsymbol{L} \to S$ such that for each $\boldsymbol{y} \in \boldsymbol{L}$, $h(F(\boldsymbol{y})) = g(h(\boldsymbol{y}))$.*

Proof. Let $h_0 : [p(x_0), \mathbf{a}] \to I_0$ be a homeomorphism such that

$$h_0(\mathbf{a}) = 0 \quad \text{and} \quad h_0(p(x_0)) = e^{i(\frac{\pi}{2})}.$$

For each $n \in \mathbb{Z}$, let $h_n : [F^n(p(x_0)), \mathbf{a}] \to I_n$ given by $h_n = g^n \circ h_0 \circ F^{-n}$. Let $h : \mathbf{L} \to S$ be the function given by

- $h(\mathbf{a}) = 0$.
- Let $\mathbf{y} \in \mathbf{L}$, $\mathbf{y} \neq \mathbf{a}$. There exists $n \in \mathbb{Z}$, $\mathbf{y} \in [F^n(p(x_0)), \mathbf{a}]$. Then $h(\mathbf{y}) = h_n(\mathbf{y})$.

It follows that $h : \mathbf{L} \to S$ is a homeomorphism.

Let $\mathbf{y} \in [F^n(p(x_0)), \mathbf{a}]$, then

$$\begin{aligned} h(F(\mathbf{y})) &= g^{n+1} \circ h_0 \circ F^{-(n+1)}(F(\mathbf{y})) = g^{n+1} \circ h_0 \circ F^{-n}(\mathbf{y}) \\ &= g \circ g^n \circ h_0 \circ F^{-n}(\mathbf{y}) = g(h(\mathbf{y})). \end{aligned}$$

This completes the proof. □

Corollary 6.5. *The topological entropy of $C(F|_{\boldsymbol{L}}) : C(\boldsymbol{L}) \to C(\boldsymbol{L})$ is ∞.*

Proof. Let S be the dendrite and let $g : S \to S$ be the homeomorphism both described in Example 6.1.

According to Proposition 6.4, there exists a homeomorphism $h : \mathbf{L} \to S$ such that for each $\mathbf{y} \in \mathbf{L}$, $h(F(\mathbf{y})) = g(h(\mathbf{y}))$.

The induced mappings $C(F|_{\mathbf{L}}) : C(\mathbf{L}) \to C(\mathbf{L})$, $C(h) : C(\mathbf{L}) \to C(S)$, and $C(g) : C(S) \to C(S)$ are homeomorphisms with the property that for each $A \in C(\mathbf{L})$,

$$C(h)(C(F|_{\mathbf{L}})(A)) = C(g)(C(h)(A)).$$

Since $ent(C(g)) = \infty$, then $ent(C(F|_{\mathbf{L}})) = \infty$. □

Corollary 6.6. *Let $J \subset D$ be the dendrite described in Corollary 6.3,*

$$J = D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)) \cup (\cup_{n \in \mathbb{Z}}[f^n(x_0), f^n(u)]).$$

Then the topological entropy of $C(f|_J) : C(J) \to C(J)$ is ∞.

Proof. Notice that $f|_J : J \to J$ is a homeomorphism, and the natural mapping $p : J \to \mathbf{L}$ is monotone and onto. Then $C(f|_J) : C(J) \to C(J)$ is a homeomorphism and $C(p) : C(J) \to C(\mathbf{L})$ is a surjective map.

According to Corollary 6.3, for each $x \in J$, $p(f(x)) = F(p(x))$. Then for each $A \in C(J)$,

$$C(p)(C(f|_J)(A)) = C(F|_{\mathbf{L}})(C(p)(A)).$$

Since $ent(C(F|_{\mathbf{L}})) = \infty$, then $ent(C(f|_J)) = \infty$. □

Theorem 6.7 summarizes our work in this section.

Theorem 6.7. *Let $f : D \to D$ be a dendrite homeomorphism. If there exists a point $x_0 \in D \setminus R(f)$ such that*

$$x_0 \notin D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)),$$

then $ent(C(f)) = \infty$.

Proof. Let $J \subset D$ be the dendrite described in Corollary 6.3,

$$J = D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)) \cup (\cup_{n\in\mathbb{Z}}[f^n(x_0), f^n(u)]).$$

This dendrite is strongly invariant under $f : D \to D$. Then the hyperspace $C(J)$ is a closed subset of $C(D)$ strongly invariant under the induced mapping $C(f)$.

By Corollary 6.6,

$$ent(C(f)|_{C(J)}) = ent(C(f|_J)) = \infty.$$

It follows that $ent(C(f)) = \infty$. □

7. Entropy of the Induced Dendrite Homeomorphism $C(f)$ (Second Part)

Let $D = (D, d)$ denote a nondegenerate dendrite. Let $f : D \to D$ be a homeomorphism. Recall that according to Proposition 4.8, the set of recurrent points $R(f)$ is closed.

In this section, we prove the following:

- If $R(f) = D$, then $ent(C(f)) = 0$.
- If $R(f) \neq D$ and for every $x \in D \setminus R(f)$, $x \in D_{\min}(\alpha(x, f) \cup \omega(x, f))$, then $ent(C(f)) = 0$.

Theorem 7.1. *Let $f : D \to D$ be a dendrite homeomorphism such that $R(f) = D$. Then $ent(2^f) = 0$.*

Theorem 7.1 is proved in [9].

Corollary 7.2. *Let $f : D \to D$ be a dendrite homeomorphism such that $R(f) = D$. Then $ent(C(f)) = 0$.*

Proof. The hyperspace $C(D)$ is a closed subset of 2^D, and it is strongly invariant under mapping $2^f : 2^D \to 2^D$.

Hence, $0 \leq ent(C(f)) = ent(2^f|_{C(D)}) \leq ent(2^f) = 0$. □

From now on until the end of the section, we assume the following conditions:

- $f : D \to D$ is a homeomorphism.
- $R(f) \neq D$.
- For each $x_0 \in D$, $x_0 \in D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f))$.

Proposition 7.3. *Let U be a component of $D \setminus R(f)$. Then*

- *there exists* $N \in \mathbb{N}$ *such that* $f^N(U) = U$*;*
- *there exist two distinct points* $a, b \in R(f)$ *such that* $U = (a, b)$*.*

Proof. Let us prove the first part.

Let Δ be the collection of all components of $D \setminus R(f)$. If Δ is finite, the result readily follows.

Assume the cardinality of Δ is infinite. Let $U \in \Delta$. It is enough to show that there exists $N \in \mathbb{N}$ such that $f^N(U) \cap U \neq \emptyset$.

Assume that

$$\text{for every } n \in \mathbb{N}, \quad f^n(U) \cap U = \emptyset. \tag{7.1}$$

Let $x_0 \in U$. Since $x_0 \notin R(f)$,

$$x_0 \in D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)) \setminus (\alpha(x_0, f) \cup \omega(x_0, f)).$$

By Proposition 4.9, $card(\omega(x_0))$ and $card(\alpha(x_0))$ are finite.

Hence, $D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f))$ is a tree.

Condition (7.1) implies that

$$\text{for each } n, m \in \mathbb{Z}, \quad n \neq m, \quad f^n(U) \cap f^m(U) = \emptyset. \tag{7.2}$$

It follows that

$$\lim_{n\to\infty} diam(f^n(U)) = 0 \quad \text{and} \quad \lim_{n\to\infty} diam(f^n(cl(U))) = 0.$$

Therefore, for each $x \in cl(U)$, $\omega(x, f) = \omega(x_0, f)$.

Let $y_0 \in cl(U) \setminus U$. Note that $y_0 \in R(f)$.

Since $\omega(y_0, f) = \omega(x_0, f)$, $y_0 \in D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f))$ and

$$[y_0, x_0] \subset D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)).$$

Also, $f^k(y_0) = y_0$, where $k = card(\omega(x_0, f))$.

The set $U \cup \{y_0\}$ is connected. Hence, $[y_0, x_0] \subset U \cup \{y_0\}$, and $(y_0, x_0]$ is contained in U.

The collection of arcs $\{[y_0, f^{n\cdot k}(x_0)] : n \in \mathbb{Z}\}$ has the following properties:

- For each $n \in \mathbb{Z}$, $[y_0, f^{n\cdot k}(x_0)] \subset D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f))$.
- According to condition (7.2), for every $n, m \in \mathbb{Z}$, $n \neq m$,

$$[y_0, f^{n\cdot k}(x_0)] \cap [y_0, f^{m\cdot k}(x_0)] = \{y_0\}.$$

Hence,

$$A = \bigcup \{[y_0, f^{n\cdot k}(x_0)] : n \in \mathbb{Z}\} \subset D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)).$$

This is a contradiction. Thus, there exists $N \in \mathbb{N}$ such that $f^N(U) \cap U \neq \emptyset$.

Now we prove the second part.

Let U be a component of $D \setminus R(f)$. Let $N \in \mathbb{N}$ such that $f^N(U) = U$.

Note that $cl(U)$ is a dendrite strongly invariant under f^N. Let $a \in cl(U)$ such that $f^N(a) = a$.

Let $x_0 \in U$. It follows that for any $n \in \mathbb{Z}$,

$$[a, f^{N\cdot n}(x_0)] \subset U \cup \{a\}.$$

If $[a, f^N(x_0)] \subset [a, x_0]$, then by Proposition 4.15, there exists $b \in D$, $b \in Fix(f^N)$, $b \neq a$, such that

$$\lim_{n\to\infty} f^{N\cdot n}(x_0) = a \quad \text{and} \quad \lim_{n\to-\infty} f^{N\cdot n}(x_0) = b.$$

In this case, we have $[a, b] \subset U \cup \{a, b\}$ and $(a, b) \subset U$.

If $[a, x_0] \subset [a, f^N(x_0)]$, then there exists $b \in D$, $b \in Fix(f^N)$, $b \neq a$, such that

$$\lim_{n\to\infty} f^{N\cdot n}(x_0) = b \quad \text{and} \quad \lim_{n\to-\infty} f^{N\cdot n}(x_0) = a.$$

Again, we have that $[a, b] \subset U \cup \{a, b\}$ and $(a, b) \subset U$.

The case $f^N(x_0) \notin [a, x_0]$ and $x_0 \notin [a, f^N(x_0)]$ is impossible. Let us see why.

Assume that $f^N(x_0) \notin [a, x_0]$ and $x_0 \notin [a, f^N(x_0)]$. Since $f^N(x_0) \in U$, $[f^N(x_0), x_0] \subset U$. Let $u \in [a, x_0]$ such that

$$[f^N(x_0), u] \cap [a, x_0] = \{u\}.$$

Note that

$$[f^N(x_0), u] \cup [u, x_0] = [f^N(x_0), x_0]. \tag{7.3}$$

It readily follows that $u \neq a$.

By hypothesis,

$$x_0 \notin [a, f^N(x_0)] = [a, u] \cup [f^N(x_0), u].$$

Hence, $u \neq x_0$. Therefore, the point $u \in U$ has the following properties:

- $[a, u] \cap [u, x_0] = \{u\}$.
- $[a, u] \cap [u, f^N(x_0)] = \{u\}$.
- $[u, x_0] \cap [u, f^N(x_0)] = \{u\}$.

That is, u is a branch point of D.

Now, from (7.3), we know that $u \in [f^N(x_0), x_0]$. It implies that

$$u \in D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)).$$

For every $n \in \mathbb{Z}$, $f^{N\cdot n}(u) \in D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f))$. And for every $n, m \in \mathbb{Z}$, $n \neq m$, $f^{N\cdot n}(u) \neq f^{N\cdot m}(u)$. Then the tree $D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f))$ contains infinitely many branch points. This is a contradiction. Hence, we have proved that there exist two distinct points $a, b \in cl(U) \setminus U$ such that

- $(a, b) \subset U$, and $x_0 \in (a, b)$;

- for each $x \in (a, b)$,
$$\lim_{n\to\infty} f^{N\cdot n}(x) = a \quad \text{and} \quad \lim_{n\to-\infty} f^{N\cdot n}(x) = b,$$
or for each $x \in (a, b)$,
$$\lim_{n\to\infty} f^{N\cdot n}(x) = b \quad \text{and} \quad \lim_{n\to-\infty} f^{N\cdot n}(x) = a.$$
Either one of these two options implies that
$$\{a, b\} \subset D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)).$$

Finally, let $x \in U$. We claim that $x \in (a, b)$. If $x \notin (a, b)$, then with a similar argument as the one described above, we find a branch point of D in (a, b). It leads us to a contradiction. Thus, $U = (a, b)$. □

Let U be a component of $D \setminus R(f)$. It is said that

- U is *periodic of period* 1 provided that $f(U) = U$.
- U is *periodic of period* $k \geq 2$ provided that $f^k(U) = U$ and for each $1 \leq i \leq k-1$, $f^i(U) \cap U = \emptyset$.

Notice that if $U = (a, b)$ and $f(U) = U$, then $f(a) = a$ and $f(b) = b$. Furthermore, if U is of period $k \in \mathbb{N}$, then $f^k(a) = a$ and $f^k(b) = b$, and $o(a, f) \cap o(b, f) = \emptyset$.

Lemma 7.4. *Let U_0 be a component of $D \setminus R(f)$ of period $k \geq 2$. Let $U_i = f^i(U_0)$, $0 \leq i \leq k-1$.*

Then there exists a homeomorphism $l : D \to D$ with the following properties:

- *For each $x \in D \setminus (\cup_{i=0}^{k-1} U_i)$, $l(x) = f(x)$.*
- *For every $x \in \cup_{i=0}^{k-1} U_i$, $l^k(x) = x$. That is, $x \in Per(l)$.*
- *$R(l) = R(f) \cup (\cup_{i=0}^{k-1} U_i)$.*

Proof. Let $x_0 \in U_0$, $a_0 = \lim_{n\to\infty} f^{k\cdot n}(x_0)$, $b_0 = \lim_{n\to-\infty} f^{k\cdot n}(x_0)$.

Then for each $0 \leq i \leq k-1$, $U_i = (a_i, b_i)$, where $a_i = f^i(a_0)$ and $b_i = f^i(b_0)$.

Define $l : D \to D$ as follows:

- If $x \in U_{k-1}$, $l(x) = f^{1-k}(x)$.
- If $x \in D \setminus U_{k-1}$, $l(x) = f(x)$.

It is not difficult to see that $l : D \to D$ is a homeomorphism that holds the first and the second properties; in particular, for each point $x \in \cup_{i=0}^{k-1} U_i$, $l^k(x) = x$.

The third part of the lemma, $R(l) = R(f) \cup (\cup_{i=0}^{k-1} U_i)$, is immediate. □

Proposition 7.5. *There exists a homeomorphism $l : D \to D$ with the following properties:*

- *For every $x \in R(f)$, $l(x) = f(x)$.*

- *The set of recurrent points of* l *is* D, $R(l) = D$.

Proof. Let Δ be the collection of all components of $D \setminus R(f)$. There are two cases: Δ is finite or Δ is denumerable. We study the second case; the first one is similar.

Let $\Delta = \{U_1, U_2, U_3, \ldots\}$. We assume that $U_i \cap U_j = \emptyset$ if $i \neq j$.

Notice that by Proposition 3.2, $\lim_{n\to\infty} diam(U_n) = 0$.

Our argument has two parts. First, we produce a sequence of homeomorphisms $\{l_n : D \to D\}_{n=1}^{\infty}$. Then we define $l : D \to D$ as the limit of that sequence.

Let $k_1 \in \mathbb{N}$ be the period of component U_1. If $k_1 = 1$, let $\Delta_1 = \{U_1\}$ and define $l_1 : D \to D$ as follows:

- For every $x \in D \setminus U_1$, $l_1(x) = f(x)$.
- For every $x \in U_1$, $l_1(x) = x$.

Hence, $l_1 : D \to D$ is a homeomorphism with the following properties:

- For every $x \in R(f)$, $l_1(x) = f(x)$.
- $R(l_1) = R(f) \cup U_1$.

If $k_1 \geq 2$, let $\Delta_1 = \{U_1, f(U_1), f^2(U_1), \ldots, f^{k_1-1}(U_1)\}$ and $\cup\Delta_1 = \cup_{j=0}^{k_1-1} f^j(U_1)$.

Let $l_1 : D \to D$ be the homeomorphism we obtain after we follow the procedure described in Lemma 7.4. This homeomorphism has the following properties:

- For each $x \in D \setminus (\cup\Delta_1)$, $l_1(x) = f(x)$.
- For every $x \in \cup\Delta_1$, $l_1^{k_1}(x) = x$.
- $R(l_1) = R(f) \cup (\cup\Delta_1)$.
- For each $x \in R(f)$, $l_1(x) = f(x)$.

Now let $n_2 = \min\{i \in \mathbb{N} : U_i \in \Delta \setminus \Delta_1\}$. Consider the component U_{n_2} and let k_2 be its period. Let $\Delta_2 = \{U_{n_2}\}$ if $k_2 = 1$; or let

$$\Delta_2 = \{U_{n_2}, f(U_{n_2}), f^2(U_{n_2}), \ldots, f^{k_2-1}(U_{n_2})\}$$

if $k_2 \geq 2$.

From the homeomorphism $l_1 : D \to D$, by Lemma 7.4, we obtain a new homeomorphism $l_2 : D \to D$ with these properties:

- For each $x \in D \setminus (\cup\Delta_2)$, $l_2(x) = l_1(x)$.
- For every $x \in \cup\Delta_2$, $l_2^{k_2}(x) = x$.
- $R(l_2) = R(l_1) \cup (\cup\Delta_2)$.
- For each $x \in R(f)$, $l_2(x) = f(x)$.

Following this procedure, we obtain a sequence of sets $\{\Delta_n : n \in \mathbb{N}\}$ and a sequence of homeomorphisms $\{l_n : D \to D : n \in \mathbb{N}\}$. They enjoy these properties:

- For each $j \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $U_j \in \Delta_n$. Thus, $\Delta = \cup_{n=1}^{\infty} \Delta_n$.
- For each $n \in \mathbb{N}$, $R(f) \subset R(l_n) \subset R(l_{n+1})$.
- For each $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that

  $$\max \{diam(U) : U \in \Delta \setminus (\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_{j_0})\} < \varepsilon.$$

  Then, in the Hausdorff metric,

  $$\lim R(l_n) = D.$$

- For every point $x \in R(f)$ and for every $n \in \mathbb{N}$, $l_n(x) = f(x)$.

Another consequence is this one. Let $x_0 \in D \setminus R(f)$. There exists an open component $U \in \Delta$, $x_0 \in U$. Hence, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$,

$$l_n(x) = l_{n_0}(x), \quad \text{for every} \quad x \in U.$$

Then, in particular, $\lim_{n\to\infty} l_n(x_0)$ does exist.

Finally, let $l : D \to D$ be the function given in this way:

- For each $x \in R(f)$, let $l(x) = f(x)$.
- For each $x \in D \setminus R(f)$, let $l(x) = \lim_{n\to\infty} l_n(x)$.

Notice that for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$,

$$\max \{d(l_n(x), l(x)) : x \in D\} < \varepsilon.$$

It follows that $l : D \to D$ is a homeomorphism and $R(l) = D$. □

Proposition 7.6. *Let* $\Gamma = \{L \in C(D) : End(L) \subset R(f)\}$. *Then*

- Γ *is a closed subset of* $C(D)$.
- Γ *is invariant under* $C(f)$.

Proof. The set of recurrent points of f, $R(f)$, is a closed subset of D. Hence, $2^{R(f)}$ is a closed subset of hyperspace 2^D.

According to Theorem 3.4, function $\varphi : 2^D \to C(D)$, $\varphi(A) = D_{\min}(A)$, is continuous. Let

$$\Theta = \varphi(2^{R(f)}) = \{\varphi(A) : A \in 2^{R(f)}\}.$$

Claim. $\Gamma = \Theta$.

Let $L \in \Gamma$. Since $End(L) \subset R(f)$ and $L = \varphi(cl(End(L)))$, then $L \in \Theta$.

On the other hand, let us assume $L \in \Theta$. Since $L = \varphi(A)$ for some $A \in 2^{R(f)}$, then $End(L) \subset A$. Thus, $L \in \Gamma$. We conclude that $\Gamma = \Theta$.

Now we prove the second part of the Proposition. Let $L \in \Gamma$.

The set of recurrent points $R(f)$ is strongly invariant under f. Since $f : D \to D$ is a homeomorphism, $f(End(L)) = End(f(L))$. Hence, $End(f(L)) \subset R(f)$. It follows that $f(L) \in \Gamma$. □

Proposition 7.7. *Let* $\Gamma = \{L \in C(D) : End(L) \subset R(f)\}$. *Then* $ent(C(f)|_\Gamma) = 0$.

Proof. By Proposition 7.5, there exists a homeomorphism $l : D \to D$ with the following three properties:

- For each $x \in R(f)$, $l(x) = f(x)$.
- The set of recurrent points of l is D, $R(l) = D$.
- By Corollary 7.2, $ent(C(l)) = 0$.

Let $L \in \Gamma$. Since $cl(End(L)) \subset R(f)$,

$$f(cl(End(L))) = l(cl(End(L))).$$

Now, using Corollary 4.2,

$$\begin{aligned} f(L) &= D_{\min}(f(cl(End(L)))) \\ &= D_{\min}(l(cl(End(L)))) \quad = \quad l(L). \end{aligned}$$

Therefore, $C(f)|_\Gamma = C(l)|_\Gamma$.

It follows that $ent(C(f)|_\Gamma) = ent(C(l)|_\Gamma) = ent(C(l)) = 0$. □

Proposition 7.8. *Let* $\Gamma = \{L \in C(D) : End(L) \subset R(f)\}$. *Let* $A \in C(D)$ *such that* $A \notin \Gamma$. *Then there exists* $B \in \Gamma$ *such that*

$$\lim_{n\to\infty} H(C(f)^n(B), C(f)^n(A)) = 0.$$

Proof. Let $A \in C(D) \setminus \Gamma$. Let $\Delta = \{U_1, U_2, U_3, \dots\}$ be the collection of all components of $D \setminus R(f)$.

Case 1. There exists $U \in \Delta$ such that $A \subset cl(U)$.

Let $k \in \mathbb{N}$ be the period U. Let $a, b \in D$ such that $cl(U) = [a, b]$, and for every $x \in U$,

$$\lim_{n\to\infty} f^{k\cdot n}(x) = a \quad \text{and} \quad \lim_{n\to-\infty} f^{k\cdot n}(x) = b.$$

Notice that A is a point or an arc contained in $[a, b]$. Also $[a, b] \in \Gamma$.

If $b \notin A$, then

$$\lim_{n\to\infty} H(C(f)^{k\cdot n}(\{a\}), C(f)^{k\cdot n}(A)) = 0.$$

Let $B = \{a\}$. Then $\lim_{n\to\infty} H(C(f)^n(B), C(f)^n(A)) = 0$.

If $b \in A$, then $A = [c, b]$ with $c \in (a, b)$. Let $B = [a, b]$. Then

$$\lim_{n\to\infty} H(C(f)^{k\cdot n}(B), C(f)^{k\cdot n}(A)) = 0.$$

And again, it follows that $\lim_{n\to\infty} H(C(f)^n(B), C(f)^n(A)) = 0$.

Case 2. For each $U \in \Delta$, A is not contained in $cl(U)$.

In this case, we have that $A \cap R(f) \neq \emptyset$. Let $L = D_{\min}(A \cap R(f))$. Hence, $L \subset A$ and $L \in \Gamma$.

Let Δ' be the collection of all elements U of Δ such that $A \cap U \neq \emptyset$ and $cl(U)$ is not contained in A.

$$\Delta' = \{W_1, W_2, \ldots\}, \quad W_i \cap W_j = \emptyset, \quad \text{if} \quad i \neq j.$$

From now on, we assume Δ' is denumerable. The proof of the finite case follows a similar argument.

For each $j \in \mathbb{N}$ consider $W_j \in \Delta'$. Let $x \in W_j$, let k_j be the period of W_j, and let

$$a_j = \lim_{n \to \infty} f^{k_j \cdot n}(x) \quad \text{and} \quad b_j = \lim_{n \to -\infty} f^{k_j \cdot n}(x).$$

Hence, $W_j = (a_j, b_j)$.

Let

- $L_j = [a_j, b_j]$, if $b_j \in L$.
- $L_j = \{a_j\}$, if $a_j \in L$.
- $B = L \cup (\cup_{j=1}^{\infty}) L_j$.

Note the following:

- For each $j \in \mathbb{N}$, $L_j \cap L \neq \emptyset$. Since $\lim_{j \to \infty} diam(L_j) = 0$, B is a closed subset of D.
- For each $m \in \mathbb{N}$, $B_m = L \cup (\cup_{j=1}^{m}) L_j$ is a dendrite. Furthermore, $B_m \in \Gamma$.
- Since $\lim B_m = B$, $B \in \Gamma$.

Claim. $\lim_{n \to \infty} H(C(f)^n(B), C(f)^n(A)) = 0$.

Let $\varepsilon > 0$. There exist only finitely many elements of Δ, say

$$\{U_{n_1}, U_{n_2}, \ldots, U_{n_s}\},$$

with diameter $\geq \varepsilon$.

Then there are finitely many elements of collection Δ', say

$$\{W_{m_1}, W_{m_2}, \ldots, W_{m_l}\},$$

where

$$o(W_{m_t}, C(f)) \cap \{U_{n_1}, U_{n_2}, \ldots, U_{n_s}\} \neq \emptyset, \quad 1 \leq t \leq l.$$

For each $1 \leq t \leq l$, $A_{m_t} = A \cap cl(W_{m_t})$ is an arc with this property:

$$\lim_{n \to \infty} H(C(f)^n(L_{m_t}), C(f)^n(A_{m_t})) = 0.$$

There exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and for very $1 \leq t \leq l$,

$$H(C(f)^n(L_{m_t}), C(f)^n(A_{m_t})) < \varepsilon.$$

Hence, for every $n \geq n_0$,

$$H(C(f)^n(L \cup (\cup_{t=1}^{l} L_{m_t})), C(f)^n(L \cup (\cup_{t=1}^{l} A_{m_t}))) < \varepsilon.$$

Now, let $W_m \in \Delta' \setminus \{W_{m_1}, W_{m_2}, \dots, W_{m_l}\}$. Note that for every $n \in \mathbb{N}$,
$$diam(C(f)^n(cl(W_m))) < \varepsilon.$$

Since both $A_m = A \cap cl(W_m)$ and the corresponding L_m are contained in $cl(W_m)$,
$$H(C(f)^n(L_m), C(f)^n(A_m)) < \varepsilon, \quad \text{for every} \quad n \in \mathbb{N}.$$

Since $B = L \cup (\cup_{j=1}^{\infty}) L_j$ and $A = L \cup (\cup_{j=1}^{\infty}) A_j$, $A_j = A \cap cl(W_j)$, we have that for every $n \geq n_0$, $H((C(f)^n(B), C(f)^n(A))) < \varepsilon$. □

The following result summarizes our work in this section.

Theorem 7.9. *Let $f : D \to D$ be a dendrite homeomorphism such that $R(f) \neq D$. If for each $x_0 \in D \setminus R(f)$,*
$$x_0 \in D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)),$$
then $ent(C(f)) = 0$.

Proof. Let $\Gamma = \{L \in C(D) : End(L) \subset R(f)\}$. Let $A \in C(D) \setminus \Gamma$.

By Proposition 7.8, there exists $B \in \Gamma$ such that
$$\lim_{n \to \infty} H(C(f)^n(B), C(f)^n(A)) = 0.$$
Hence, $\Gamma \cup o(A, C(f))$ is a closed subset of $C(D)$ invariant under $C(f)$.

By Lemma 5.5,
$$ent(C(f)|_{\Gamma \cup o(A, C(f))}) = ent(C(f)|_{\Gamma}) = 0.$$

Notice that
$$C(D) = \cup \{\Gamma \cup o(A, C(f)) : A \in C(D) \setminus \Gamma\}.$$
Then
$$ent(C(f)) = \sup\{ent(C(f)|_{\Gamma \cup o(A, C(f))}) : A \in C(D) \setminus \Gamma\} = 0. \quad □$$

8. Coda

Let D be a nondegenerate dendrite. Let $f : D \to D$ be a homeomorphism. Our goal in this short section is to show that conditions $End(D) \subset R(f)$ and $ent(C(f)) = 0$ are equivalent.

Proposition 8.1. *If for every $x \in D$, $x \in D_{\min}(\alpha(x, f) \cup \omega(x, f))$, then $End(D) \subset R(f)$.*

Proof. Let $x \in End(D)$. Since $x \in D_{\min}(\alpha(x, f) \cup \omega(x, f))$, there exist a and b in the union $\alpha(x, f) \cup \omega(x, f)$ such that $x \in [a, b]$.

It follows that $x = a$ or $x = b$. Thus, $x \in R(f)$. □

Proposition 8.2. *If $End(D) \subset R(f)$, then for every $x \in D$,*
$$x \in D_{\min}(\alpha(x, f) \cup \omega(x, f)).$$

Proof. Assume, to the contrary, that there exists $x_0 \in D$ such that

$$x_0 \notin D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)).$$

It follows that $x_0 \notin R(f)$ and $x_0 \notin End(D)$.

By Corollary 6.3, there exists $a \in D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f))$ such that

- $[x_0, a] \cap D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)) = \{a\}$, and
- for each pair $n, m \in \mathbb{Z}$, $n \neq m$,

$$[f^n(x_0), f^n(a)) \cap [f^m(x_0), f^m(a)) = \emptyset.$$

Let $u, v \in End(D)$ such that $x_0 \in [u, v]$. Then

$$[u, x_0] \cap [x_0, a] = \{x_0\} \quad \text{or} \quad [v, x_0] \cap [x_0, a] = \{x_0\}.$$

Let us assume that $[u, x_0] \cap [x_0, a] = \{x_0\}$. Then $[u, x_0] \cup [x_0, a] = [u, a]$ is an arc such that $x_0 \in (u, a)$. It follows that $u \notin D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f))$.

Since for each $n \in \mathbb{Z}$, $f^n(x_0) \notin D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f))$ and

$$f^n(D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f))) = D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)),$$

we have that

- $[u, a] \cap D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)) = \{a\}$, and
- for each pair $n, m \in \mathbb{Z}$, $n \neq m$,

$$[f^n(u), f^n(a)) \cap [f^m(u), f^m(a)) = \emptyset.$$

Therefore, $u \in R(f)$, $u \notin D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f))$ and

$$\alpha(u, f) \cup \omega(u, f) \subset D_{\min}(\alpha(x_0, f) \cup \omega(x_0, f)).$$

This is a contradiction. □

Corollary 8.3. *$End(D) \subset R(f)$ if and only if $ent(C(f)) = 0$.*

Proof. The result is an immediate consequence of theorems 6.7 and 7.9 and of propositions 8.1 and 8.2. □

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(Hernández) Departamento de Matemáticas; Facultad de Ciencias; UNAM; Ciudad Universitaria, C.P. 04510, Cd. Mx., MEXICO.

Email address: `paloma_hz@yahoo.com.mx`

(Méndez) Departamento de Matemáticas; Facultad de Ciencias; UNAM; Ciudad Universitaria, C.P. 04510, Cd. Mx., MEXICO.

Email address: `hml@ciencias.unam.mx`