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PRODUCTS OF REGULAR LOCALLY COMPACT SPACES ARE $K_{\mathbb{R}}$-SPACES

HELGE GLÖCKNER AND NIKU MASBOUGH

Abstract. A theorem by N. Noble from 1970 asserts that every cartesian product of completely regular, locally pseudo-compact $k_{\mathbb{R}}$-spaces is a $k_{\mathbb{R}}$-space. As a consequence, all products of locally compact Hausdorff spaces are $k_{\mathbb{R}}$-spaces. We provide an accessible proof for this fact. More generally, we show that all products of not necessarily Hausdorff, regular locally compact spaces are $k_{\mathbb{R}}$-spaces.

1. Introduction

Recall that a (not necessarily Hausdorff) topological space X is called *compact* if every open cover of X has a finite subcover, and *locally compact* if every neighborhood of a point in X contains a compact neighborhood. If every neighborhood contains a closed neighborhood, then X is called *regular*; it is *completely regular* if the topology on X is initial with respect to the set $C(X,\mathbb{R})$ of all continuous real-valued functions on X. A function $f\colon X \to Y$ to a topological space Y is called *k-continuous* if its restriction $f|_K\colon K \to Y$ is continuous for each compact subset $K \subseteq X$. If every k-continuous real-valued function on X is continuous, then X is called a *$k_{\mathbb{R}}$-space*. Every k-continuous function from a $k_{\mathbb{R}}$-space to a completely regular topological space is continuous. We shall always endow cartesian products of topological spaces with the product topology. Stimulated by [4], we prove the following theorem:

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Theorem 1.1. *Every product $X = \prod_{\alpha \in A} X_\alpha$ of a family $(X_\alpha)_{\alpha \in A}$ of regular locally compact spaces is a $k_\mathbb{R}$-space. Moreover, every k-continuous function $f\colon X \to Y$ to a regular topological space Y is continuous.*

Notably, every product of locally compact Hausdorff spaces is a $k_\mathbb{R}$-space, the latter being (completely) regular. Previously, this fact was known as a special case of [4, Theorem 5.6 (ii)] mentioned in our abstract,[1] and it is also subsumed by part (i) of the cited theorem dealing with products for a class of spaces including all Čech-complete spaces (and hence all locally compact Hausdorff spaces). Noble left the proof of [4, Theorem 5.6 (ii)] to the reader, and asserted that it follows by an "obvious adaptation of the proof of 5.3." We did not find this assertion convincing in the general case of *loc. cit.* Yet, the proof of our theorem is an adaptation of the one of [4, Theorem 5.3]. It varies a less general version in [2]. The theorem has been used in recent research, [3].

We mention related results: Recall that a topological space X is a *k-space* if all k-continuous functions from X to topological spaces are continuous; every k-space is a $k_\mathbb{R}$-space. A topological space X is of *pointwise countable type* if it is a union of compact subsets K which have a countable base $(U_n)_{n \in \mathbb{N}}$ of open neighborhoods $U_n \supseteq K$ in X. Every space of pointwise countable type is a k-space; countable products of spaces of pointwise countable type are of pointwise countable type (cf. [1]).

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2. Proof of Theorem 1.1

Let $x = (x_\alpha)_{\alpha \in A} \in X$. To see that a k-continuous function $f\colon X \to Y$ to a regular topological space Y is continuous at x, let us show that $f^{-1}(U)$ is a neighborhood of x in X for each neighborhood U of $f(x)$ in Y. By hypothesis, there exists an open neighborhood V of $f(x)$ in Y whose closure $\overline{V}$ is contained in U. For each $\alpha \in A$, let K_α be a compact neighborhood of x_α in X_α; since X_α is regular, after shrinking K_α we may assume that K_α is closed in X_α. Since $K := \prod_{\alpha \in A} K_\alpha$ is compact by Tychonoff's Theorem, $f|_K$ is continuous by hypothesis. Hence $(f|_K)^{-1}(V)$ is a neighborhood of x in K. As K is endowed with the product topology, we find neighborhoods L_α of x_α in K_α such that $\prod_{\alpha \in A} L_\alpha \subseteq f^{-1}(V)$ and $N := \{\alpha \in A\colon L_\alpha \neq K_\alpha\}$ is finite. After shrinking L_α for $\alpha \in N$, we may assume that each L_α is closed in X_α and compact. After replacing K_α with L_α for all α, we may assume that

$$f(K) \subseteq V. \tag{2.1}$$

[1]Compare also [4, p. 187, lines 11-13 of the introduction].

For $y = (y_\alpha)_{\alpha\in A} \in X$, let us write $\mathrm{supp}_K(y) := \{\alpha \in A \colon y_\alpha \notin K_\alpha\}$. For each finite subset $F \subseteq A$, let $\Sigma_F(K)$ be the set of all $y = (y_\alpha)_{\alpha\in A} \in X$ such that $\mathrm{supp}_K(y)$ is finite and $\mathrm{supp}_K(y) \cap F = \emptyset$, i.e., $y_\alpha \in K_\alpha$ for all $\alpha \in F$.

Claim $(*)$: We claim that there exists a finite subset $F \subseteq A$ such that

$$\Sigma_F(K) \subseteq f^{-1}(\overline{V}). \tag{2.2}$$

If this is true, then

$$\prod_{\alpha\in F} K_\alpha \times \prod_{\alpha\in A\setminus F} X_\alpha \subseteq f^{-1}(U), \tag{2.3}$$

whence $f^{-1}(U)$ is a neighborhood of x. In fact, if $y = (y_\alpha)_{\alpha\in A}$ is an element of the left-hand side of (2.3), then

$$y_\alpha^G := \begin{cases} y_\alpha & \text{if } \alpha \in G, \\ x_\alpha & \text{else} \end{cases}$$

defines an element $y^G := (y_\alpha^G)_{\alpha\in A} \in \Sigma_F(K)$ for each G in the set of finite subsets of A containing F (which is directed via inclusion). Since $C := \prod_{\alpha\in A}(K_\alpha \cup \{y_\alpha\})$ is compact, $f|_C$ is continuous. As the net y^G converges to y in C with respect to the product topology, we deduce that $f(y^G) \to f(y)$. Since $f(y^G) \in \overline{V}$ for all G (see (2.2)), we deduce that $f(y) \in \overline{V} \subseteq U$, establishing (2.3).

To prove Claim $(*)$, let us suppose it was false and derive a contradiction. If the claim was false, we could obtain a sequence $(x^n)_{n\in\mathbb{N}}$ of elements $x^n = (x_\alpha^n)_{\alpha\in A} \in X$ such that the following holds for all $n \in \mathbb{N}$:

(a) $\mathrm{supp}_K(x^n)$ is finite;

(b) $\mathrm{supp}_K(x^m) \cap \mathrm{supp}_K(x^\ell) = \emptyset$ for all $m, \ell \leq n$ such that $m \neq \ell$;

(c) $f(x^n) \in Y \setminus \overline{V}$.

If this is true, we define $K'_\alpha := K_\alpha \cup \{x_\alpha^n \colon n \in \mathbb{N}\}$ and note that $K'_\alpha \setminus K_\alpha$ is either empty or a singleton, by (b). Hence K'_α is compact (like K_α). As $K' := \prod_{\alpha\in A} K'_\alpha$ is compact and $(x^n)_{n\in\mathbb{N}}$ is a sequence in K', we see that $(x^n)_{n\in\mathbb{N}}$ has a convergent subnet $(x^{n(j)})_{j\in J}$ for some directed set J. Let $z = (z_\alpha)_{\alpha\in A} \in K'$ be a limit of the subnet. Given α, condition (b) implies that $x_\alpha^n \in K_\alpha$ for all sufficiently large n, whence also $x_\alpha^{n(j)} \in K_\alpha$ eventually. Since K_α is closed in X_α and $x_\alpha^{n(j)} \to z_\alpha$, we deduce that $z_\alpha \in K_\alpha$ for all α and hence $z \in K$. But

$$f(x^{n(j)}) \in Y \setminus \overline{V} \subseteq Y \setminus V. \tag{2.4}$$

Since $f|_{K'}$ is continuous, we have $f(x^{n(j)}) \to f(z)$. As the right-hand side $Y \setminus V$ of (2.4) is closed, we deduce that $f(z) \in Y \setminus V$. But $f(z) \in f(K) \subseteq V$ by (2.1), contradiction.

It only remains to construct $x^1, x^2, \ldots$, which we achieve by recursion. As we suppose that Claim $(*)$ is false, $\Sigma_\emptyset(K)$ is not a subset of $f^{-1}(\overline{V})$; we therefore find an element $x^1 \in \Sigma_\emptyset(K)$ such that $f(x^1) \notin \overline{V}$. If $x^1, \ldots, x^n$ satisfying (a)–(c) have been found, then $F := \mathrm{supp}_K(x^1) \cup \cdots \cup \mathrm{supp}_K(x^n)$ is a finite set. As we assume that Claim $(*)$ is false, there exists $x^{n+1} \in \Sigma_F(K)$ such that $f(x^{n+1}) \notin \overline{V}$. Since $\mathrm{supp}_K(x^m) \subseteq F$ for $m \in \{1, \ldots, n\}$ but $\mathrm{supp}_K(x^{n+1}) \subseteq A \setminus F$ as $x^{n+1} \in \Sigma_F(K)$, we have $\mathrm{supp}_K(x^m) \cap \mathrm{supp}_K(x^{n+1}) = \emptyset$. As condition (b) already holds for n, the preceding shows that it also holds for $n+1$ in place of n. Note that $\mathrm{supp}_K(x^{n+1})$ is finite since $x^{n+1} \in \Sigma_F(K)$. This completes the recursive construction.

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