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TWO GRAPHICAL MODELS OF THE KAUFFMAN POLYNOMIAL AND THEIR RELATIONSHIP

XIAN'AN JIN AND QI YAN

Abstract. We introduce two oriented 4-valent graphical models for the Kauffman polynomial: one (called HJ) is obtained by combining Jaeger's formula and Kauffman-Vogel model for the Homflypt polynomial; the other (called WF) is obtained by combining Kauffman-Vogel model for the Kauffman polynomial and Wu's formula. The main goal of this paper is to explore the relationship between the two models. We find that there is a one-to-many correspondence between the terms of HJ model and the terms of WF model.

1. Introduction

In [14, 17], Kauffman and Vogel generalized the Homflypt and Kauffman polynomials from links to 4-valent rigid vertex spatial graphs. Conversely, an unoriented (resp. oriented) 4-valent plane graph expansion for the Kauffman (resp. Homflypt) polynomial of unoriented (resp. oriented) links was obtained, which is implicit in [17]. In 1989, Jaeger announced a relation [16], we shall call it Jaeger's formula, between the Kauffman polynomial of an unoriented link diagram and the Homflypt polynomials of some oriented link diagrams constructed from the unoriented link diagram. Recently, Wu generalized Jaeger's formula from link diagrams to 4-valent rigid vertex spatial graph diagrams [22]. We shall call it Wu's formula. Note that 4-valent rigid vertex spatial graph diagrams include 4-valent plane graphs as a special case. In this paper we shall confine ourselves in 4-valent plane graphs.

2010 *Mathematics Subject Classification.* 57M25, 57M15.

Key words and phrases. Kauffman polynomial, 4-valent graph, Jaeger's formula, Homflypt polynomial, Relationship.

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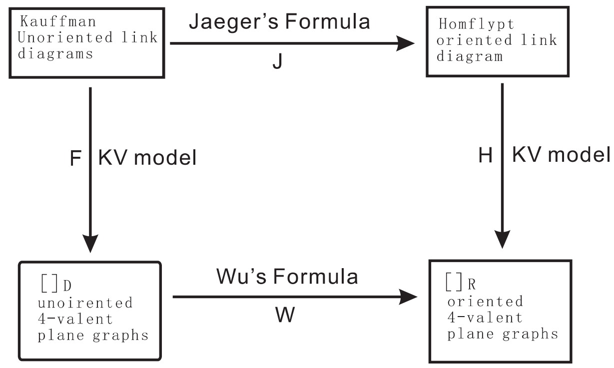

FIG. 1. The relation diagram.

We illustrate above descriptions by a relation diagram as shown in Fig. 1. According to the diagram, two oriented 4-valent graphical models for the Kauffman polynomial will be produced: one (HJ) is obtained by combining Jaeger's formula and Kauffman-Vogel model for the Homflypt polynomial; the other (WF) is obtained by combining Kauffman-Vogel model for the Kauffman polynomial and Wu's formula. The main goal of this paper is to explore the relationship between the two models. We modify the Jaeger's formula and then find that there is a one-to-many correspondence between the terms of HJ model and the terms of WF model.

As a result, we actually verified the consistency of the two Kauffman-Vogel models, Jaeger's formula and Wu's formula. We pointed out that in [9], Huggett found and verified a relationship (i.e. the replacement of a crossing by a clasp) between a famous Thistlethwaite's result [20] which expresses the Jones polynomial as a special parametrization of the Tutte polynomial [21] and a Jaeger's result relating the Homflypt polynomial with the Tutte polynomial.

2. Homflypt and Kauffman polynomials

To proceed rigorously, it is necessary to recall the definition of the Homflypt and Kauffman polynomials. The Homflypt polynomial was introduced in [7] and [19], independently, which is a writhe-normalization of its regular isotopy counterpart: the R polynomial. Let L be an oriented link diagram. We denote by $R_L(z,a) \in \mathbf{Z}[z^{\pm}, a^{\pm}]$ the R polynomial of L.

Axioms for the R polynomial

(1) $R_{\bigcirc} = 1$.
(2) R_L is invariant under Reidemeister moves II and III.
(3) (the kink formulae) The effect of Reidemeister move I on R is to multiply by a or a^{-1} according to the type of Reidemeister move I:

$$R_{L(+)} = aR_L, \quad R_{L(-)} = a^{-1}R_L, \tag{2.1}$$

where $L(+)$ (resp. $L(-)$) denotes diagrams with a positive (resp. negative) curl and L denotes the result of removing this curl by Reidemeister move I.
(4) (the skein relation)

$$R_{L_+} - R_{L_-} = zR_{L_0}, \tag{2.2}$$

where L_+, L_- and L_0 are link diagrams which are identical except near one crossing where they are as in Fig. 2 and are called a skein triple.

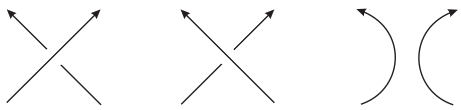

FIG. 2. A skein triple: L_+, L_- and L_0.

The Alexander-Conway [1, 5] and Jones [11] polynomials are both special cases of the Homflypt polynomial.

The Kauffman polynomial was introduced in [15]. We work with its Dubrovnik version [18]. Let L be an unoriented link diagram. We denote by $D_L(z,a) \in \mathbf{Z}[z^{\pm}, a^{\pm}]$ the Dubrovnik (briefly, D) polynomial of L. The Dubrovnik polynomial satisfies the following axioms:

Axioms for the D polynomial

(1) $D_{\bigcirc} = 1$.
(2) D_L is invariant under Reidemeister moves II and III.
(3) (the kink formulae) The effect of Reidemeister move I on D is to multiply by a or a^{-1} according to the type of Reidemeister move I:

$$D_{L(+)} = aD_L, \quad D_{L(-)} = a^{-1}D_L, \tag{2.3}$$

where $L(+)$ (resp. $L(-)$) denotes diagrams with a positive (resp. negative) curl and L denotes the result of removing this curl by Reidemeister move I.

(4) (the switching formula)

$$D_{L_+} - D_{L_-} = z(D_{L_0} - D_{L_\infty}), \tag{2.4}$$

where L_+, L_-, L_0 and L_∞ are link diagrams which are identical except near one crossing where they are as shown in Fig. 3.

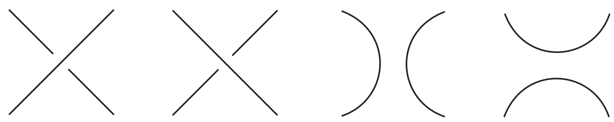

FIG. 3. The quadruple: L_+, L_-, L_0 and L_∞ (from left to right).

The Kauffman polynomial is a writhe-normalization of the Dubrovnik polynomial, which is the generalization of both the Jones polynomial [11] and the BLM-Ho's Q polynomial [3, 8]. The D polynomial specializes to the Kauffman bracket polynomial [12] by putting $a = -A^3$ and $z = A - A^{-1}$ [16].

We point out that, when we mention the Homflypt and Kauffman polynomials we sometimes mean the R and D polynomials, respectively.

3. Kauffman-Vogel models

In this section, we explain the 4-valent graphical models of the R and D polynomials. A graph is *planar* if it can be embedded in the plane, that is, it can be drawn on the plane so that no two edges intersect. The embedding of a planar graph is called a *plane* graph. It is well known that any graph can be embedded in the 3-dimensional Euclidean space [2], and such an embedding is called a *spatial* graph.

A *4-valent graph* is a graph whose each vertex is of degree 4. We always consider simple closed curves called *free loops* as special cases of 4-valent graphs, in other words, a free loop is a graph having one edge and having no vertices.

In [14], Kauffman defined 4-valent graphs with *rigid vertices* and introduced the notion of *rigid vertex ambient isotopy* for 4-valent rigid vertex spatial graphs. In [17], Kauffman and Vogel introduced two 3-variable (A, B, a) polynomials for 4-valent rigid vertex spatial graphs in terms of the R and D polynomials, respectively. When G has no vertices, i.e. G is a link, The two 3-variable polynomials of 4-valent rigid vertex spatial graph G will specialize to R and D polynomials, respectively.

Conversely, the R and D polynomials of link diagrams can be expressed as the sum of such 3-variable polynomials of 4-valent plane graphs constructed from link diagrams. In [4], Carpentier proved that such 3-variable polynomials can be computed recursively completely within the category of planar graphs without resorting to links. Thus we obtain 4-valent plane graphical models for both the R and D polynomials. Now we give a detailed account of the two models.

3.1. R **polynomial.** By an *oriented* 4-valent plane graph, we mean a 4-valent plane graph together with an edge orientation of the graph such that at each vertex, the four (not necessarily distinct) edges incident with the vertex are oriented like a crossing of an oriented link diagram as shown in Fig. 4.

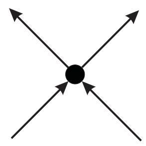

FIG. 4. Crossing-like oriented vertex.

The 3-variable polynomial $[G]_R = [G]_R(A, B, a)$ for an oriented 4-valent plane graph G can be defined via the following graphical calculus [17].

Graphical calculus for the $[]_R$ polynomial

(1) $[\circlearrowleft]_R = 1$, where $\circlearrowleft$ is a free loop and its orientation is actually irrelevant.
(2) $[G \sqcup \circlearrowleft]_R = \delta[G]_R$, where $G \sqcup \circlearrowleft$ is the disjoint union of an oriented 4-valent plane graph G and $\circlearrowleft$, and $\delta = \frac{a-a^{-1}}{A-B}$.
(3) Let

$$\begin{aligned}\lambda &= \frac{Aa^{-1} - Ba}{A - B},\\ \theta &= \frac{B^2a - A^2a^{-1}}{A - B},\\ \eta &= \frac{B^3a - A^3a^{-1}}{A - B}.\end{aligned}$$

Then identities as shown in Fig. 5 hold:

The following theorem is implicit in [17].

Theorem 3.1. *Let L be an oriented link diagram. Then*

$$R_L(A - B, a) = \sum_G A^{i(G)}B^{j(G)}[G]_R, \tag{3.1}$$

where the summation is over all oriented 4-valent plane graphs: G's, obtained from L by applying to each crossing (positive or negative) one of the *two replacements as shown in Fig. 6, and* $i(G)$ *and* $j(G)$ *are the numbers of positive and negative crossings of* L *smoothed to form* G*, respectively.*

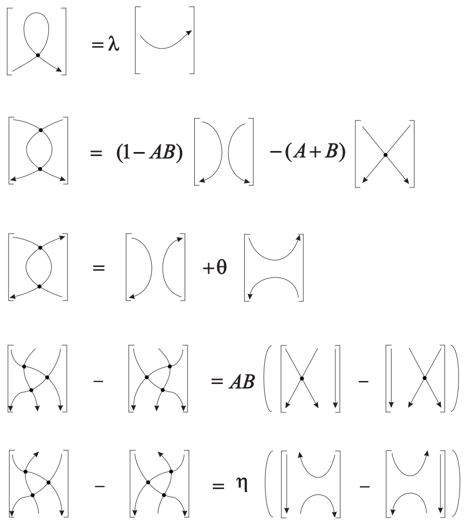

FIG. 5. Identities for $[]_R$.

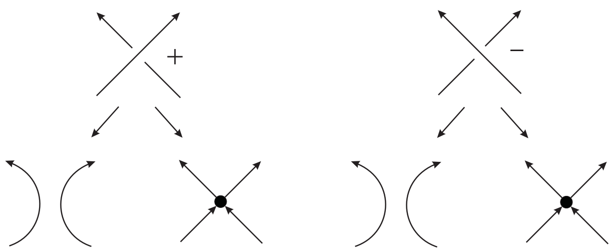

FIG. 6. Two types of replacements of an oriented crossing.

By a little abuse of notations, if we write $[L] = R_L$, then we have the following recursive equation as shown in Fig. 7. Note that the subscript R of $[]_R$ is omitted.

3.2. D **polynomial.** The 3-variable polynomial $[G]_D = [G]_D(A, B, a)$ for an unoriented 4-valent plane graph G can be defined via the following graphical calculus [17, 4].

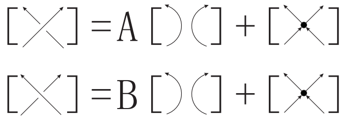

FIG. 7. Recursive equation for R_L.

Graphical calculus for the $[]_D$ polynomial

(1) $[\bigcirc]_D = 1$, where $\bigcirc$ is a free loop.
(2) $[G \sqcup \bigcirc]_D = \mu[G]_D$, where $G \sqcup \bigcirc$ is the disjoint union of an unoriented 4-plane graph G and $\bigcirc$, and $\mu = \frac{a-a^{-1}}{A-B} + 1$.
(3) Let

$$\begin{aligned} o &= \frac{Aa^{-1} - Ba}{A - B} - (A + B), \\ \gamma &= \frac{B^2a - A^2a^{-1}}{A - B} + AB, \\ \xi &= \frac{B^3a - A^3a^{-1}}{A - B}. \end{aligned}$$

Then identities as shown in Fig. 8 hold:

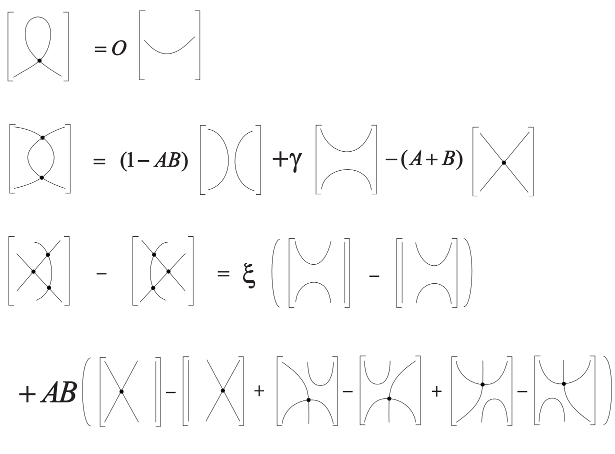

FIG. 8. Identities for $[]_D$.

The following theorem is implicit in [17], and explicit in [4].

Theorem 3.2. *Let L be an unoriented link diagram. Then*

$$D_L(A - B, a) = \sum_G A^{i(G)} B^{j(G)} [G]_D, \tag{3.2}$$

where the summation is over all (unoriented) 4-valent plane graphs: G's, obtained from L by applying to each crossing one of the three types of replacements as shown in Fig. 9, and $i(G)$ and $j(G)$ are the numbers of crossings of L of A-smoothings and B-smoothings used to form G, respectively.

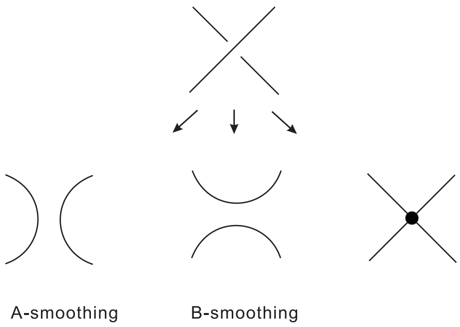

FIG. 9. Three types of replacements of an unoriented crossing.

Similarly, if we write $[L] = D_L$, then we have the following recursive equation as shown in Fig. 10.

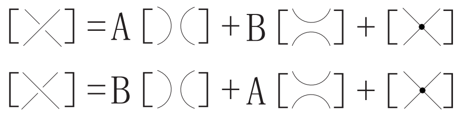

FIG. 10. Recursive equation for D_L.

4. Jaeger and Wu's formulae

Let L be an oriented link diagram, the *rotation number* (also called Whitney degree, see [13], p. 170) $\mathrm{rot}(L)$ of L is equal to the sum of signs for all Seifert circles of L with the convention that the sign is $+1$ if the circle is counterclockwise oriented and the sign is -1 if the circle is clockwise oriented. It actually measures the total turn of the unit tangent vector to the underlying plane curves of the link diagram. The rotation number of an oriented 4-valent plane graph is thus equal to that of any oriented link diagram obtained from the graph by converting each vertex into a crossing.

4.1. **Jaeger's formula.** Jaeger (see [16], pp. 219-222) established a relation between Kauffman polynomial and Homflypt polynomial, which expresses the D polynomial of a link diagram as the certain weighted sum of Homflypt polynomials of some oriented link diagrams obtained by firstly "splicing" some crossings of the link diagram and then assigning an orientation. Jaeger's formula can also be found in [6, 22]. Here we give it a slightly different formulation. Note that in this paper we use the normalized versions of the R and D polynomials, while in [16, 6, 22], the authors all dealt with unnormalized versions.

Let L be an unoriented link diagram. We call a segment of the diagram between two adjacent crossings an *edge* of L. An edge orientation of L is *balanced* if, at each crossing, among four (not necessarily distinct) edges around the crossing, two edges are "in" and two edges are "out". Up to rotation, there are four possible balanced edge orientations near a crossing: two are crossing-like oriented and the other two are alternatingly oriented. (See Fig. 11.)

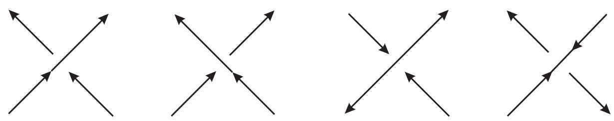

FIG. 11. Balanced edge orientation near a crossing: the first is a positive crossing-like oriented crossing; the second is a negative crossing-like oriented crossing; the third and the fourth are both alternatingly oriented called *top outward* crossing and *top inward* crossing, respectively in [22].

Denote by $\mathcal{O}(L)$ the set of all balanced edge orientations of L.

Given an $o \in \mathcal{O}(L)$, equipping L with o, we obtain L_o. To obtain oriented link diagrams from L_o we need to be "splicing" all alternatingly oriented crossings. There are two ways of smoothing a top outward crossing and a top inward crossing of L_o as shown in Fig. 12. A *resolution* r of L_o is a choice of A or B smoothing for every top outward and top inward crossing of L_o. Denote by $\sum(L_o)$ the set of all resolutions of L_o. Given a $r \in \sum(L_o)$, for each top outward and top inward crossing c of L_o, we define the weight to be

$$[L_o, r; c] = \begin{cases} q - q^{-1} & \text{if } c \text{ is a top outward crossing and } r \\ & \text{applies } A \text{ smoothing to } c, \\ q^{-1} - q & \text{if } c \text{ is a top outward crossing and } r \\ & \text{applies } B \text{ smoothing to } c, \\ 0 & \text{if } c \text{ is a top inward crossing.} \end{cases}$$

We can take the weight of the unspliced crossing L_o to be 1. The total weight $[L_o, r]$ of the resolution r (applied to L_o) is defined to be the product of weights of all crossings of L_o. Denote by $L_{o,r}$ the oriented link diagram obtained by applying r to L_o.

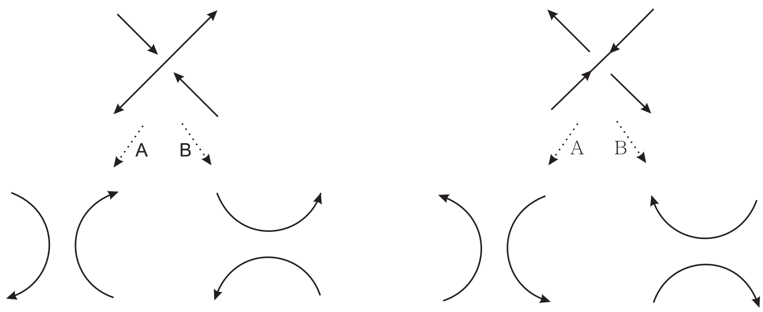

FIG. 12. Two ways of smoothing a top outward crossing and a top inward crossing.

Theorem 4.1. ***(Jaeger's Formula)*** *Let L be an unoriented link diagram. Then*

$$D_L(q-q^{-1}, a^2q^{-1}) = J \sum_{o\in\mathcal{O}(L)} \sum_{r\in\sum(L_o)} (qa^{-1})^{\mathrm{rot}(L_{o,r})}[L_o, r]R_{L_{o,r}}(q-q^{-1}, a),$$

where $J = \frac{1}{qa^{-1}+q^{-1}a}$.

Remark 4.2. We have two remarks on Theorem 4.1.

(1) There is a coefficient J in Theorem 4.1, for we use the normalized Homflypt and Kauffman polynomials in this paper. So

$$J = \frac{\delta}{\mu} = \frac{\frac{a-a^{-1}}{q-q^{-1}}}{\frac{a^2q^{-1}-a^{-2}q}{q-q^{-1}}+1} = \frac{1}{qa^{-1}+q^{-1}a}.$$

(2) Since, for L_o which contains a top inward crossing and any $r \in \sum(L_o)$, we have $[L_o, r] = 0$, there are some terms in the right hand of Theorem 4.1 which are equal to 0. In Theorem 4.1, actually we can only consider balanced edge orientations of L which do not contain a top inward crossing. We add some 0 terms in the summation, which, you will see, is important to prove our main Theorem 5.1.

4.2. **Wu's formula.** In [22], Wu built a relation between the 3-variable KV polynomial and the 3-variable MOY polynomial of 4-valent rigid vertex spatial graphs. We shall only restrict ourselves to 4-valent plane graphs.

Let G be a 4-valent plane graph. An edge orientation of G is *balanced* if, at each vertex, among four edges incident with the vertex, two edges are "in" and two edges are "out". Up to rotation, there are two possible balanced edge orientations near a vertex: one is crossing-like oriented and the other is alternatingly oriented. (See Fig. 13.)

FIG. 13. Balanced edge orientation near a vertex: the first is a crossing-like oriented vertex; the second is an alternatingly oriented vertex.

Denote by $\mathcal{O}(G)$ the set of all balanced edge orientation of G.

Given an $o \in \mathcal{O}(G)$, equipping G with o, we obtain G_o. To obtain oriented 4-plane graphs from G_o we need to splice all alternatingly oriented vertices. There are two ways of smoothing an alternatingly oriented vertex of G_o as shown in Fig. 14. A *resolution* r of G_o is a choice of L or R smoothing of every alternatingly oriented vertex of G_o. Denote by $\sum(G_o)$ the set of all resolutions of G_o. Given a $r \in \sum(G_o)$, for each alternatingly oriented vertex v of G_o, we define the weight to be

$$[G_o, r; v] = \begin{cases} -q & \text{if } r \text{ applies } L \text{ smoothing to the vertex } v, \\ -q^{-1} & \text{if } r \text{ applies } R \text{ smoothing to the vertex } v. \end{cases}$$

We can take the weight of crossing-like oriented vertex of G_o to be 1. The total weight $[G_o, r]$ of the resolution r is defined to be the product of weights of all vertices of G_o. Denote by $G_{o,r}$ the oriented 4-valent plane graph obtained by applying r to G_o.

Theorem 4.3. ***(Wu's Formula)*** *Let G be an unoriented 4-valent plane graph. Then*

$$[G]_D(q, q^{-1}, a^2q^{-1}) = J \sum_{o\in\mathcal{O}(G)} \sum_{r\in\sum(G_o)} (qa^{-1})^{\mathrm{rot}(G_{o,r})}[G_o, r][G_{o,r}]_R(q, q^{-1}, a),$$

where $J = \frac{1}{qa^{-1}+q^{-1}a}$.

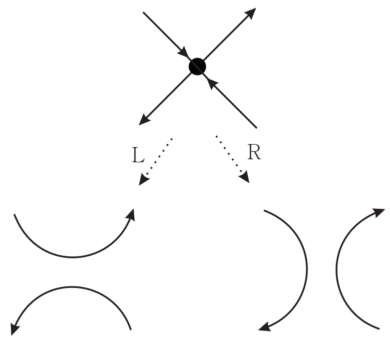

FIG. 14. Two ways of smoothing an alternatingly oriented vertex.

Remark 4.4. Note that $[G_o, r; v] = -[D_e, z; v]$ in [22], since by comparing Figs. 7, 10 with Eqs. (1.11) and (1.9) in [22], we know that the $[G]_R$ (resp. $[G]_D$) is equal to the result of multiplying $(-1)^{|V(G)|}$ and $R(G)$ (resp. $P(G)$) in [22].

5. Two models and their relationship

Now we are in a position to derive two oriented 4-valent graphical models for the Kauffman polynomial. We call the HJ model the summation obtained by applying Jaeger's formula (J) firstly and then Kauffman-Vogel model for the Homflypt polynomial (H). Similarly, we call WF model the summation obtained by applying Kauffman-Vogel model for the Kauffman polynomial (F) and Wu's formula (W). Let L be an unoriented link diagram.

(1) The HJ model: $D_L(q - q^{-1}, a^2q^{-1})$

$$\begin{aligned} &= J \sum_{o\in\mathcal{O}(L)} \sum_{r\in\sum(L_o)} (qa^{-1})^{\mathrm{rot}(L_{o,r})}[L_o, r] \sum_{{}_{o,r}I} q^{i({}_{o,r}I)-j({}_{o,r}I)}[{}_{o,r}I]_R(q, q^{-1}, a) \\ &= J \sum_{o\in\mathcal{O}(L)} \sum_{r\in\sum(L_o)} \sum_{{}_{o,r}I} (qa^{-1})^{\mathrm{rot}({}_{o,r}I)}[L_o, r] q^{i({}_{o,r}I)-j({}_{o,r}I)}[{}_{o,r}I]_R(q, q^{-1}, a), \end{aligned}$$

where the third summation runs over all oriented 4-valent plane graphs: ${}_{o,r}I$'s, obtained from $L_{o,r}$ by applying H. The second " = " holds since $\mathrm{rot}(L_{o,r})=\mathrm{rot}({}_{o,r}I)$ for any ${}_{o,r}I$.

(2) The WF model: $D_L(q-q^{-1}, a^2q^{-1})$

$$\begin{aligned} &= \sum_G q^{i(G)-j(G)} J \sum_{o\in\mathcal{O}(G)} \sum_{r\in\sum(G_o)} (qa^{-1})^{\mathrm{rot}(G_{o,r})}[G_o, r][G_{o,r}]_R(q, q^{-1}, a) \\ &= J\sum_G \sum_{o\in\mathcal{O}(G)} \sum_{r\in\sum(G_o)} q^{i(G)-j(G)}(qa^{-1})^{\mathrm{rot}(G_{o,r})}[G_o, r][G_{o,r}]_R(q, q^{-1}, a), \end{aligned}$$

where the first summation runs over all unoriented 4-plane graphs: G's, obtained from L by applying F.

Note that in the HJ model, for different orientation o, resolution r and different ways of applying H to $L_{o,r}$, the obtained oriented 4-valent plane graphs: ${}_{o,r}I$'s, are all different in the sense that the crossings and edges of L are labeled differently and kept unchanged after splicing some crossings. Let S be the set of all oriented 4-valent plane graphs constructed from L in the HJ model. In other words, the terms in the HJ model are all different. However, in the WF model, there exist many terms whose corresponding $G_{o,r}$'s are the same. In other words, for different G, different orientation o and resolution r, the obtained $G_{o,r}$'s are not all different.

Furthermore, note that the set of different oriented 4-valent planes in both models are the same: they are both the set of 4-valent plane graphs obtained from L by replacing each with an unoriented crossing c of L by one of the following twelve types of configurations: V_1, V_2, V_3, V_4, C_1, C_2, C_3, C_4, A_1, A_2, A_3 and A_4 as shown in Fig. 15. Of course, we demand orientations of all local replacements be compatible on each edge of the underlying 4-valent plane graph of L when we construct oriented 4-valent plane graphs from L.

Let $\mathcal{T} = \{G; o, r\}$ be the set of different ways: $G; o, r$'s, of constructing oriented 4-valent plane graphs from L, which correspond all terms in the WF model. Now we rewrite the HJ and WF models as the sum

$$\sum_{s\in S} c_s[s]_R \quad \text{and} \quad \sum_{G;o,r\in\mathcal{T}} d_{G;o,r}[G_{o,r}]_R.$$

Then we have

Theorem 5.1. *For each $s \in S$, there exists a subset $T_s = \{G; o, r \in \mathcal{T} | s = G_{o,r}\}$ such that $T_{s_1} \cap T_{s_2} = \emptyset$ for any $s_1, s_2 \in s$, $\cup_{s\in S} T_s = \mathcal{T}$ and $c_s = \sum_{G;o,r\in T_s} d_{G;o,r}$.*

Proof. We have shown the existence of T_s such that $T_{s_1} \cap T_{s_2} = \emptyset$ for any $s_1, s_2 \in s$ and $\cup_{s\in S} T_s = \mathcal{T}$ in the preceding several paragraphs. Hence, it suffices for us to prove that $c_s = \sum_{G;o,r\in T_s} d_{G;o,r}$. Let $s =_{o,r} I$.

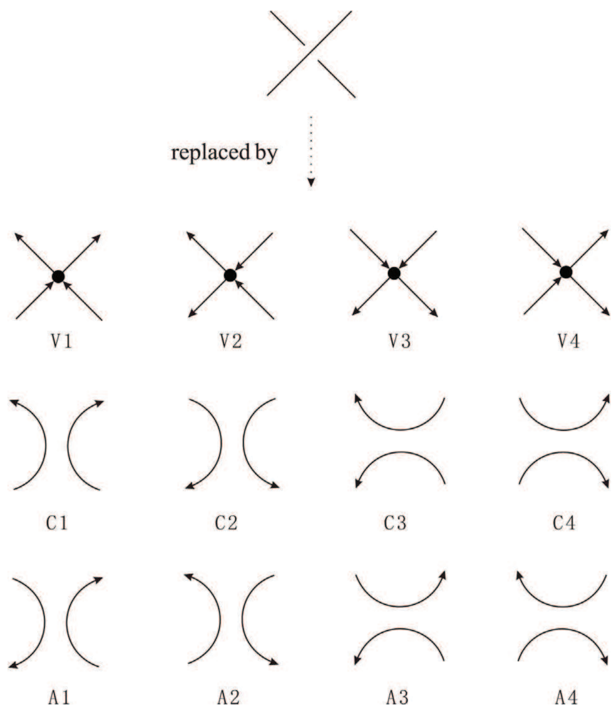

FIG. 15. 12 configurations used to replace an unoriented crossing.

Recall that

$$\begin{aligned} c_s &= J(qa^{-1})^{\mathrm{rot}(s)}[L_o, r]q^{i(_o,_rI)-j(o,rI)}, \\ d_{G;o,r} &= J(qa^{-1})^{\mathrm{rot}(G_{o,r})}q^{i(G)-j(G)}[G_o, r] \\ &= J(qa^{-1})^{\mathrm{rot}(s)}q^{i(G)-j(G)}[G_o, r]. \end{aligned}$$

Now let

$$\begin{aligned} c_{HJ} &= [L_o, r]q^{i(_o,_rI)-j(_o,_rI)}, \\ c_{WF} &= \sum_{G;o,r\in T_s} q^{i(G)-j(G)}[G_o, r]. \end{aligned}$$

Then we only need to prove $c_{HJ} = c_{FW}$. We suppose that s is the 4-valent plane graph obtained from L by v_i replacements of the configuration V_i, c_i replacements of the configuration C_i and a_i replacements of the configuration A_i for $i = 1, 2, 3, 4$.

By applying J firstly then H, an unoriented crossing c of L will be replaced by one of eight oriented configurations firstly, then each of the four crossing-like oriented configurations is replaced by one of two configurations (see Fig. 16). The corresponding weight in the product $[L_o, r]q^{i({}_o,{}_rI)-j({}_o,{}_rI)}$ is labeled as the subscript in that figure.

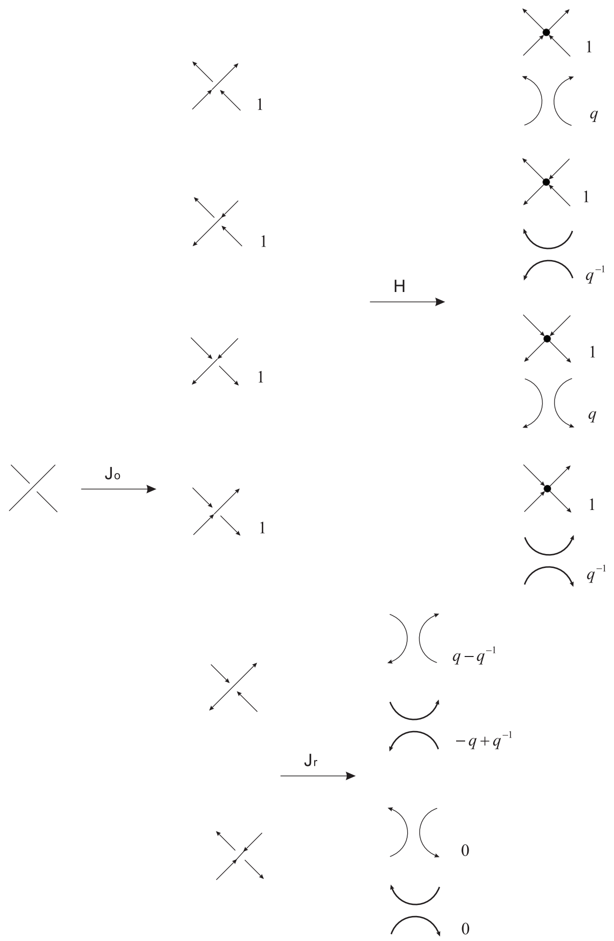

FIG. 16. The HJ model expansion, J_o means orientation by applying J, J_r means resolution by applying J.

Similarly, applying F firstly then W, an unoriented crossing c of L will be replaced by one of three unoriented configurations: A-smoothing, B-smoothing and the vertex replacement firstly, then each of the two smoothings is replaced by one of four oriented configurations and the vertex replacement is replaced by one of eight oriented configurations (see Fig. 17). The corresponding weight in $q^{i(G)-j(G)}[G_o, r]$ is also labeled as the subscript in the figure.

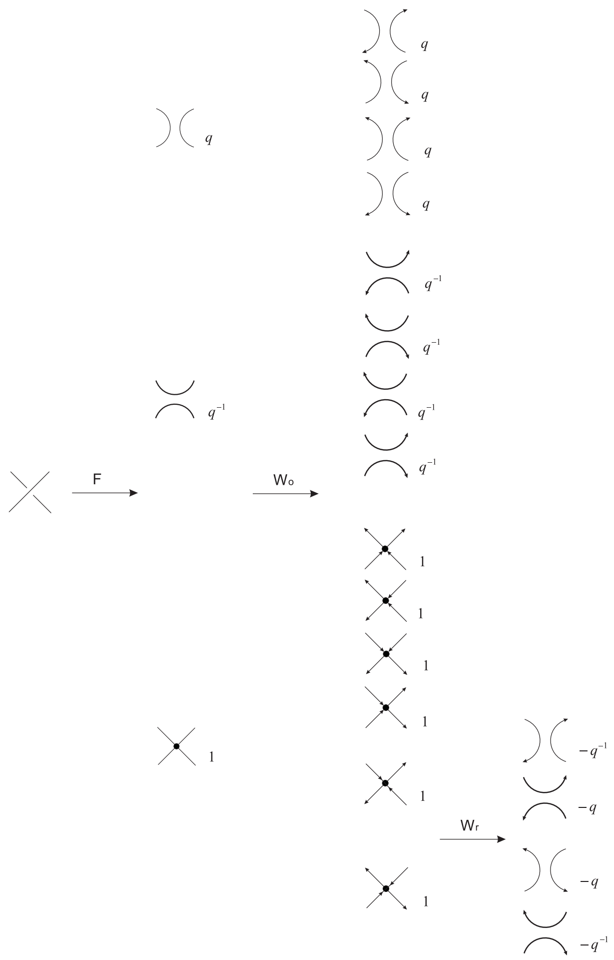

FIG. 17. The WF model expansion, W_o means orientation by applying W, W_r means resolution by applying W.

Note that in Figs. 16 and 17, the corresponding weights of V_1, V_2, V_3, V_4, C_1, C_2, C_3 and C_4 in the two figures are the same. Now we analyze the remaining four configurations: A_1, A_2, A_3, A_4. There are two cases:
Case 1. If $a_2 \neq 0$ or $a_4 \neq 0$, it is clear that $c_{HJ} = 0$. Now we consider c_{WF}. Without loss of generality we suppose that $a_2 \neq 0$. Note that A_2

appears twice in Fig. 17. This means we can obtain a_2 A_2 configurations by selecting k A_2 configurations via applying F and then selecting the remaining $a_2 - k$ A_2 configurations via applying W for any $k = 0, 1, \cdots, a_2$. Note that $\sum_{k=0}^{a_2} q^k(-q)^{a_2-k} = 0$. Hence $c_{WF} = 0$.

Case 2. Otherwise, it means that s does not contain A_2 and A_4 configurations. Similarly, A_1 and A_3 appears twice in Fig. 17. Since $(q - q^{-1})^{a_1} = \sum_{k=1}^{a_1} q^k(-q^{-1})^{a_1-k}$ and $(-q + q^{-1})^{a_1} = \sum_{k=1}^{a_3}(-q)^{a_3-k}(q^{-1})^k$, we have $c_{HJ} = c_{WF}$.

This completes the proof of Theorem 5.1. □

Theorem 5.1 tells us that many terms of the WF model add up to one term of the HJ model, hence, the HJ model is more efficient than WF model. Now we provide an example to illustrate Theorem 5.1.

Example 5.2. The Hopf link.

We first expand the R polynomial of the Hopf link based on the HJ model. There are six different balanced orientations for the Hopf link, and twenty four oriented 4-valent plane graphs (i.e. states) are constructed from the Hopf link (see Fig. 18).

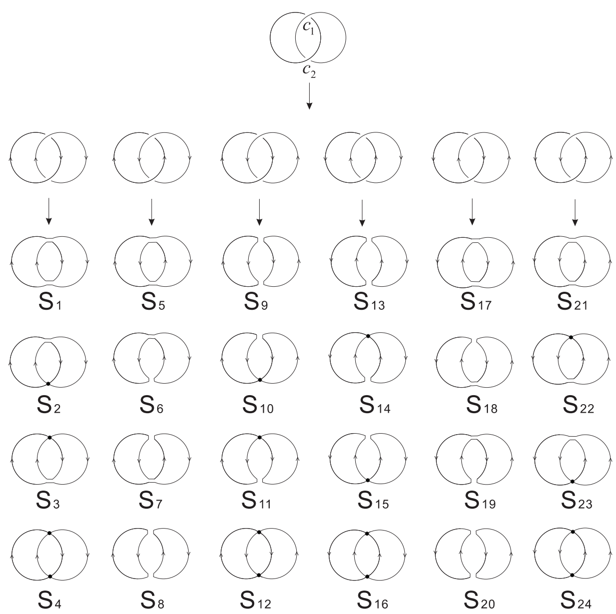

FIG. 18. Hopf link, 6 balanced orientations and 24 states in the HJ model.

We then expand its R polynomial based on the WF model. Nine unoriented 4-valent plane graphs by applying F are obtained firstly, then forty eight terms all together are obtained by applying W to each 4-valent plane graph (see Fig. 19).

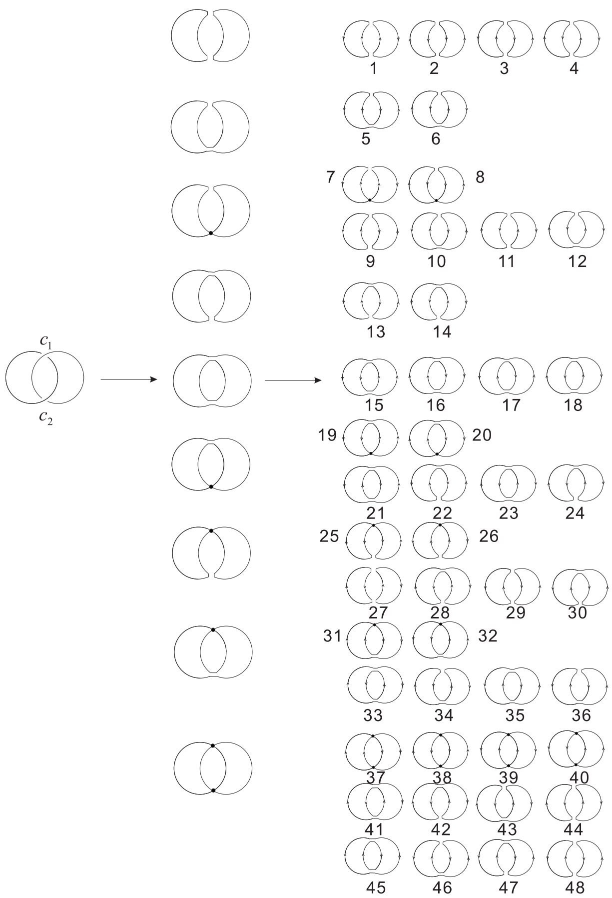

FIG. 19. Hopf link, 9 corresponding unoriented 4-valent plane graphs and 48 terms in the WF model.

For each s_i, its corresponding weights of the two crossings in HJ model, oriented 4-valent plane graphs corresponding to elements of T_{s_i} and the their corresponding weights of two crossings are listed in Table 1. It is easy to verify that $c_{HJ} = c_{WF}$.

s and its weight $[w(c_1), w(c_2)]$	Graphs corresponding to elements of T_s and their corresponding weights of crossings c_1 and c_2
s_1 $[q^{-1}, q^{-1}]$	16 $[q^{-1}, q^{-1}]$
s_2 $[q^{-1}, 1]$	20 $[q^{-1}, 1]$
s_3 $[1, q^{-1}]$	31 $[1, q^{-1}]$
s_4 $[1, 1]$	37 $[1, 1]$
s_5 $[q^{-1} - q, q^{-1} - q]$	18 $[q^{-1}, q^{-1}]$, 23 $[q^{-1}, -q]$, 33 $[-q, q^{-1}]$, 41 $[-q, -q]$
s_6 $[q^{-1} - q, q - q^{-1}]$	14 $[q^{-1}, q]$, 24 $[q^{-1}, -q^{-1}]$, 28 $[-q, q]$, 42 $[-q, -q^{-1}]$
s_7 $[q - q^{-1}, q^{-1} - q]$	06 $[q, q^{-1}]$, 12 $[q, -q]$, 34 $[-q^{-1}, q^{-1}]$, 43 $[-q^{-1}, -q]$
s_8 $[q - q^{-1}, q - q^{-1}]$	04 $[q, q]$, 11 $[q, -q^{-1}]$, 27 $[-q^{-1}, q]$, 44 $[-q^{-1}, -q^{-1}]$
s_9 $[q, q]$	02 $[q, q]$
s_{10} $[q, 1]$	08 $[q, 1]$
s_{11} $[1, q]$	25 $[1, q]$
s_{12} $[1, 1]$	38 $[1, 1]$
s_{13} $[q, q]$	01 $[q, q]$
s_{14} $[1, q]$	26 $[1, q]$
s_{15} $[q, 1]$	07 $[q, 1]$
s_{16} $[1, 1]$	39 $[1, 1]$
s_{17} $[0, 0]$	17 $[q^{-1}, q^{-1}]$, 21 $[q^{-1}, -q^{-1}]$, 35 $[-q^{-1}, q^{-1}]$, 45 $[-q^{-1}, -q^{-1}]$
s_{18} $[0, 0]$	05 $[q, q^{-1}]$, 10 $[q, -q^{-1}]$, 36 $[-q, q^{-1}]$, 46 $[-q, -q^{-1}]$
s_{19} $[0, 0]$	13 $[q^{-1}, q]$, 22 $[q^{-1}, -q]$, 30 $[-q^{-1}, q]$, 47 $[-q^{-1}, -q]$
s_{20} $[0, 0]$	03 $[q, q]$, 09 $[q, -q]$, 29 $[-q, q]$, 48 $[-q, -q]$
s_{21} $[q^{-1}, q^{-1}]$	15 $[q^{-1}, q^{-1}]$
s_{22} $[1, q^{-1}]$	32 $[1, q^{-1}]$
s_{23} $[q^{-1}, 1]$	19 $[q^{-1}, 1]$
s_{24} $[1, 1]$	40 $[1, 1]$

Table 1. s, T_s and corresponding weights.

Now we simplify the HJ model by deleting the 0-terms and obtain

Theorem 5.3. *Let L be an unoriented link diagram. Then*

$$D_L(q - q^{-1}, a^2q^{-1}) = J \sum_{\sigma} \left\{ \prod_{c} w(\sigma, c) \right\} (qa^{-1})^{\mathrm{rot}(\sigma)} [\sigma]_R(q, q^{-1}, a),$$

where $J = \frac{1}{qa^{-1}+q^{-1}a}$, the summation runs over all oriented 4-valent plane graphs obtained from L by replacing each crossing by one of the V_1, V_2, V_3, V_4, C_1, C_2, C_3, C_4, A_1, and A_3 (see Fig. 15), the product is over all crossings of L, and the weight $w(\sigma, c)$ depends on the replacement of c in σ and is shown as the subscript in Fig. 16.

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