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Aura Lucina Kantún-Montiel

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Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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CANONICAL PROJECTIONS OF LIE GROUPS AS EQUIVARIANT FIBRATIONS

AURA LUCINA KANTÚN-MONTIEL

Abstract. Given a Lie group G and its compact subgroup H, we consider G as an H-space endowed with the conjugation action and prove that the quotient projection $G \to G/H$ is an equivariant H-fibration. As a consequence, every G-map $E \to G/H$ is a G-fibration.

1. Introduction

In equivariant homotopy theory, G-fibrations (the equivariant version of a Hurewicz fibration) play such an important role as Hurewicz fibrations do in usual homotopy theory.

Generally speaking, equivariant homotopy theory is well developed for the case when the acting group G is a compact Lie group. For example, one of the notable results is that if H is a closed subgroup of a compact Lie group G, then every G-map $p : E \to G/H$ is a G-fibration for any G-space E (see [12, p. 53]).

A natural question is whether this result remains valid when the acting group G is non-compact.

In [8, Theorem 5.1], it is shown that the projection $G/K \to G/H$ is a G-fibration provided that G is a compact (not necessarily Lie) group, K and H are closed subgroups of G such that $K \subset H$, and G/K is metrizable. Furthermore, in [6, Corollary 6.5], it is proved that if G is a compact metrizable group and H is its closed subgroup, then any G-map $E \to G/H$ is a G-fibration.

In this paper, the above-mentioned results are extended to the case of non-compact Lie groups. Our main results are theorems 4.2 and 4.4.

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2. Notation and Preliminaries

By the letter G, we will denote a locally compact Hausdorff topological group.

By a *G-space*, we mean a topological space X, together with a fixed continuous action $(g, x) \mapsto gx$ of G on X.

Let X be a G-space. For $x \in X$, the subgroup $G_x = \{g \in G \mid gx = x\}$ is called the *isotropy group* at x and the G-space $G(x) = \{gx \mid g \in G\}$ is called the *G-orbit* of x.

A subset $A \subset X$ of a G-space X is called a *G-invariant* or just *invariant* if $G(a) \subset A$ for every $a \in A$.

Let X and Y be G-spaces. A continuous map $f : X \to Y$ is called a *G-map* or *equivariant map* if $f(gx) = gf(x)$ for every $(g, x) \in G \times X$. If a G-map $f : X \to Y$ is a homeomorphism, then it is called a *G-homeomorphism*, and we say that X and Y are *G-homeomorphic* whenever there exists a G-homeomorphism between them.

A homotopy $F : X \times I \to Y$, where $I = [0, 1]$, is called a *G-homotopy* if it is a G-map where $X \times I$ is a G-space with the diagonal action $g(x, t) = (gx, t)$.

For a topological group G and its closed subgroup H, we will denote $G/H = \{gH \mid g \in G\}$ the coset space, endowed with the action of G defined by left translations, i.e., $g \cdot g'H = gg'H$ for every $g \in G$ and $g'H \in G/H$.

Proposition 2.1 ([4, Ch. I, Proposition 4.1])**.** *Let G be a compact group, let X be a Hausdorff G-space, and let $x \in X$. Then the G-map $G/G_x \to G(x)$, $gG_x \mapsto gx$ is a G-homeomorphism, and hence, G/G_x and $G(x)$ are G-homeomorphic.*

A *G-ANR-space* is the equivariant version of an absolute neighborhood retract, that is, a G-space X is G-ANR if for any metrizable G-space Y and any equivariant closed embedding $i : X \hookrightarrow Y$, there is an invariant neighborhood U of $i(X)$ in Y such that $i(X)$ is a G-retract of U.

The following significant result, in particular, shows that if G is a compact Lie group, then each one of the orbits of any G-space is a G-ANR.

Proposition 2.2 ([10, Corollary 1.6.7])**.** *Let H be a closed subgroup of a compact Lie group G. Then G/H is a G-ANR.*

More results and all the basic notions of equivariant topology can be found in [4] and [12]. For the equivariant theory of retracts we refer the reader to [1], [2], and [3].

3. G-Fibrations

By a *G-fibration*, we mean a natural equivariant version of a Hurewicz fibration: a G-map $p : E \to B$ is called a G-fibration if it has the equivariant homotopy lifting property (EHLP) with respect to every G-space X (see [12, p. 53]). That is to say, for every G-space X, every G-map $f : X \to E$, and every G-homotopy $F : X \times I \to B$ such that $F(x,0) = p \circ f(x)$, $x \in X$, there exists a G-homotopy $\widetilde{F} : X \times I \to E$ as a *filler* of the following diagram (i.e., $\widetilde{F} \circ \partial_0 = f$ and $p \circ \widetilde{F} = F$):

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & E \\ {\scriptstyle \partial_0}\big\downarrow & \overset{\widetilde{F}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[F]{} & B \end{array}$$

where $\partial_0(x) = (x, 0)$.

Clearly, every G-space can be considered as an H-space for any subgroup $H \subset G$. That is, we have a restriction functor from the category of G-spaces to the one of H-spaces. This functor preserves equivariant fibrations.

Proposition 3.1 ([9, Proposition 2.2]). *Let H be a closed subgroup of a compact group G. If $p : E \to B$ is a G-fibration, then p is an H-fibration.*

In the case of compact Lie group actions, G-fibrations appear in the following natural way.

Proposition 3.2 ([8, Proposition 3.1]). *Let H be a closed subgroup of a compact Lie group G and let E be a metrizable G-space. Then every G-map $E \to G/H$ is a G-fibration.*

Also, we are going to need another kind of G-fibration with stronger conditions.

A G-map $p : E \to B$ of metrizable G-spaces will be called a *regular G-fibration* if for any closed G-subset A of a metrizable G-space X and any diagram of G-maps

$$\begin{array}{ccc} X \times \{0\} \cup A \times I & \xrightarrow{\ f\ } & E \\ \big\downarrow & \overset{\widetilde{F}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[F]{} & B \end{array}$$

there exists a G-homotopy $\widetilde{F} : X \times I \to E$ as a filler.

It is not difficult to see that every regular G-fibration is a G-fibration, taking $A = \emptyset$ in the definition of regular G-fibrations and recalling that any G-map between metrizable G-spaces, having the EHLP with respect to metrizable G-spaces, also has the EHLP with respect to all G-spaces (see [8, Remark A.1]).

Proposition 3.3 ([7, Theorem 3.4])**.** *Let G be a compact group and $p : E \to B$ a G-map of metrizable G-spaces. If E and B are G-ANR-spaces and p is a G-fibration, then p is a regular G-fibration.*

Regular G-fibrations are locally characterized in the following way.

Proposition 3.4 ([8, Proposition 3.6])**.** *Let G be a compact group and $p : E \to B$ be a G-map of metrizable G-spaces. Suppose that there is a covering $\mathcal{U}$ of B by open G-invariant sets such that the restriction of p, $p^{-1}(U) \to U$ is a regular G-fibration for each $U \in \mathcal{U}$. Then p is a regular G-fibration.*

4. G-Fibrations by Conjugation

One of the important examples of G-fibrations is the canonical projections of topological groups onto their coset spaces.

Proposition 4.1. *Let H be a closed subgroup of a locally compact group G. Then the projection $\pi : G \to G/H$, $g \mapsto gH$, is a G-fibration where both spaces are endowed with the actions defined by left translations.*

Proof. The proof is essentially the same as in [5, Proposition 2.2]. In the proof of that proposition, in order to guarantee that the projection $G \to G/H$ is a (non-equivariant) fibration, it is required that the subgroup H be a compact Lie group. However, we observe that it is not necessary to assume that H is a Lie group; instead, we can apply E. G. Skljarenko's result [11, Theorem 15] which claims that the projection $G \to G/H$ is a non-equivariant fibration for every closed subgroup H of a locally compact group G. Further, continuing exactly in the same way as in [5, Proposition 2.2], we get the desired result. □

The main object of this paper is the quotient projection $q : G \to G/H$ where H is a compact subgroup of a Lie group G. Here, G is considered as an H-space endowed with the conjugation action, i.e., $h * g = hgh^{-1}$ for every $h \in H$ and $g \in G$. At the same time, we let H act on G/H by the rule $h \cdot gH = hgH$. Below, we will show that the projection $q : G \to G/H$ is an equivariant H-fibration. This is going to be a consequence of our main result.

Theorem 4.2. *Let H be a compact subgroup of a Lie group G and let $\Gamma = H \times H$. If Γ acts on G by conjugation, $(h_1, h_2) * g = h_1 g h_2^{-1}$, and on G/H by $(h_1, h_2) \cdot gH = h_1 gH$, then the projection $q : G \to G/H$, $g \mapsto gH$, is a regular Γ-fibration.*

Proof. It is easily seen that q is a Γ-map, because $q((h_1, h_2) * g) = q(h_1 g h_2^{-1}) = h_1 g h_2^{-1} H = h_1 gH = (h_1, h_2) \cdot gH = (h_1, h_2) \cdot q(g)$.

We are going to apply Proposition 3.4. Let $g_0 H \in G/H$ and note that $q^{-1}(\Gamma(g_0 H)) = \Gamma(g_0)$.

Since Γ is a compact Lie group, by Proposition 2.2, there is a Γ-invariant neighborhood U of $\Gamma/\Gamma_{g_0} \cong \Gamma(g_0)$ in G and a Γ-retraction $r : U \to \Gamma(g_0)$. Let $V = q(U)$, then $U = q^{-1}(V)$, and we have the following commutative diagram of Γ-maps

$$\begin{array}{ccc} U & \xrightarrow{\;r\;} & \Gamma(g_0) \\ {\scriptstyle q|_U}\downarrow & & \downarrow{\scriptstyle q|_{\Gamma(g_0)}} \\ V & \xrightarrow[\;r_H\;]{} & \Gamma(g_0 H) \end{array} \tag{4.1}$$

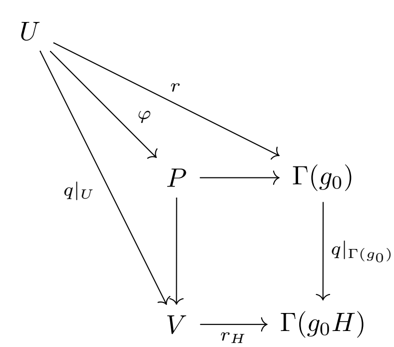
where $r_H(gH) = r(g)H$ for every $gH \in V$.

Now we will see that, in fact, this is a pull-back diagram.

Let P be the pull-back of $V \xrightarrow{r_H} \Gamma(g_0 H) \xleftarrow{q|_{\Gamma(g_0)}} \Gamma(g_0)$. This is

$$\begin{aligned} P &= \{(gH, \gamma(g_0)) \in V \times \Gamma(g_0) \mid r_H(gH) = q(\gamma(g_0))\} \\ &= \{(gH, \gamma(g_0)) \mid r(g)H = \gamma(g_0)H\}. \end{aligned}$$

Then there is a unique Γ-map $\varphi : U \to P$ such that the following diagram commutes



and φ is defined by $\varphi(g) = (q(g), r(g))$ for $g \in U$.

If $\varphi(g) = \varphi(g')$, that is, $(q(g), r(g)) = (q(g'), r(g'))$, then $gH = g'H$ and $r(g) = r(g')$. So there is an $h \in H$ such that $g' = gh$ and $r(g) = r(g') = r(gh) = r((e, h^{-1}) * g) = (e, h^{-1}) * r(g) = r(g)h$, which means that $h = e$, and hence, $g' = g$. Therefore, φ is injective.

If $(gH, \gamma(g_0)) \in P$, then $(r(g))^{-1}\gamma(g_0) \in H$ implying $g(r(g))^{-1}\gamma(g_0) \in U$, and therefore, φ is surjective.

We claim that $\varphi(g(r(g))^{-1}\gamma(g_0)) = (gH, \gamma(g_0))$. Indeed,

$$\begin{aligned}\varphi(g(r(g))^{-1}\gamma(g_0)) &= (q(g(r(g))^{-1}\gamma(g_0)), r(g(r(g))^{-1}\gamma(g_0))) \\ &= (q((e, ((r(g))^{-1}\gamma(g_0))^{-1}) * g), r((e, ((r(g))^{-1}\gamma(g_0))^{-1}) * g)) \\ &= ((e, ((r(g))^{-1}\gamma(g_0))^{-1}) \cdot q(g), (e, ((r(g))^{-1}\gamma(g_0))^{-1}) * r(g)) \\ &= (q(g), r(g)(r(g))^{-1}\gamma(g_0)) = (gH, \gamma(g_0)).\end{aligned}$$

Now, let us define $\bar{\varphi} : P \to U$ by $\bar{\varphi}(gH, \gamma(g_0)) = g(r(g))^{-1}\gamma(g_0)$ for $(gH, \gamma(g_0)) \in P$.

Suppose $(g'H, \gamma'(g_0)) = (gH, \gamma(g_0)) \in P$. Then $\gamma' = \gamma$ and there is an $h \in H$ such that $g' = gh$. So

$$\begin{aligned}\bar{\varphi}(g'H, \gamma'(g_0)) &= g'(r(g'))^{-1}\gamma'(g_0) \\ &= gh(r(gh))^{-1}\gamma(g_0) \\ &= gh(r((e, h^{-1}) * g))^{-1}\gamma(g_0) \\ &= gh((e, h^{-1}) * r(g))^{-1}\gamma(g_0) \\ &= gh(r(g)h)^{-1}\gamma(g_0) \\ &= ghh^{-1}(r(g))^{-1}\gamma(g_0) \\ &= g(r(g))^{-1}\gamma(g_0) \\ &= \bar{\varphi}(gH, \gamma(g_0)).\end{aligned}$$

Let $g \in U$, then

$$\begin{aligned}\bar{\varphi} \circ \varphi(g) &= \bar{\varphi}(q(g), r(g)) = \bar{\varphi}(gH, r(g)) \\ &= g(r(g))^{-1}r(g) = g.\end{aligned}$$

Now, let $(gH, \gamma(g_0)) \in P$, then

$$\begin{aligned}\varphi \circ \bar{\varphi}(gH, \gamma(g_0)) &= \varphi(g(r(g))^{-1}\gamma(g_0)) \\ &= (q(g(r(g))^{-1}\gamma(g_0)), r(g(r(g))^{-1}\gamma(g_0))).\end{aligned}$$

Since $(r(g))^{-1}\gamma(g_0) \in H$, we see that $q(g(r(g))^{-1}\gamma(g_0)) = gH$. And we have

$$\begin{aligned}r(g(r(g))^{-1}\gamma(g_0)) &= r((e, (\gamma(g_0))^{-1}r(g)) * g) \\ &= (e, (\gamma(g_0))^{-1}r(g)) * r(g) \\ &= r(g)(r(g))^{-1}\gamma(g_0) = \gamma(g_0).\end{aligned}$$

Hence, $\varphi \circ \bar{\varphi}(gH, \gamma(g_0)) = (gH, \gamma(g_0))$, as required. Consequently, $\bar{\varphi}$ is a well-defined, continuous inverse map of φ. Therefore, φ is a Γ-homeomorphism, and we conclude that (4.1) is a pull-back diagram.

By Proposition 2.1, $q|_{\Gamma(g_0)}$ is equivalent to $\Gamma/\Gamma_{g_0} \to \Gamma/\Gamma_{g_0 H}$, which, by propositions 3.2 and 2.2, is a Γ-fibration of Γ-ANRs. Then, by Proposition 3.3, $q|_{\Gamma(g_0)}$ is a regular Γ-fibration and, since (4.1) is a pull-back diagram, so is $q|_U$.

Finally, due to the arbitrary choice of $g_0 H$, applying Proposition 3.4, we conclude that q is a regular Γ-fibration. □

Corollary 4.3. *Let H be a compact subgroup of a Lie group G. If G is considered as an H-space by conjugation with the action $h * g = hgh^{-1}$, and H acts on G/H by $h \cdot gH = hgH$, then the projection $q : G \to G/H$, $g \mapsto gH$, is an H-fibration.*

Proof. H is isomorphic to $\Delta = \{(h, h) \mid h \in H\} < \Gamma = H \times H$. Since by Theorem 4.2, q is a regular Γ-fibration, we infer that it is also a Γ-fibration, and by Proposition 3.1, that it is a Δ-fibration. This means that $q : G \to G/H$ is an H-fibration by conjugation. □

Generalizing Proposition 3.2, we are going to show below that every G-map $E \to G/H$ is a G-fibration for arbitrary Lie group actions even for a G-space E which is not necessarily metrizable.

Theorem 4.4. *Let H be a compact subgroup of a Lie group G and let E be a G-space. Then any G-map $p : E \to G/H$ is a G-fibration.*

Proof. Suppose that the following commutative diagram of G-maps is given:

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & E \\ {\scriptstyle \partial_0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[F]{} & G/H \end{array}$$

Let $S = f^{-1}p^{-1}(eH)$, then $X = GS$.

Now consider the commutative diagram of H-maps

$$\begin{array}{ccc} S & \xrightarrow{\;c_e\;} & G \\ {\scriptstyle \partial_0}\big\downarrow & \overset{\varphi}{\nearrow} & \big\downarrow{\scriptstyle q} \\ S \times I & \xrightarrow[F]{} & G/H \end{array}$$

where G is considered as an H-space with the action $h * g = hgh^{-1}$ and c_e is the constant H-map $s \mapsto e$ for all $s \in S$.

Since by Corollary 4.3, $q : G \to G/H$ is an H-fibration, there is an H-map $\varphi : S \times I \to G$ such that $\varphi \circ \partial_0(s) = \varphi(s,0) = c_e(s) = e$ and $q \circ \varphi(s,t) = F(s,t)$, where $\partial_0(s) = (s,0)$.

Note that $p(gf(s)) = gp(f(s)) = gH = q(g)$ for each $g \in G$ and $s \in S$.

Define $\widetilde{F} : X \times I \to E$ as $\widetilde{F}(x) = g\varphi(s,t)f(s)$, for $x = gs \in X$. Then for each $x = gs \in X$ and $t \in I$, we have

$$\widetilde{F}(x,0) = \widetilde{F}(gs,0) = g\varphi(s,0)f(s) = gf(s) = f(gs) = f(x),$$

and

$$\begin{aligned} p\widetilde{F}(x,t) &= p(g\varphi(s,t)f(s)) = gp(\varphi(s,t)f(s)) \\ &= gq(\varphi(s,t)) = gF(s,t) = F(gs,t) \\ &= F(x,t). \end{aligned}$$

Clearly, $\widetilde{F}$ is a G-map; therefore, p is a G-fibration. □

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Universidad del Papaloapan; Av. Ferrocarril SN; Ciudad Universitaria; CP 68400, Loma Bonita; Oaxaca, México.

Email address: alkantun@unpa.edu.mx