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PROPERTIES OF ENDPOINTS AND OPPOSITE ENDPOINTS OF HEREDITARILY DECOMPOSABLE CHAINABLE CONTINUA (HDCC)

AMEEN A. ALHASSAN

Abstract. Let X be an hereditarily decomposable chainable continuum with $X = H \cup K$ where H and K are proper subcontinua and $H \cap K$ is nondegenerate. In this paper we consider properties of endpoints and opposite endpoints of such continuum X and the subcontinua H and K.

1. Introduction

The Knaster continuum K is an example of chainable continuum with exactly one endpoint k and no opposite endpoints [5]. The double Knaster continuum is decomposable with no endpoints and no opposite endpoints [2]. Let H be an arc with endpoints h_1 and h_2. If Y is a chainable continuum with $Y = H \cup K$ and k and h_1 lie in $H \cap K$ as shown in Figure 3 (see [3, p. 44]), then Y is decomposable continuum with exactly one endpoint h_2 and no opposite endpoints. Does there exist an hereditarily decomposable chainable continuum X with either zero or one endpoint? In 2017, Adikari [1] provided an answer to this question as presented here in Theorem 1.6. In Section 2 of this paper, we provide results on the possibility of endpoints and opposite endpoints of such a continuum X.

2010 *Mathematics Subject Classification.* Primary 54F15, 54F50; Secondary 54F55, 54B05.

Key words and phrases. Hereditarily decomposable continuum, chainable continuum, nondegenerate, unicoherent, irreducibility, endpoint, opposite endpoints.

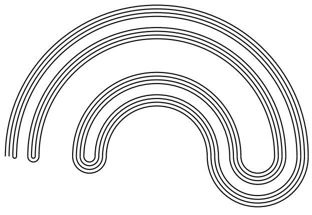

FIGURE 1. The Knaster Continuum

A *continuum* is a nonempty compact connected metric space. A *non-degenerate* continuum consists more than one point. A continuum is *decomposable* if it can be written as union of two of its proper subcontinua. A continuum is *hereditarily decomposable* if each of its nondegenerate subcontinua is decomposable. A continuum is *irreducible* between two distinct points if there is no proper subcontinuum of it containing these points. A continuum M is *unicoherent* if, whenever H and K are subcontinua of M with $M = H \cup K$, then $H \cap K$ is connected. A continuum M is *hereditarily unicoherent* if every subcontinuum of M is unicoherent. A continuum M is a *triod* if it contains a subcontinuum H such that $M - H$ has at least three components. The continuum M is *atriodic* if it contains no triod. The *composant of* $p \in M$ is the set $C_p = \{q \in M |$ there exists a proper subcontinuum H of M with $p, q \in H\}$. In a hereditarily unicoherent space, if three continua have a common point and each of them contains a point that is not in the union of the other two, then the union of the three continua is a triod.

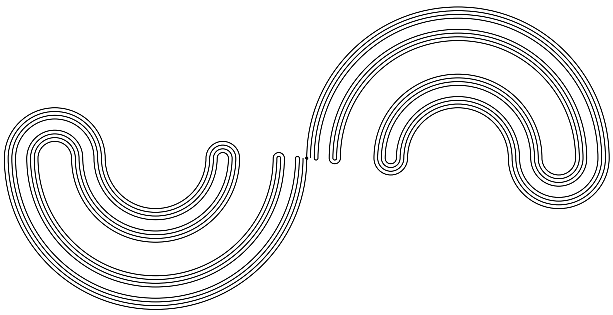

FIGURE 2. The Double Knaster Continuum

Let $C = \{C_1, ..., C_n\}$ be an indexed collection of open sets. C is a *chain* if $C_i \cap C_j \neq \emptyset$ if and only if $|j - i| \leq 1$. The mesh of chain C is defined by $mesh(C) = max\{diam(C_i) : i = 1, 2, ..., n\}$. The chain C is an

$\epsilon - chain$ if $mesh(C) < \epsilon$. A continuum M is *chainable* if for every $\epsilon > 0$ there exists an ϵ-chain C covering M. It is known that every chainable continuum is atriodic and hereditarily unicoherent.

A point p of a chainable continuum M is an *endpoint* if, for every $\epsilon > 0$, there is an ϵ-chain C covering M with $p \in C_1$ where C_1 is the first link of the chain C. Points p and q are *opposite endpoints* of a chainable continuum M if, for every $\epsilon > 0$, there exists an ϵ-chain C covering M with $p \in C_1$ and $q \in C_n$ where C_1 is the first link of the chain C and C_n is the last link of the chain C. In 1951, Bing [2] provided a characterization of endpoints for chainable continua.

Theorem 1.1 (Bing's Characterization of Endpoints)**.** *For any nondegenerate chainable continuum M and $p \in M$, the following are equivalent:*

(A) p is an endpoint of M.

(B) If each of two subcontinua of M contains p, then one of the subcontinua contains the other.

(C) Each nondegenerate subcontinuum of M containing p is irreducible from p to some other point.

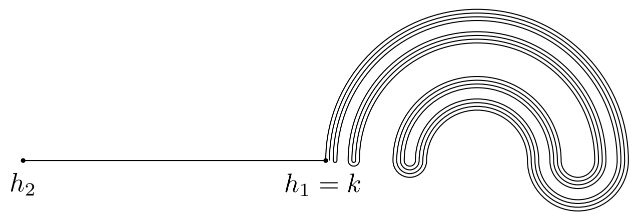

FIGURE 3. 1-Endpoint Decomposable Continuum

Remark 1.2. Opposite endpoints of a chainable continuum M are always endpoints of M, but the converse need not be true in general.

Example 1.3. Let M be the topological sin $\frac{1}{x}$-curve as shown in Figure 4. M is a decomposable chainable continuum with three endpoints x_1, x_2 and x_3. The endpoints x_1 and x_3 in the limit part of M are not opposite endpoints.

Theorem 1.4. *Let M be a nondegenerate chainable continuum with endpoints p and q. The points p and q are opposite endpoints of M if and only if M is irreducible between p and q.*

Proof. The first direction is immediate from the definition of opposite endpoints. The converse has been proved by Bing [2, Theorem 14, p. 661]. $\square$

Theorem 1.5 (The Boundary Bumping Theorem)**.** *Let X be a continuum and let A be a nonempty proper subset of X. Let C denote a component of A. Then $\bar{C} \cap Bd(A) \neq \emptyset$* [6, Theorem 5.6, p. 74].

Theorem 1.6. [1] *Every nondegenerate hereditarily decomposable chainable continuum must have at least a pair of opposite endpoints.*

Lemma 1.7. [7] *Let M be any continuum (not necessarily chainable). Then the following are equivalent:*

(a) Each subcontinuum of M is unicoherent.
(b) The intersection of any collection of subcontinua of M is connected.

Theorem 1.8. *Points x and y are in distinct composants of a nondegenerate indecomposable continuum X if and only if X is irreducible between x and y.*

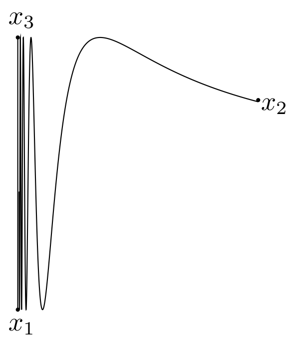

FIGURE 4. The Topological $\sin \frac{1}{x}$-curve

2. Properties of Endpoints of HDCC

Lemma 2.1. *Let M be a nondegenerate decomposable chainable continuum with $M = P \cup Q$ where P and Q are proper subcontinua of M. If points $p \in P - Q$ and $q \in Q - P$ are endpoints of M, then p and q are opposite endpoints of M.*

Proof. Suppose that H is a subcontinuum of M containing both p and q. Since p is an endpoint of M and $q \in H - P$, then P must be contained in H. Similarly, Q must be contained in H. Therefore, $H = M$ and M is irreducible between p and q. By Theorem 1.4, p and q are opposite endpoints of M. $\square$

Theorem 2.2. *Let X be an hereditarily decomposable chainable continuum with $X = H \cup K$ where H and K are proper subcontinua of X and $H \cap K$ has nonempty interior. Then, for all h_1 and h_2 in $H - K$, h_1 and h_2 are not opposite endpoints of H.*

Proof. By contradiction, suppose that h_1 and h_2 are opposite endpoints of H. By irreducibility, there is no proper subcontinuum of H containing h_1 and h_2. By Theorem 1.6, X must have at least a pair of opposite endpoints $h \in H - K$ and $k \in K - H$. Let x be a point in U where $U = int(H \cap K)$. Let C denote the component of $H - U$ containing h (Figure 5). By the Boundary Bumping Theorem, $\bar{C} \cap Bd(H - U) \neq \emptyset$. Thus, C is a proper subcontinuum of H with $h \in C$ and h_1 and h_2 cannot be both in C because otherwise a contradiction with irreducibility appears. Therefore, $C \cup K$ is a proper subcontinuum of X containing both h and k. Then h and k are not opposite endpoints of X. □

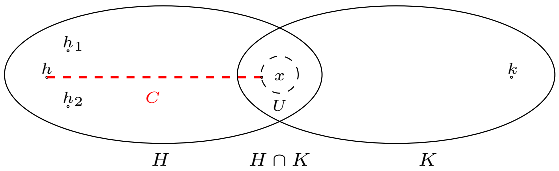

FIGURE 5

Corollary 2.3. *Let X be an hereditarily decomposable chainable continuum with $X = H \cup K$ where H and K are proper subcontinua of X and $H \cap K$ has nonempty interior. If there exist points h_1 and h_2 in H such that h_1 and h_2 are endpoints of X, then h_1 and h_2 cannot be opposite endpoints of H.*

Proof. Since $h_1, h_2 \in H$ are endpoints of X, then h_1 and h_2 cannot be in $H \cap K$. By Theorem 2.2, h_1 and h_2 cannot be opposite endpoints of H. □

Remark 2.4. Theorem 2.2 and Corollary 2.3 need not to be true if X is not hereditarily decomposable chainable continuum. For example, let H be a pseudo-arc and let K be a nondegenerate proper subcontinuum of X. Every point in H is an endpoint of H (see [2, Theorem 16 , p. 662]). Let h_1 and h_2 be endpoints of H such that they are in distinct composants of H. By Theorem 1.8, H is irreducible between h_1 and h_2. By Theorem 1.4, h_1 and h_2 are opposite endpoints of H.

Theorem 2.5. *Let X be an hereditarily decomposable chainable continuum with $X = H \cup K$ where H and K are proper subcontinua of X and $H \cap K$ is nondegenerate. If there exist points h_1 and h_2 in H such that h_1 and h_2 are endpoints of X, then h_1 and h_2 cannot be opposite endpoints of H.*

Proof. By contradiction, suppose that h_1 and h_2 are points of H which are endpoints of X and opposite endpoints of H. Since each of h_1 and h_2 is an endpoint of X, neither of them is a point of $H \cap K$. Since H is decomposable, $H = H_1 \cup H_2$ where H_1 and H_2 are proper subcontinua of H. Then neither of h_1 and h_2 is a point of $H_1 \cap H_2$ since each of h_1 and h_2 is an endpoint H. Since h_1 and h_2 are opposite endpoints of H, then H is irreducible between h_1 and h_2. Thus, neither H_1 nor H_2 contains both h_1 and h_2. Without loss of generality, assume $h_1 \in H_1 - H_2$ and $h_2 \in H_2 - H_1$. One of H_1 or H_2 say H_1, intersects K (Figure 6). Therefore, $H = H_1 \cup H_2$ and $H_1 \cup K$ are two subcontinua of X, each containing h_1 and neither a subset of the other. This contradicts the assumption that h_1 is an endpoint of X.

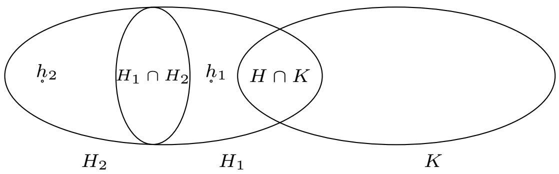

FIGURE 6

□

Theorem 2.6. *Let X be an hereditarily decomposable chainable continuum such that $X = H \cup K$ where H and K are proper subcontinua of X and $H \cap K$ is nondegenerate. Then there must exist points $h_0, k_0 \in H \cap K$ such that h_0 is an endpoint of H and k_0 is an endpoint of K.*

Proof. Since $H \cap K$ is a nondegenerate hereditarily decomposable chainable continuum, then $H \cap K$ must have at least a pair of opposite endpoints x_1 and x_2.

Claim: One of x_1 or x_2 must be an endpoint of H. A similar argument will show that one of them must be an endpoint of K.

By contradiction, if x_1 is not an endpoint of H, then there exist two subcontinua A_1 and B_1 of H both containing x_1 but neither of them containing the other. Thus, there exist points $a_1 \in A_1 - B_1$ and $b_1 \in B_1 - A_1$. Since x_1 is an endpoint of $H \cap K$, then one of $A_1 \cap K$ and

$B_1 \cap K$ contains the other. Since neither A_1 nor B_1 contains the other, it follows that $A_1 - K$ and $B_1 - K$ are not equal. Hence, at least one of $(A_1 - B_1) - K$ or $(B_1 - A_1) - K$ is nonempty. The continuum X contains no triod since X is chainable. Hence, only one of $(A_1 - B_1) - K$ or $(B_1 - A_1) - K$ can be nonempty because otherwise $A_1 \cup B_1 \cup K$ is a triod. Without loss of generality, assume $(A_1 - B_1) - K$ is nonempty and $(B_1 - A_1) - K$ is empty. Then b_1 must be contained in $H \cap K$. Thus, $A_1 \cap H \cap K$ is a proper subcontinuum of $H \cap K$ since $b_1 \notin A_1$. Therefore, $x_2 \notin A_1$ since x_1 and x_2 are opposite endpoints of $H \cap K$.

If x_2 is not an endpoint of H, then there similarly exist two subcontinua A_2 and B_2 of H both containing x_2 but neither of them contains the other with $(A_2 - B_2) - K$ nonempty and $x_1 \notin A_2$.

If both of the above occur, then $A_1 \cup A_2 \cup K$ is a triod, which is a contradiction. Therefore, one of x_1 or x_2 must be an endpoint of H. Similarly, one of x_1 or x_2 must be an endpoint of K. □

Remark 2.7. Theorem 2.6 need not be true in general if X is not hereditarily decomposable chainable continuum, i.e. there exists a decomposable chainable continuum X with $X = H \cup K$ where H and K are proper subcontinua of X and there is no endpoint in $H \cap K$ of either H or K. For example, let X be a chainable continuum as shown in Figure 7 with $X = H \cup K$ where H and K are proper subcontinua of X and $X = H \cap K$ is the double Knaster continuum which does not have any endpoint.

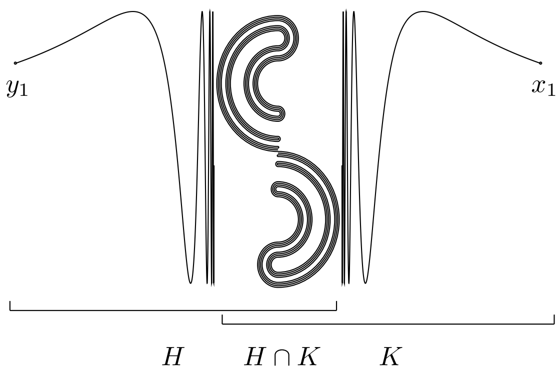

FIGURE 7. 2-Endpoints Decomposable Continuum

Question 2.1. Let X be an hereditarily decomposable chainable continuum with $X = H \cup K$ where H and K are proper subcontinua of X. Are there subcontinua A and B and a point $x \in A \cap B$ such that $A \subset H$, $B \subset K$, $X = A \cup B$ and x is an endpoint of each of A and B?

Question 2.1 has negative answer. For example, let H and K be as the following proper subcontinua (Figure 8).

$H = \{(x, sin(\frac{1}{x}))|0 < x \leq 1\} \cup \{(0, y)| -1 \leq y \leq 1\}$ and

$K = \{(x, sin(\frac{1}{x}) - 1)| -1 \leq x < 0\} \cup \{(0, y)| -2 \leq y \leq 0\}$.

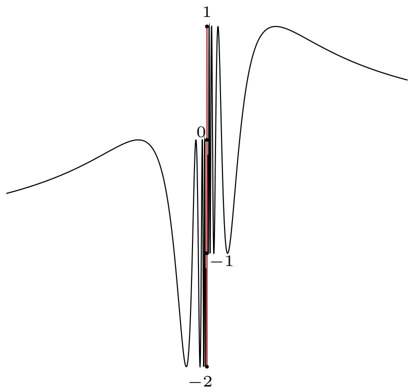

FIGURE 8

Theorem 2.8. *Let X be a nondegenerate decomposable chainable continuum with $X = H \cup K$ where H and K are proper subcontinua of X and there exists a point $h_0 \in H \cap K$ such that h_0 is an endpoint of H. If there exists a point $h \in H$ such that h is an endpoint of X, then h_0 and h are opposite endpoints of H.*

Proof. Assume h_0 and h are not opposite endpoints of H. Then there exists a proper subcontinuum H_1 of H containing both h_0 and h. There exists a point $\tilde{h} \in H$ such that $\tilde{h} \in H - H_1$. We consider two possible cases for $\tilde{h}$ as follows:

Case 1: If $\tilde{h} \in H \cap K$, then H_1 and $H \cap K$ are subcontinua of H both containing h_0, and neither of them contains the other. Therefore, h_0 is not an endpoint of H.

Case 2: If $\tilde{h} \notin H \cap K$, let U be a very small nonempty proper open subset of H with $\overline{U} \subset H - (H_1 \cup K)$ and $\tilde{h} \in U$. Let C denote the component of $H - U$ containing h as shown in Figure 9. By the Boundary Bumping Theorem, $\bar{C} \cap Bd(H - U) \neq \emptyset$, hence $C = \bar{C}$ is a proper subcontinuum of H that contains a point of $H - (H_1 \cup K)$. Thus, C and $H_1 \cup K$ are subcontinua of X both containing h, but neither of them contains the other. Then h is not an endpoint of X. $\square$

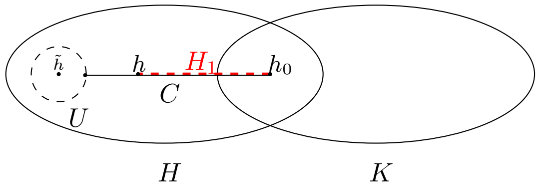

FIGURE 9

Theorem 2.9. *Let X be an hereditarily decomposable chainable continuum with $X = H \cup K$ where H and K are proper subcontinua of X. Let $h \in H$ be an endpoint of H with $h \notin H \cap K$. Then h is also an endpoint of X if and only if there exists a point $h_0 \in H \cap K$ such that h and h_0 are opposite endpoints of H.*

Proof. Suppose that there exists a point $h_0 \in H \cap K$ such that h and h_0 are opposite endpoints of H, but h is not an endpoint of X. Then there exist subcontinua X_1 and X_2 of X such that $h \in X_1 \cap X_2$ and neither of them contains the other. There exist points x_1 and x_2 in X with $x_1 \in X_1 - X_2$ and $x_2 \in X_2 - X_1$. We consider the following cases:

Case 1: If each of X_1 and X_2 is contained in H, then h is not an endpoint of H.

Case 2: Assume X_i is contained in H for one of $i = 1$ or $i = 2$. Without loss of generality, let X_1 be contained in H. Since X is hereditarily unicoherent, each of $X_2 \cap H$, $X_2 \cap K$ and $H \cap K$ is a subcontinuum of X. If $H \cap K$ is not contained in X_2, then $(X_2 \cap H) \cup (X_2 \cap K) \cup (H \cap K)$ is a triod, but X is chainable. Then $H \cap K$ must be contained in X_2. Furthermore, $X_2 \cap H \neq H$ since $x_1 \in X_1 \subset H$, but $x_1 \notin X_2$. Therefore, $X_2 \cap H$ is a proper subcontinuum of H containing h and h_0 which implies h and h_0 are not opposite endpoints of H.

Case 3: Suppose that X_i intersects each of H and K for each $i = 1$ and $i = 2$. If $H \cap K$ is not contained in X_1, then $(X_1 \cap H) \cup (X_1 \cap K) \cup (H \cap K)$ is a triod. Similarly, $(X_2 \cap H) \cup (X_2 \cap K) \cup (H \cap K)$ is a triod if $H \cap K$ is not contained in X_2. Hence, $X_1 \cap X_2$ must contain $H \cap K$. Now, we consider the following subcases:

Case 3.1: Assume $X_i \cap H = H$ for one of $i = 1$ or $i = 2$. Without loss of generality, suppose that $X_1 \cap H = H$, hence $X_2 \cap H \neq H$ because otherwise $H \cup (X_1 \cap K) \cup (X_2 \cap K)$ is a triod. Therefore, $X_2 \cap H$ is a proper subcontinuum of H containing h and h_0.

Case 3.2: If $X_i \cap H \neq H$ for each of $i = 1$ and $i = 2$, then $X_i \cap H$ is a proper subcontinuum of H containing h and h_0 for $i = 1, 2$.

To prove the converse, suppose that h is an endpoint of X. By Theorem 2.6, there must exist an endpoint $\tilde{h} \in H \cap K$ of H. By Theorem 2.8, h and $\tilde{h}$ are opposite endpoints of H. Set $h_0 = \tilde{h}$. $\square$

Example 2.10. Let H be a topological $sin\frac{1}{x}$-curve with endpoints h, h_0 and h_1. Let K be an arc with endpoints k_0 and k_1. Let X be the hereditarily decomposable chainable continuum with $X = H \cup K$, $h_0, k_0 \in H \cap K$ and $h, h_1, k_1 \notin H \cap K$ as shown in Figure 10. The following observations hold:

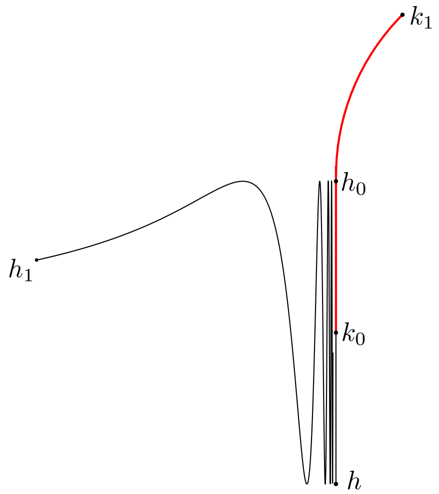

FIGURE 10

i. The point h is not an endpoint of X since H and K union the limit part of H are subcontinua of X both containing h, but neither of them contains the other.
ii. Points h and $\tilde{h}$ are not opposite endpoints of H for all $\tilde{h} \in H \cap K$ since the limit part of H is a proper subcontinuum of X.

3. QUESTIONS

Question 3.1. Let X be an hereditarily decomposable chainable continuum with $X = H \cup K$ where H and K are proper subcontinua of X and $H \cap K$ is nondegenerate. Let x be a point in $H \cap K$. Under what conditions is x an endpoint of each of H and K?

Question 3.2. Let A be a nondegenerate compact 0-dimensional subset of $\mathbb{R}^2$. Is there an hereditarily decomposable chainable continuum X in $\mathbb{R}^2$ such that the set of the endpoints of X equal to A?

Question 3.3. Let X be a continuum with $X = H \cup K$ where H and K are proper chainable subcontinua of X and $H \cap K$ is nondegenerate. When is the continuum X chainable?

In 1966, Fugate [4, Theorem 1 p. 466] proved that for chainable continua H and K with $H \cap K \neq \emptyset$, $H \cup K$ is chainable if and only if $H \cup K$ is atriodic and unicoherent.

On the other hand, it is not easy to indicate when $H \cup K$ is atriodic.

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