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A BANACH-STONE TYPE THEOREM AND TOPOLOGICAL DIMENSION

KAZUHIRO KAWAMURA

Abstract. We prove a Banach-Stone type theorem for linear isometries of vector-valued continuous function spaces in connection with topological dimensions of underlying spaces, generalizing a previous result of the author. Some related problems are discussed.

1. Introduction and Preliminaries

This paper is a continuation of [9] and studies linear isometries between vector-valued continuous function spaces. The classical Banach-Stone theorem states that every surjective linear isometry between the Banach spaces of all complex-valued continuous functions on compact Hausdorff spaces is a unimodular weighted composition operator (see [5], [6] and [8] for background). Seeking for analogous theorems for isometries between vector-valued continuous function spaces and being motivated by [1], [2], [7] and [15], we ask the following question. For a compact Hausdorff space X and a Banach space E, $C(X, E)$ denotes the Banach space of all E-valued continuous functions on X with the sup norm $\|f\|_\infty = \sup_{x\in X} \|f(x)\|$. Undefined notation will be explained later.

Question 1.1. Let X and Y be compact Hausdorff spaces, let E be a real or complex Banach space and let A and B be linear subspaces of $C(X, E)$ and $C(Y, E)$ respectively. Find a set of conditions on X, Y, A, B and E which implies that every surjective linear isometry $T : A \to B$

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between A and B is a *generalized weighted composition operator* in the sense that there exists a homeomorphism $\varphi : \mathrm{Ch}_E(B) \to \mathrm{Ch}_E(A)$ between the Chouqet boundaries $\mathrm{Ch}_E(B)$ and $\mathrm{Ch}_E(A)$, and a family of linear isometries $(V_y : E \to E)_{y \in \mathrm{Ch}_E(B)}$ on E such that

$$Tf(y) = V_y(f(\varphi(y))), \ f \in A, \ y \in \mathrm{Ch}_E(B), \tag{1.1}$$

and the map $\mathrm{Ch}_E(B) \to U(E); y \mapsto V_y$ is continuous with respect to the strong operator topology on the space $U(E)$ of all linear isometries on E.

Where the *Choquet boundary* $\mathrm{Ch}_E(A)$ of A is defined ([1], [5], [15]) by:

$$\mathrm{Ch}_E(A) = \{x \in X \mid \xi \circ \delta_x \in \mathrm{ext}(A^*) \text{ for some } \xi \in \mathrm{ext}(E^*)\}.$$

The space $\mathrm{Ch}_{\mathbb{C}}(A)$ coincides with the usual Choquet boundary. We assume that A and B satisfy the following conditions:

- (S1) For each $u \in E$, the constant map $c_u : X \to E$ taking value u belongs to A, and
- (S2) For each pair of distinct points x_1 and x_2 of X and for each $u \in E$, there exists a function $f \in A$ such that $f(x_1) = u$ and $f(x_2) = 0$,

and the same for B. Also we consider the following condition ([9]):

- (ext) For each $x \in \mathrm{Ch}_E(A)$, we have $\xi \circ \delta_x \in \mathrm{ext}(A^*)$ for each $\xi \in \mathrm{ext}(E^*)$, and the same for B as well.

With this setup we give an answer to the above question as follows, generalizing [9, Theorem 4.6]. For a separable metrizable space X, $\dim X$ denotes the topological dimension of X ([4]).

Theorem 1.2. *Let X, Y be compact metrizable spaces and let E be a Hilbert space. Let $A \subset C(X,E), B \subset C(Y,E)$ be linear subspaces of $C(X,E)$ and $C(Y,E)$ respectively satisfying the conditions (S1),(S2) and (ext). If either*

1. $\dim_{\mathbb{R}} E = 1$, *or*
2. $2\dim X + 1 \leq \dim_{\mathbb{R}} E < \infty$, *or*
3. $\dim X < \dim_{\mathbb{R}} E = \infty$,

then every surjective linear isometry $T : A \to B$ between A and B is a generalized weighted composition operator.

A few remarks are in order. First, the statement that both (S1) and (S2) hold for A and B indicates that these spaces contain "enough functions." It is a vector valued-function space-analogue of the requirement for scalar-valued function spaces in the sense of [12]. While by [9, Lemma 2.5], a subspace A of $C(X,E)$ satisfies the condition (ext) when, for example,

($*$) E is a Hilbert space and a subgroup $\Gamma \subset U(E)$ acts transitively on $S(E)$ so that A is a Γ-invariant subspace. In particular $E = \mathbb{C}$ and A is a complex-linear subspace.

See also [9, Proposition 2.4] for other linear subspaces satisfying the condition (ext).

Secondly, the above theorem does not apply to E and X if

$$(\dim_{\mathbb{R}} E, \dim X) = (2, 1), (3, 2).$$

We prove in section 2 that the same conclusion as Theorem 1.2 holds if

(i) $\dim_{\mathbb{R}} E = 2$ and X contains no simple closed curves, or
(ii) $\dim_{\mathbb{R}} E = 3$ and X contains no topological copies of the 2-sphere and the projective plane.

In particular when X and Y are connected closed surfaces other than the 2-sphere and the projective plane, we have the same conclusion as that of Theorem 1.2. The case (i) above is a natural real-linear analogue of [14, Lemma 5]. See also [11]. This leads to a natural question whether the same holds if the underlying space is the 2-sphere or the projective plane. We adopt the idea of [12] to construct a linear subspace $A \subset C(M, \mathbb{R}^3)$, M being the 2-sphere or the projective plane, with $\mathrm{Ch}_{\mathbb{R}^3}(A) = M$, and a linear isometry $S : A \to A$ which is not a generalized weighted composition operator. Our subspace A satisfies neither (S1), (S2) nor ($*$) and it is still an open question whether such an isometry exists on a subspace satisfying (S1), (S2) and (ext).

In the rest of this section we fix notation and recall some basic facts. For a real or complex Banach space E, the closed unit ball and the unit sphere are denoted by $B(E)$ and $S(E)$ respectively. For a subspace M of E, M^* denotes the dual space of M with the operator norm. An *extreme point* of a convex set C of a linear space L is a point $\xi \in C$ with the property that the equality $\xi = \frac{\eta+\zeta}{2}$ with $\eta, \zeta \in C$ implies $\eta = \zeta = \xi$. The set of all extreme points of C is denoted by $\mathrm{ext}(C)$. By abuse of notation, the extreme points of the closed unit ball of a Banach space E is simply denoted by $\mathrm{ext}(E)$. Under this notation, a Banach space E is strictly convex if and only if $\mathrm{ext}(E) = S(E)$ (see [13]). The space of all linear isometries on E is denoted by $U(E)$ and is assumed to be endowed with the strong operator topology. For a linear subspace M of $C(X, E)$ and a point $x \in X$, let $\delta_x : M \to E$ be the evaluation map defined by $\delta_x(f) = f(x)$, $f \in M$. For a Banach space E, $\dim_{\mathbb{R}} E$ denotes the dimension of E as a $\mathbb{R}$-linear space, while for a topological space X, $\dim X$ denotes the topological dimension of X (see [4]).

We say that a collection $\mathcal{D}$ of nonempty closed subsets of a compact Hausdorff space X is an *upper semi-continuous decomposition* of X if

(d1) $\cup_{D\in\mathcal{D}} D = X$ and $D_1 \cap D_2 = \emptyset$ for each $D_1, D_2 \in \mathcal{D}$ with $D_1 \neq D_2$,

(d2) For each $D \in \mathcal{D}$ and for each open set U of X with $D \subset U$, there exists an open set V with $D \subset V \subset U$ such that

$$\cup\{D' \in \mathcal{D} \mid D' \cap V \neq \emptyset\} \subset U.$$

Let $X/\mathcal{D}$ be the space obtained from X by shrinking each element of $\mathcal{D}$ into a point, endowed with the quotient topology, and let $p : X \to X/\mathcal{D}$ be the projection. The map p is continuous and the space $X/\mathcal{D}$ is compact Hausdorff by virtue of (d2) ([3]). The space $X/\mathcal{D}$ is called the *decomposition space* of X by the decomposition $\mathcal{D}$. For a continuous surjection $f : X \to Y$ between compact metrizable spaces X and Y, the collection $\mathcal{D}_f = \{f^{-1}(y) \mid y \in Y\}$ is an upper semi-continuous decomposition of X such that the decomposition space $X/\mathcal{D}_f$ is homeomorphic to Y.

Throughout the n-dimensional sphere and real-projective space are denoted by S^n and P^n respectively.

2. Proofs

The next theorem is a consequence of [9, Theorem 3.1] and supplies a framework to obtain a Banach-Stone type theorem in Hilbert-space-valued continuous function spaces. Let E be a real or complex Hilbert space and observe that $\mathrm{ext}(E^*) = S(E^*)$ by the strict convexity of $E^* \cong E$. Let $A \subset C(X, E)$ be a subspace satisfying (S1), (S2) and (ext), and define $\Phi_A : S(E^*) \times \mathrm{Ch}_E(A) \to \mathrm{ext}(A^*)$ by

$$\Phi_A(\xi, x) = \xi \circ \delta_x,\ \xi \in S(E^*),\ x \in \mathrm{Ch}_E(A).$$

By the conditions (S1), (S2) and (ext), the map Φ_A is a bijection and further is a homeomorphism [9, Lemma 2.2]. For a surjective linear isometry $T : A \to B$, let $\tau_T : S(E^*) \times \mathrm{Ch}_E(B) \to S(E^*) \times \mathrm{Ch}_E(A)$ be the map defined by

$$\tau_T = \Phi_A^{-1} \circ T^* \circ \Phi_B$$

written by

$$\tau_T(\eta, y) = (\alpha_y(\eta), \varphi_\eta(y)),\ \eta \in S(E^*),\ y \in \mathrm{Ch}_E(B), \tag{2.1}$$

where $\alpha_y : S(E^*) \to S(E^*)$ and $\varphi_\eta : \mathrm{Ch}_E(B) \to \mathrm{Ch}_E(A)$ are continuous maps. In what follows $\mathrm{Map}(M, N)$ denotes the set of continuous maps from a space M to a space N.

Theorem 2.1. *(a consequence of* [9, Theorem 3.1]*) Let X, Y be compact Hausdorff spaces and let E be a Hilbert space. Also let A and B be subspaces of $C(X, E)$ and $C(Y, E)$ respectively satisfying the conditions (S1), (S2) and (ext). For a surjective linear isometry $T : A \to B$, we have the following.*

(1) *For an arbitrary* $y \in \mathrm{Ch}_E(B)$*, let* $V_y : E \to E$ *be the linear operator defined by*

$$V_y(u) = Tc_u(y), \ u \in E. \tag{2.2}$$

Then we have $\alpha_y = V_y^*|S(E^*)$*.*

(2) *Let* $\phi_T : S(E^*) \to \mathrm{Map}(\mathrm{Ch}_E(B), \mathrm{Ch}_E(A))$ *be the map defined by*

$$(\phi_T(\eta))(y) = \varphi_\eta(y), \ \eta \in S(E^*), \ y \in \mathrm{Ch}_E(B). \tag{2.3}$$

Suppose that

(stab) ϕ_T *is a constant map*

and let $\phi_T(\eta) \equiv \varphi$ *for each* $\eta \in S(E^*)$*. Then* $\varphi : \mathrm{Ch}_E(B) \to \mathrm{Ch}_E(A)$ *is a homeomorphism and* $V_y : E \to E$ *is a linear isometry for each* $y \in \mathrm{Ch}_E(B)$ *such that*

$$Tf(y) = V_y(f(\varphi_*(y))), \ y \in \mathrm{Ch}_E(B), \ f \in A. \tag{2.4}$$

(3) *The map* $\mathrm{Ch}_E(B) \to U(E); y \mapsto V_y$ *is continuous with respect to the strong operator topology on* $U(E)$*.*

Thus the detection of generalized weighted composition operators among a class of isometries is reduced to decide whether the map ϕ_T defined above satisfies the condition (stab). Much of the previous research assumes strong separation conditions on function spaces to derive that the map ϕ_T satisfies (stab), while what is shown in [14, Lemma 5] is rephrased that the condition (stab) is satisfied if $\mathrm{Ch}_{\mathbb{C}}(B)$ contains no simple closed curves (see also [11]). Our proof of Theorem 1.2 also comprises verifying the condition (stab) under appropriate assumptions.

In what follows we keep the notation of Theorem 2.1: X, Y are compact Hausdorff spaces, E is a Hilbert space, $A \subset C(X, E)$ and $B \subset C(Y, E)$ are linear subspaces satisfying the conditions (S1), (S2) and (ext), and $T : A \to B$ is a surjective linear isometry.

For each $y \in \mathrm{Ch}_E(B)$, let $\phi_y : S(E^*) \to \mathrm{Ch}_E(A)$ be the map defined by

$$\phi_y(\eta) = (\phi_T(\eta))(y)(= \varphi_\eta(y)), \ \eta \in \mathrm{ext}(E^*). \tag{2.5}$$

For $\eta_1, \dots, \eta_r \in S(E^*)$, let $H(\eta_1, \dots, \eta_r)$ be the linear subspace of E^* spanned by $\{\eta_1, \dots, \eta_r\}$ and define the subset $S(\eta_1, \dots, \eta_r)$ of $S(E^*)$ by $S(\eta_1, \dots, \eta_r) = H(\eta_1, \dots, \eta_r) \cap S(E^*)$.

Lemma 2.2. [9, Lemma 4.1] *For an arbitrary point* $x \in \phi_y(S(E^*))$ *and for each finite subset* $\{\eta_1, \dots, \eta_r\} \subset \phi_y^{-1}(x)$*, we have*

$$S(\eta_1, \dots, \eta_r) \subset \phi_y^{-1}(x).$$

In particular, $\{\eta, -\eta\} \subset \phi_y^{-1}(x)$ *for each* $\eta \in \phi_y^{-1}(x)$*.*

Let L be a subspace of E^* and notice that $S(L) = L \cap S(E^*)$. Also define the map $\phi_{y,L}$ by $\phi_{y,L} = \phi_y|S(L) : S(L) \to \mathrm{Ch}_E(A)$. For a point $x \in \mathrm{Ch}_E(A)$, let $H_{x,L}$ be the subspace of E^* spanned by the set $\phi_{y,L}^{-1}(x)$. In particular, H_{x,E^*} is simply denoted by H_x.

Lemma 2.3. [9, Proposition 4.2, Proof of Lemma 4.3]

(1) *For each* $x \in \phi_{y,L}(S(L))$, *we have*

$$\phi_{y,L}^{-1}(x) = H_x \cap S(L).$$

(2) *Assume that* $\dim_{\mathbb{R}} L < \infty$. *For each pair* x_1, x_2 *of distinct points of* $\phi_{y,L}^{-1}(S(L))$, *we have*

$$\dim_{\mathbb{R}} H_{x_1,L} + \dim_{\mathbb{R}} H_{x_2,L} \leq \dim_{\mathbb{R}} L.$$

As [9, Theorem 4.6], we apply the Hurewicz Theorem for the proof of Theorem 1.2.

Theorem 2.4. *(Hurewicz,* [4, Theorem 1.12.2]*) Let* $f : M \to N$ *be a closed map between separable metrizable spaces* M *and* N. *Then we have*

$$\dim M \leq \dim N + \sup_{q \in N} \dim f^{-1}(q).$$

Proof of Theorem 1.2. The conclusion under the hypothesis (1) has been proved in [9, Theorem 4.5]. Let $m = \dim_{\mathbb{R}} E \geq 2$ and $k = \dim X$.

First we assume that $2k + 1 \leq m < \infty$ and take an arbitrary point $y \in \mathrm{Ch}_E(B)$. We show

Claim. There exists a point $x_0 \in \phi_y(S(E^*))$ such that

$$\dim_{\mathbb{R}} H_{x_0} \geq m - k.$$

Suppose that $\dim_{\mathbb{R}} H_x < m - k$ for each $x \in \phi_y(S(E^*))$. Then we have $\dim \phi_y^{-1}(x) = \dim(H_x \cap S(E^*)) < m - k - 1$ and by Theorem 2.4 we have a contradiction as follows.

$$\begin{aligned} m - 1 &= \dim S(E^*) \\ &\leq \dim \mathrm{Ch}_E(A) + \sup_{x \in \phi_y(S(E^*))} \dim \phi_y^{-1}(x) \\ &< k + (m - k - 1) = m - 1. \end{aligned}$$

This proves Claim. Suppose that ϕ_y is not a constant map. Then for each $x \in \phi_y(S(E^*)) \setminus \{x_0\}$ we have $\dim_{\mathbb{R}} H_x \leq m - (m - k) = k$ by Lemma 2.3 (2) and hence

$$\dim \phi_y^{-1}(x) \leq k - 1, \ x \in \phi_y(S(E^*)) \setminus \{x_0\}. \tag{2.6}$$

The space $S(E^*) \setminus \phi_y^{-1}(x_0)$, as an open subset of the sphere $S(E^*)$, has topological dimension $(m-1)$. Using this and noticing that the restriction $\phi_y| : S(E^*) \setminus \phi_y^{-1}(x_0) \to \phi_y(S(E^*)) \setminus \{x_0\}$ is a closed map, we can apply Theorem 2.4 to obtain

$$\begin{aligned} m-1 &= \dim(S(E^*) \setminus \phi_y^{-1}(x_0)) \\ &\leq \dim \mathrm{Ch}_E(A) + \sup_{x \neq x_0} \dim \phi_y^{-1}(x) \\ &\leq k + k - 1 = 2k - 1, \end{aligned}$$

which contradicts the hypothesis $m \geq 2k+1$. Thus we see ϕ_y is a constant map for each $y \in \phi_y(S(E^*))$ and hence the condition (stab) is verified.

When $\dim_{\mathbb{R}} E = \infty > \dim X$, take an arbitrary finite dimensional subspace L of E^* such that $\dim_{\mathbb{R}} L \geq 2\dim X + 1$. The previous argument shows that $\phi_y|L$ is constant map. Since L is arbitrarily chosen under the dimension constraint $\dim_{\mathbb{R}} L \geq 2\dim X + 1$, we see that ϕ_y itself is a constant map. Hence the condition (stab) is verified and Theorem 2.1 finishes the proof of Theorem 1.2.

The case $\dim_{\mathbb{R}} E = 2$ or 3 is dealt with in the next theorem which is a higher dimensional analogue of [14, Lemma 5]. Let $S^m = \{x \in \mathbb{R}^{m+1} \mid |x| = 1\}$, the unit sphere of the Euclidean space $\mathbb{R}^{m+1}$. For a linear subspace H of $\mathbb{R}^m$, let $D_H = H \cap S^m$. We consider a collection $\mathcal{H}$ of linear subspaces of $\mathbb{R}^{m+1}$ and the collection $\mathcal{D}_{\mathcal{H}} = \{D_H \mid H \in \mathcal{H}\}$ associated with $\mathcal{H}$ such that

(h1) $1 \leq \dim H \leq m$,
(h2) $H_1 \cap H_2 = \{0\}$,
(h3) $\mathcal{D}_{\mathcal{H}}$ is an upper semi-continuous decomposition of S^m.

For an integer $m \geq 0$, let $\mathcal{P}_m$ be the family of the decomposition spaces $S^m/\mathcal{D}_{\mathcal{H}}$ of S^m by the decompositions $\mathcal{D}_{\mathcal{H}}$ induced by the families $\mathcal{H}$ of linear subspaces of $\mathbb{R}^{m+1}$ satisfying (h1)-(h3). We see $\mathcal{P}_0$ consists of a single one-point space and it is straightforward to see that $\mathcal{P}_1 = \{\text{a simple closed curve}\}$. We observe

Lemma 2.5. $\mathcal{P}_2 = \{S^2, P^2\}$.

Proof. Let $\mathcal{H}$ be a collection of linear subspaces of $\mathbb{R}^{m+1}$ satisfying (h1)-(h3). If $\dim H = 1$ for each $H \in \mathcal{H}$, then $\mathcal{D}_{\mathcal{H}}$ is the decomposition of S^2 by antipodal points, and the resulting composition space is the projective plane. If $\dim H_0 = 2$ for some $H_0 \in \mathcal{H}$, then the other elements of $\mathcal{H}$ is one-dimensional due to (h1) and (h2). Hence $S^2/\mathcal{D}_{\mathcal{H}}$ is the projective plane modulo the projective circle P^1, hence is homeomorphic to S^2. $\square$

The result stated in Section 1 is a consequence of the next theorem and the above lemma.

Theorem 2.6. *Let X, Y be compact Hausdorff spaces and let E be a finite-dimensional Hilbert space with $m = \dim_{\mathbb{R}} E \geq 2$. Let $A \subset C(X, E)$, $B \subset C(Y, E)$ be linear subspaces of $C(X, E)$ and $C(Y, E)$ respectively satisfying the conditions (S1),(S2) and (ext). If $\mathrm{Ch}_E(A)$ does not contain any topological copy of any space in $\mathcal{P}_{m-1}$, then every surjective linear isometry $T : A \to B$ between A and B is a generalized weighted composition operator.*

Corollary 2.7. *Let X, Y, A, B be spaces as in the above theorem. If either*

(1) $\dim_{\mathbb{R}} E = 2$ *and X contains no simple closed curves,or*
(2) $\dim_{\mathbb{R}} E = 3$ *and X contains no topological copies of the 2-sphere and the real-projective plane,*

then every surjective linear isometry $T : A \to B$ is a generalized weighted composition operator.

Proof of Theorem 2.6. Let $T : A \to B$ be a surjective linear isometry and suppose that T is not a generalized weighted composition operator. As in Theorem 1.2 we consider the map $\phi_T : S(E^*) \to \mathrm{Map}(\mathrm{Ch}_E(B), \mathrm{Ch}_E(A))$ defined in (2.3). It is not a constant map by Theorem 2.1 and the hypothesis. Then the map $\phi_y : S(E^*) \to \mathrm{Ch}_E(A)$ defined in (2.5) is not a constant map for some $y \in \mathrm{Ch}_E(B)$.

The fibers $\{\phi_y^{-1}(x) \mid x \in \phi_y(S(E^*))\}$ of ϕ_y forms an upper semi-continuous decomposition $\mathcal{D}$ of the $(m-1)$-dimensional sphere $S(E^*)$. By (1) of Lemma 2.3, we have $\phi_y^{-1}(x) = H_x \cap S(E^*)$ and $\dim H_x \geq 1$. Since ϕ_y is not a constant map, we see that $H_x \neq E^*$ and thus $\dim H_x \leq m-1$. Hence $\mathcal{D}$ is a decomposition given by a family of subspaces of E^* satisfying (h1)-(h3) and the decomposition space $S(E^*)/\mathcal{D}$ belongs to the class $\mathcal{P}_{m-1}$. Also the space is homeomorphic to the image $\phi_y(S(E^*))$ by the compactness of $S(E^*)$ and thus $\mathrm{Ch}_E(A)$ contains a topological copy of $S(E^*)/\mathcal{D}$, contradicting the hypothesis.

Therefore the map ϕ_y is a constant map for each $y \in \mathrm{Ch}_E(B)$ and the condition (stab) is verified, which concludes the proof by Theorem 2.1. $\square$

3. Isometries which are not generalized weighted composition operators

In this section we give two examples of isometries between $\mathbb{R}^3$-valued continuous function spaces over the 2-sphere and the projective plane that are not generalized weighted composition operators. The construction is based on the idea of [12]. The subspaces constructed satisfies neither (S1), (S2) nor $(*)$ and it is an open question whether such an isometry exists on a subspace satisfying the conditions (S1), (S2) and (ext) . It should be mentioned that such isometries do exist on some subspaces of $C(S^1,\mathbb{R}^2)$ ([12]) and $C(S^3,\mathbb{R}^4)$ ([10]). The failure of the conditions (S1), (S2) and $(*)$ makes the present example of secondary importance and we will omit some details of the argument.

First we introduce notation. For a 3×3-real metrix M and $d\in\mathbb{R}^3$ let $f_{M,d}:\mathbb{R}^3\to\mathbb{R}^3$ be the affine map defined by

$$f_{M,d}(x)=Mx+d,\ x\in\mathbb{R}^3$$

and the restriction $f_{M,d}|S^2$ to the unit sphere $S^2=\{x\in\mathbb{R}^3\mid \|x\|=1\}$ is also denoted by $f_{M,d}$. For $a,b,c\in\mathbb{R}$, let $P(a,b;c)$ be the 3×3-matrix given by

$$P(a,b;c)=\begin{pmatrix} a & -b & 0\\ b & a & 0\\ 0 & 0 & c\end{pmatrix} \tag{3.1}$$

We define a linear subspace A of $C(S^2,\mathbb{R}^3)$ by

$$A=\{f_{P(a,b,;c),d}\mid a,b,c\in\mathbb{R},d\in\mathbb{R}^2\times\{0\}\}.$$

For a vector $d={}^t(d_1,d_2,0)\in\mathbb{R}^2\times\{0\}$, let $\bar{d}={}^t(d_1,-d_2,0)$. The map $T:A\to A$ defined by

$$T(f_{P(a,b;c),d})=f_{P(a,b;c),\bar{d}} \tag{3.2}$$

is a linear isometry such that $T\circ T=\mathrm{id}_A$.

Proposition 3.1. *Under the above notation, A is a linear subspace of $C(S^2,\mathbb{R}^3)$ with $\mathrm{Ch}_{\mathbb{R}^3}(A)=S^2$ which satisfies neither (S1), (S2) nor $(*)$. The map $T:A\to A$ is an isometry which is not a generalized weighted composition operator.*

Sketch of Proof. The constant functions that belong to A take values in $\mathbb{R}^2\times\{0\}$. If $x={}^t(x_1,x_2,x_3)$ is a point of S^2 with $x_3\neq 0$, then the equality $f_{P(a,b;c),d}(x)=0$ implies $c=0$. Also we see $f_{P(a,b;0),d}(S^2)\subset\mathbb{R}^2\times\{0\}$. These observations show that A fails to satisfy (S1) and (S2). If A is invariant under the action of a subgroup Γ of the orthogonal group $O(3)$, then for each $\gamma\in\Gamma$, for each $a,b,c\in\mathbb{R}$ and for each $d\in\mathbb{R}^2\times\{0\}$, the function $\gamma(f_{P(a,b;c),d})$ must be of the form $f_{P(l,m;n),e}$ for some $(l,m,n)\in\mathbb{R}^3$ and $e\in\mathbb{R}^2\times\{0\}$. This implies that each $\gamma\in\Gamma$ is of the form $\gamma=R\oplus(\pm 1),R\in O(2)$. Hence Γ does not act transitively on S^2. Thus A fails to satisfy the condition $(*)$. In order to verify that T is an isometry, we first observe that $f_{P(a,b;c),d}(S^2)$ for $d={}^t(d_1,d_2,0)$ is (the boundary of) the ellipsoid such that

(i) $f_{P(a,b;c),d}(S^2)\cap(\mathbb{R}^2\times\{0\})$ is the circle of radius $\sqrt{a^2+b^2}$ centered at d, and
(ii) $\{(d_1,d_2)\}\times[-|c|,|c|]$ is an axis of $f_{P(a,b;c),d}(S^2)$.

From this we see directly that $\|f_{P(a,b;c),d}\|_\infty = \sup_{x\in S^2}\|f_{P(a,b;c),d}(x)\| = \|f_{P(a,b;c),\bar{d}}\|_\infty$ and hence S is an isometry.

In order to verify $\mathrm{Ch}_{\mathbb{R}^3}(A) = S^2$, we use the fact that the Choquet boundary is a boundary ([6, p.37]), that is,

(iii) for each $f \in A$, there exists a point $x \in \mathrm{Ch}_E(A)$ such that $\|f\|_\infty = \|f(x)\|$.

Let $r : S^2 \to S^2$ be the reflection given by

$$r(x_1, x_2, x_3) = (x_1, x_2, -x_3),\ (x_1, x_2, x_3) \in \mathbb{R}^3.$$

In view of the symmetry of the ellipsoid $f_{P(a,b;c),d}(S^2)$ with respect to the plane $\mathbb{R}^2 \times \{0\}$, we see that the supremum $\sup_{z\in S^2}\|f_{P(a,b;c),d}(z)\|$ $(= \|f_{P(a,b;c),d}\|_\infty)$ is attained at exactly two points x, y with $y = r(x)$. Hence it suffices to prove that

$$r(\mathrm{Ch}_{\mathbb{R}^3}(A)) = \mathrm{Ch}_{\mathbb{R}^3}(A). \tag{3.3}$$

Let $R : A \to A$ be the linear map defined by

$$R(f_{P(a,b;c),d}) = f_{P(a,b;-c),d}.$$

From (i) and (ii), it follows that R is an isometry. A direct computation shows

$$\delta_x \circ R = \delta_{r(x)},\ x \in S^2. \tag{3.4}$$

Since R is an isometry, so is its dual map $R^* : A^* \to A^*$. Hence if $\xi \circ \delta_x \in \mathrm{ext}(A^*)$, $\xi \in S((\mathbb{R}^3)^*)$, then by (3.4), we have $R^*(\xi \circ \delta_x) = \xi \circ \delta_{r(x)} \in \mathrm{ext}(A^*)$. This proves (3.3).

Finally we derive a contradiction by assuming that T is a generalized weighted composition operator of the form

$$T(f_{P(a,b;c),d})(x) = V_x(f_{P(a,b;c),d}(\varphi(x))),\ x \in S^2 (= \mathrm{Ch}_{\mathbb{R}^3}(A)),\ f_{P(a,b;c),d} \in A$$

for a homeomorphism $\varphi : S^2 \to S^2$ and a continuous map $S^2 \to O(3); x \mapsto V_x$. The above implies that

$$P(a, b; c)x + \bar{d} = V_x(P(a, b; c)\varphi(x) + d)$$

which reduces to

$$\begin{aligned} P(a, b; c)x &= V_x(P(a, b; c)\varphi(x)) && (3.5)\\ \bar{d} &= (V_x P(a, b; c))d && (3.6) \end{aligned}$$

for each $a, b, c \in \mathbb{R}$, $d \in \mathbb{R}^2 \times \{0\}$ and $x \in S^2$. Using (3.5) for $P(1, 0; 1) = I_3$ we see $\varphi(x) = V_x^{-1}$ and then by (3.5) we have

$$P(a, b; c)V_x = V_x P(a, b; c),\ a, b, c \in \mathbb{R},\ x \in S^2. \tag{3.7}$$

Using (3.6) to $d = {}^t(1,0,0), {}^t(0,1,0)$ and using the continuity of the map $x \mapsto V_x$ and the connectedness of S^2, we find $\epsilon = \pm 1$ such that for each $x \in S^2$,

$$V_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \; x \in S^2. \tag{3.8}$$

However we see, by a direct computation, (3.7) and (3.8) are not consistent.

This completes the sketch of the proof. □

The next example is an isometry of a $\mathbb{R}^3$-valued continuous function space over the projective plane. The example is constructed using the previous one. Here the projective plane P^2 is given by the identification space

$$P^2 = \{(x_1, x_2, x_3) \in S^2 \mid x_3 \geq 0\}/[(x_1, x_2, 0) \sim (-x_1, -x_2, 0), (x_1, x_2, 0) \in S^2]$$

and the point of P^2 represented by a point $(x_1, x_2, x_3) \in S^2$ is written as $[x_1 : x_2 : x_3]$. Let $L = \{[x_1, x_2, 0] \mid (x_1, x_2, 0) \in S^2\}$, which is homeomorphic to $P^1 \approx S^1$. Let $\alpha : P^2 \to S^2$ be the map given by

$$\begin{aligned} \alpha([0:0:1]) &= [0:0:1] \\ \alpha([x_1 : x_2 :, x_3]) &= \left(2x_1\sqrt{\frac{x_3}{1+x_3}}, 2x_2\sqrt{\frac{x_3}{1+x_3}}, 2x_3 - 1\right), \\ & \qquad [x_1 : x_2 : x_3] \neq [0:0:1]. \end{aligned}$$

The map α is a well-defined continuous surjection such that $\alpha^{-1}(0,0,1) = \{[0:0:1]\}$ and $\alpha^{-1}(0,0,-1) = L$. It induces a linear map

$$\alpha^\sharp : C(S^2, \mathbb{R}^3) \to C(P^2, \mathbb{R}^3)$$

given by $\alpha^\sharp(f) = f \circ \alpha, \; f \in C(S^2, \mathbb{R}^3)$. It follows from the surjectivity of α that

$$\|\alpha^\sharp(f_{P(a,b;c),d})\|_\infty = \|f_{P(a,b;c),d}\|_\infty$$

for each $f_{P(a,b;c),d} \in A$. Let $B = \alpha^\sharp(A)$ and let $g_{P(a,b;c),d} = \alpha^\sharp(f_{P(a,b;c),d}) = f_{P(a,b;c),d} \circ \alpha : P^2 \to \mathbb{R}^3$ and define $T : B \to B$ by

$$T(g_{P(a,b;c),d}) = g_{P(a,b;c),\bar{d}}.$$

We can make use of the previous proposition and its proof to show the following result. We omit the detail.

Proposition 3.2. *Under the above notation, B is a linear subspace of $C(P^2, \mathbb{R}^3)$ with $\mathrm{Ch}_{\mathbb{R}^3}(B) = P^2$ which satisfies neither (S1), (S2), nor $(*)$. The map $T : B \to B$ is an isometry which is not a generalized weighted composition operator.*

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