http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 55, 2020

Pages 77–85

http://topology.nipissingu.ca/tp/

# On some questions of Sack and Watson

by

Mehdi Parsinia

Electronically published on May 8, 2019

**Topology Proceedings**

**Web:** http://topology.auburn.edu/tp/
**Mail:** Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
**E-mail:** topolog@auburn.edu
**ISSN:** (Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

**TOPOLOGY PROCEEDINGS**
Volume 55 (2020)
Pages 77-85

E-Published on May 08, 2019

# ON SOME QUESTIONS OF SACK AND WATSON

MEHDI PARSINIA

ABSTRACT. In this paper, we investigate answers to the two questions raised in [J. Sack and S. Watson, Characterizations of ideals in intermediate $C$-rings $A(X)$ via $A$-compactifications of $X$, Internat. J. Math. Math. Sci, Volume 2013, 1-6] and the problem raised in [J. Sack and S. Watson, Characterizing $C(X)$ among intermediate $C$-rings on $X$, *Topology Proc.*, **45** (2015), 301-313].

## 1. INTRODUCTION

Throughout this article all topological spaces are assumed to be Tychonoff. For a given topological space X, $C(X)$ denotes the algebra of all real-valued continuous functions on $X$ and $C^*(X)$ denotes the subalgebra of $C(X)$ consisting of all bounded elements. A subring $A(X)$ of $C(X)$ is called an intermediate ring, if $C^*(X) \subseteq A(X)$. It is proved in [3, Proposition 2.1] that intermediate subrings are precisely absolutely convex subrings of $C(X)$ where by an absolutely convex subring of $C(X)$ we mean a subring $R$ for which if $f \in R$, $g \in C(X)$ and $|g| \leq |f|$, then $g \in R$. A subring of $C(X)$ which is isomorphic with $C(Y)$ for some Tychonoff space $Y$ is called a $C$-ring. An intermediate $C$-ring is a $C$-ring which is also an intermediate ring; for details about intermediate $C$-rings see [11]. The reader is referred to [8] and [2] for undefined terms and notations concerning $C(X)$ and intermediate rings, respectively. In this article,

2010 *Mathematics Subject Classification.* 54C30, 46E25.
*Key words and phrases.* Intermediate ring, $z$-ultrafilter, maximal ideal.

we answer questions from Sack and Watson [18] and [20] concerning mappings between ideals in intermediate rings $A(X)$ and $z$-filters on $X$. The first two, from [18] concern the relationships between mappings $\Im_A$ and $\mathcal{R}_A$ defined in Theorem 2.1 and after Remark 2.5, respectively. The first question asks for what rings $A(X)$ the two correspondences agree. We answer this in Theorem 2.6 to be precisely the intermediate $C$-rings. The second asks whether there exists an intermediate ring $A(X)$ such that $\mathcal{R}_A$ does map ideals in $A(X)$ to $z$-filter on $X$ but where the two correspondences $\mathcal{R}_A$ and $\Im_A$ do not agree. We describe why, in Remark 2.7, the answer to this question is negative. The third question is from [20, Problem 5.3] and asks to characterize those rings $A(X)$ such that $\Im_A$ maps maximal ideals to $z$-ultrafilters. We address this question in Theorem 2.9, by giving an alternative characterization in terms of $\Im_A$-ultrafilters.

## 2. The problems

The mappings $\mathcal{Z}_A$ and $\Im_A$ on an intermediate ring $A(X)$ are first introduced in [17] and [16], respectively, as follows: $\mathcal{Z}_A(f) = \{E \in Z(X) : \exists g \in A(X), fg|_{X\setminus E} = 1\}$ and $\Im_A(f) = \{E \in Z(X) : \forall H \in Z(X) \text{ and } H \subseteq X \setminus E, \exists g \in A(X), fg|_H = 1\}$. It is stated in [17, Theorem 1] (resp., [16, Proposition 2.2]) that an element $f \in A(X)$ is a non-unit of $A(X)$ if and only if $\mathcal{Z}_A(f)$ (resp., $\Im_A(f)$) is a $z$-filter on $X$. Moreover, the mapping $S_A$ is first introduced in [15] which assigns to each element $f$ of $A(X)$ the subset $S_A(f) = \{p \in \beta X : (fg)^*(p) = 0, \forall g \in A(X)\}$ of $\beta X$. It is stated in [15] that $S_A(fg) = S_A(f) \cup S_A(g)$, $S_A(f^2+g^2) = S_A(f) \cap S_A(g)$ and $S_A(f^n) = S_A(f)$ for each $f, g \in A(X)$ and each positive integer $n$. Also, $S_C(f) = \mathrm{cl}_{\beta X} Z(f)$ for each $f \in C(X)$ and $S_{C^*}(f) = Z(f^\beta)$ for each $f \in C^*(X)$. It follows from [10, Lemma 2.3] that $f \in A(X)$ is a unit element of $A(X)$ if and only if $S_A(f) = \emptyset$. In [13, Theorem 2.1 and Theorem 3.1] the mappings $\mathcal{Z}_A$ and $\Im_A$ are redefined in terms of the mapping $S_A$ and we cite them here for the sake of completeness.

**Theorem 2.1.** *For each element $f$ of an intermediate ring $A(X)$, we have*

$$\mathcal{Z}_A(f) = \{E \in Z(X) : S_A(f) \subseteq int_{\beta X}\, cl_{\beta X} E\}$$

*and*

$$\Im_A(f) = \{E \in Z(X) : \forall H \in Z(X) \text{ and } H \subseteq X \setminus E, \\ S_A(f) \subseteq int_{\beta X}\, cl_{\beta X}(X \setminus H)\}.$$

Recall that an ideal $I$ of a commutative ring $R$ is called a $z$-ideal, if $M_a(R) \subseteq I$ for each $a \in I$ in which $M_a(R)$ denotes the intersection of all the maximal ideals in $R$ containing $a$. It is well-known that $M_f(C(X)) = \{g \in C(X) : Z(f) \subseteq Z(g)\}$ and thus an ideal $I$ in $C(X)$ is a $z$-ideal if and only if $Z(f) \subseteq Z(g)$ where $f \in I$ and $g \in C(X)$ implies that $g \in I$.

Moreover, from [10, Proposition 2.7] it follows that $M_f(A(X)) = \{g \in A(X) : S_A(f) \subseteq S_A(g)\}$ for each element $f$ of an intermediate ring $A(X)$. Hence, an ideal $I$ in an intermediate ring $A(X)$ is a $z$-ideal if and only if $S_A(f) \subseteq S_A(g)$ where $f \in I$ and $g \in A(X)$ implies that $g \in I$. The next theorem investigates more topological characterizations of $z$-ideals in intermediate rings with respect to the mappings $\mathcal{Z}_A$ and $\Im_A$ which follows from Theorem 2.1.

**Theorem 2.2.** *The following statements are equivalent for an ideal $I$ in an intermediate ring $A(X)$.*

*(a) $I$ is a $z$-ideal.*

*(b) If $\mathcal{Z}_A(g) \subseteq \mathcal{Z}_A(f)$, $f \in I$ and $g \in A(X)$, then $g \in I$.*

*(c) If $\Im_A(g) \subseteq \Im_A(f)$, $f \in I$ and $g \in A(X)$, then $g \in I$.*

*Proof.* It is clear that whenever $S_A(f) \subseteq S_A(g)$, then, by Theorem 2.1, $\mathcal{Z}_A(g) \subseteq \mathcal{Z}_A(g)$ and $\Im_A(g) \subseteq \Im_A(f)$. Thus, the implications (b⇒a) and (c⇒a) are evident.

(a⇒b) Let $\mathcal{Z}_A(g) \subseteq \mathcal{Z}_A(f)$ where $f \in I$ and $g \in A(X)$. If $p \notin S_A(g)$, then there exists $E \in Z(X)$ such that $p \notin \mathrm{cl}_{\beta X} E$ and $S_A(g) \subseteq \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} E$ which, by Theorem 2.1, implies that $E \in \mathcal{Z}_A(g)$ and hence, by our hypothesis, $E \in \mathcal{Z}_A(f)$. It follows that $S_A(f) \subseteq \mathrm{cl}_{\beta X} E$. Therefore, $p \notin S_A(f)$ which means $S_A(f) \subseteq S_A(g)$. Consequently, as $I$ is a $z$-ideal, we have $g \in I$.

(a⇒c) Let $\Im_A(g) \subseteq \Im_A(f)$ where $f \in I$ and $g \in A(X)$. If $p \notin S_A(g)$, then, similar to the proof of (a⇒b), there exists $E \in Z(X)$ such that $p \notin \mathrm{cl}_{\beta X} E$ and $E \in \mathcal{Z}_A(g)$. By our hypothesis and the fact that $\mathcal{Z}_A(g) \subseteq \Im_A(g)$, $E \in \Im_A(f)$. Also, as $p \notin \mathrm{cl}_{\beta X} E$, there exists $F \in Z(X)$ such that $p \in \mathrm{cl}_{\beta X} F$ and $\mathrm{cl}_{\beta X} E \cap \mathrm{cl}_{\beta X} F = \emptyset$. It follows that $E \cap F = \emptyset$ and hence, as $E \in \Im_A(f)$, by Theorem 2.1 we would have $S_A(f) \subseteq \beta X \setminus \mathrm{cl}_{\beta X} F$. Therefore, $p \notin S_A(f)$ which means $S_A(f) \subseteq S_A(g)$ and hence, as $I$ is a $z$-ideal, we would have $g \in I$. □

It is well-known that $z$-ideals of $C(X)$ are in a one-one correspondence with $z$-filters on $X$, in the sense that whenever $I$ is a $z$-ideal in $C(X)$, then $Z(I) = \{Z(f) : f \in I\}$ is a $z$-filter on $X$ and whenever $\mathcal{F}$ is a $z$-filter on $X$, then $Z^{-1}(\mathcal{F}) = \{f \in C(X) : Z(f) \in \mathcal{F}\}$ is a $z$-ideal in $C(X)$. However, the same statement is not true for intermediate rings, in general. In fact, for a $z$-ideal $I$ in $A(X)$, $Z(I)$ is not necessarily a $z$-filter on $X$. It follows from [10, Proposition 2.7] that $z$-ideals of an intermediate ring $A(X)$ are in a one-one correspondence with $z_A^{\beta}$-filters on $\beta X$, see [10] for more details. It is inferred from [3, Theorem 2.2] that $z$-ideals of an intermediate ring $A(X)$ are in one-one correspondence with $z$-filters on $X$ if and only if $A(X) = C(X)$. By restricting the class of $z$-ideals to the class of $\mathcal{Z}_A$-ideals and the class of $z$-filters on $X$ to the class of $\mathcal{Z}_A$-filters, a correspondence between these subclasses could be defined, see [19]. Recall that an ideal $I$ in $A(X)$ is called a $\mathcal{Z}_A$-ideal, if $\mathcal{Z}^{-1}\mathcal{Z}_A(I) = I$ and a $z$-filter $\mathcal{F}$ is called a $\mathcal{Z}_A$-filter, if $\mathcal{Z}_A\mathcal{Z}_A^{-1}(\mathcal{F}) = \mathcal{F}$. The class of $\Im_A$-ideals and $\Im_A$-filters are defined similar to $\mathcal{Z}_A$-ideals and $\mathcal{Z}_A$-filters, respectively. Note that, for a $z$-filter $\mathcal{F}$ we designate by $\mathcal{Z}_A^{-1}(\mathcal{F})$ and $\Im_A^{-1}(\mathcal{F})$ the sets $\{f \in A(X) : \mathcal{Z}_A(f) \subseteq \mathcal{F}\}$ and $\{f \in A(X) : \Im_A(f) \subseteq \mathcal{F}\}$, respectively. Also, for a subset $S$ of $A(X)$, $\mathcal{Z}_A(S)$ and $\Im_A(S)$ denote the sets $\bigcup_{f\in S} \mathcal{Z}_A(f)$ and $\bigcup_{f\in S} \Im_A(f)$, respectively. It is shown in [9, Theorem 2.12] that every $z$-filter on $X$ is a $\mathcal{Z}_A$-filter if and only if $X$ is a $P$-space. It follows that the mapping $\mathcal{Z}_A^{-1}$ is a one-one correspondence between $z$-filters on $X$ and $\mathcal{Z}_A$-ideals of $A(X)$ if and only if $X$ is a $P$-space. We recall that a topological space $X$ is called a $P$-space, if $Z(f)$, for each $f \in C(X)$, is open in $X$, or equivalently, $C(X)$ is a regular ring, see [8, 14.29] for more details. It is easy to see that whenever $X$ is a $P$-space, then every $z$-filter on $X$ is a $\Im_A$-filter. However, the converse of this fact does not hold, in general, see the next example which investigates an intermediate ring $A(X)$ for which every $z$-filter is a $\Im_A$-filter, however, $X$ is not a $P$-space.

**Example 2.3.** Let $X = (0, 1) \cup \{2, 3, 4, ...\}$ and let $A(X)$ be the ring of all continuous functions on $X$ which are bounded on $\{2, 3, 4, ...\}$. As shown in [9, Example 3.13], every $z$-filter on $X$ is a $\Im_A$-filter. However, $X$ is not a $P$-space.

The next statement gives some sufficient conditions under which every $z$-filter on $X$ would be a $\Im_A$-filter. Note that, for each $A \subseteq C(X)$, we set $\upsilon_A X = \{p \in \beta X : f^*(p) < \infty, \forall f \in A\}$. It follows that $\upsilon_C X = \upsilon X$ and $\upsilon_{C^*} X = \beta X$. Also, $\upsilon X \subseteq \upsilon_A X$ for each $A \subseteq C(X)$. The spaces $\upsilon_A X$ are of particular importance in the context of intermediate $C$-rings, see [5] and [11]. For more details about the spaces $\upsilon_A X$ see [14].

**Proposition 2.4.** *Let $X$ be a $P$-space and $A(X)$ be an intermediate $C$-ring. Then every $z$-filter on $X$ is a $\Im_A$-filter.*

*Proof.* Let $\mathcal{F}$ be a $z$-filter on $X$ and $Z \in \mathcal{F}$. Clearly, there exists some $f \in A(X)$ such that $Z = Z(f)$. As $X$ is a $P$-space, $Z(f)$ is a clopen subset of $X$ and thus $\mathrm{cl}_{\beta X} Z(f)$ is a clopen subset of $\beta X$. Thus, $\mathrm{cl}_{\upsilon_A X} Z(f) = \mathrm{cl}_{\beta X} Z(f) \cap \upsilon_A X$ is a clopen subset and hence a zero-set of $\upsilon_A X$. Thus, there exists $g \in A(X)$ such that $\mathrm{cl}_{\upsilon_A X} Z(f) = Z(g^{\upsilon_A})$. It follows that $Z(f) = Z(g)$ and hence $\mathrm{cl}_{\upsilon_A X} Z(g) = \mathrm{cl}_{\upsilon_A X} Z(f) = Z(g^{\upsilon_A})$. Using this, our hypothesis and Theorem 2.1 we could observe that $\Im_A(g) = (Z(g)) = (Z(f))$. Hence, $g \in \Im_A^{-1}(\mathcal{F})$ and $Z = Z(f) \in \Im_A(g) \subseteq \Im_A \Im_A^{-1}(\mathcal{F})$. Therefore, $\mathcal{F}$ is a $\Im_A$-filter. $\square$

*Remark* 2.5. Note that, in Proposition 2.4, the condition that $X$ is a $P$-space could not be removed for being every $z$-filter a $\Im_A$-filter whenever $A(X)$ is an intermediate $C$-ring. For instance, let $X = [0,\infty)$, $A(X) = C^*(X)$ and $\mathcal{U}_{\mathbb{N}}$ be any free $z$-ultrafilter on $X$ containing $\mathbb{N}$. Also, let $M = \Im_A^{-1}(\mathcal{U}_{\mathbb{N}})$. As is proved in [16, Example 4.5], $\Im_A(M)$ is properly contained in $\mathcal{U}_{\mathbb{N}}$ which implies that $\mathcal{U}_{\mathbb{N}}$ is not a $\Im_A$-filter. It should be noted that from [9, Theorem 2.12] it follows that whenever every $z$-filter on $X$ is a $\mathcal{Z}_A$-filter, then every $z$-ideal in $A(X)$ would be a $\mathcal{Z}_A$-ideal, since, by [6, Theorem 3.13], every $\mathcal{Z}_A$-ideal is an intersection of maximal ideals. However, the same fact is not true for the case that every $z$-filter on $X$ is a $\Im_A$-filter, in general. For example, let $X = \mathbb{N}$ and $A(X) = C^*(\mathbb{N})$. By Proposition 2.4, every $z$-filter on $X$ is a $\Im_A$-filter, however, if we let $I = (\frac{1}{n})$ be the ideal in $A(X)$ generated by $\frac{1}{n}$. Then, as stated in [9, Example 2.15], $I$ is not a $\Im_A$-ideal. We show that $I$ is a $z$-ideal in $A(X)$. Let $S_A(f) \subseteq S_A(g)$ where $f \in I$ and $g \in A(X)$. We are to show that $g \in I$. As $f \in I$, we have $\beta\mathbb{N} \setminus \mathbb{N} = Z((\frac{1}{n})^\beta) \subseteq S_A(f)$. It follows that $Z((\frac{1}{n})^\beta) \subseteq \mathrm{int}_{\beta X} Z(g^\beta)$ and thus there exists $h \in A(X)$ such that $g = \frac{1}{n}h$; i.e., $g \in I$. Note that $S_A(f) = S_{C^*}(f) = Z(f^\beta)$ for each $f \in A(X)$. Also, we can easily observe that, as $C^*(X) \cong C(\beta X)$, whenever $f, g \in C^*(X)$ are such that $Z(f^\beta) \subseteq \mathrm{int}_{\beta X} Z(g^\beta)$, then there exists some $h \in C^*(X)$ such that $g = fh$.

In [18, Theorem 6] it is proved that whenever $A(X)$ is an intermediate $C$-ring, then $\Im_A(f) = \mathcal{R}_A(f)$ for each $f \in A(X)$ in which $\mathcal{R}_A(f) = \{E \in Z(X) : Z(f^{\upsilon_A}) \subseteq \mathrm{cl}_{\upsilon_A X} E\}$. The authors then left open the following two questions:

- precisely what rings $A(X)$ are such that $\Im_A = \mathcal{R}_A$?
- whether there exists a ring $A(X)$ such that $\mathcal{R}_A$ does map ideals in $A(X)$ to $z$-filters on $X$, but $\Im_A \neq \mathcal{R}_A$?

By the next statement, which is, in fact, a characterization of intermediate $C$-rings, we answer the first question, answer of the second question then follows. We note that the left to right side of this statement has also been proved in [18, Theorem 6] and we investigate here a new proof for this fact by means of the mapping $S_A$.

**Theorem 2.6.** *Let $A(X)$ be an intermediate ring of $C(X)$. Then $A(X)$ is a $C$-ring if and only if $\Im_A(f) = \mathcal{R}_A(f)$ for each $f \in A(X)$.*

*Proof.* ($\Rightarrow$) Let $E \in \Im_A(f)$. If $Z(f^{\upsilon_A}) \not\subseteq \mathrm{cl}_{\upsilon_A X} E$. Then there exists some $p \in Z(f^{\upsilon_A})$ such that $p \notin \mathrm{cl}_{\upsilon_A X} E$. Thus, there exists some $g \in A(X)$ such that $p \in \mathrm{cl}_{\beta X} Z(g^{\upsilon_A})$ and $\mathrm{cl}_{\beta X} Z(g^{\upsilon_A}) \cap \mathrm{cl}_{\upsilon_A X} E = \emptyset$. It follows that $Z(g) \subseteq X \setminus E$ and thus, by [11, Theorem 2.2], $S_A(f) = \mathrm{cl}_{\beta X} Z(f^{\upsilon_A}) \subseteq \mathrm{cl}_{\beta X}(X \setminus Z(g))$, since $E \in \Im_A(f)$. This is a contradiction, since, $p \in Z(f^{\upsilon_A}) \subseteq S_A(f)$. Hence, $\Im_A(f) \subseteq \mathcal{R}_A(f)$. Now, assume that $E \in \mathcal{R}_A(f)$ but $E \notin \Im_A(f)$. Thus, there exists some zero-set $H \subseteq X \setminus E$

such that $S_A(f) \not\subseteq \beta X \setminus \mathrm{cl}_{\beta X} H$ which by [11, Theorem 2.2] implies that $\mathrm{cl}_{\beta X} Z(f^{\upsilon_A}) \cap \mathrm{cl}_{\beta X} H \neq \emptyset$. But, by the hypothesis, $Z(f^{\upsilon_A}) \subseteq E$ and hence $Z(f^{\upsilon_A}) \cap H = \emptyset$ which implies that $\mathrm{cl}_{\beta X} Z(f^{\upsilon_A}) \cap \mathrm{cl}_{\beta X} H = \emptyset$. This contradiction means that $\mathcal{R}_A(f) \subseteq \Im_A(f)$ and the equality follows.

($\Leftarrow$) Assume on the contrary that $A(X)$ is not a $C$-ring. Thus, by [11, Theorem 2.2], there exists $f \in A(X)$ such that $S_A(f) \neq \mathrm{cl}_{\beta X} Z(f^{\upsilon_A})$. Hence, there exists some $p \in \beta X$ such that $p \in S_A(f)$ but $p \notin \mathrm{cl}_{\beta X} Z(f^{\upsilon_A})$. Thus, there exists a zero-set $E$ such that $p \notin \mathrm{cl}_{\beta X} E$ and $\mathrm{cl}_{\beta X} Z(f^{\upsilon_A}) \subseteq \mathrm{cl}_{\beta X} E$. As $p \notin \mathrm{cl}_{\beta X} E$, there exists a zero-set $F$ such that $p \in \mathrm{cl}_{\beta X} F$ and $\mathrm{cl}_{\beta X} F \cap \mathrm{cl}_{\beta X} E = \emptyset$. It follows that $Z(f^{\upsilon_A}) \subseteq \mathrm{cl}_{\upsilon_A X} E$. Hence, by the hypothesis, $E \in \Im_A(f)$. This implies by the definition of $\Im_A(f)$ and the fact $F \in Z(X)$ where $F \subseteq X \setminus E$ that $S_A(f) \subseteq \beta X \setminus \mathrm{cl}_{\beta X} F$ which is a contradiction, since, $p \in S_A(f) \cap \mathrm{cl}_{\beta X} F$. $\square$

*Remark* 2.7. Let $A(X)$ be an intermediate ring which is not a $C$-ring. Thus, by [11, Theorem 2.2], there exists some $f \in A(X)$ such that $S_A(f) \neq \mathrm{cl}_{\beta X} Z(f^{\upsilon_A})$. Let $p \in \beta X$ be such that $p \in S_A(f)$ and $p \notin \mathrm{cl}_{\beta X} Z(f^{\upsilon_A})$. Hence, there exists some $g \in A(X)$ such that $p \in S_A(g)$ and $S_A(g) \cap \mathrm{cl}_{\beta X} Z(f^{\upsilon_A}) = \emptyset$. As $\mathrm{cl}_{\beta X} Z(g^{\upsilon_A}) \subseteq S_A(g)$, we would have $\mathrm{cl}_{\beta X} Z(f^{\upsilon_A}) \cap \mathrm{cl}_{\beta X} Z(g^{\upsilon_A}) = \mathrm{cl}_{\beta X}(Z(f^{\upsilon_A}) \cap Z(g^{\upsilon_A})) = \mathrm{cl}_{\beta X} Z((f^2 + g^2)^{\upsilon_A}) = \emptyset$. We set $h = f^2 + g^2$. It clearly follows that $h \in A(X)$, $Z(h^{\upsilon_A}) = \emptyset$ and $S_A(h) \neq \emptyset$, since, $p \in S_A(f) \cap S_A(g) = S_A(h)$. This means that $h$ is a non-unit in $A(X)$. Thus, the principal ideal in $A(X)$ generated by $h$, $(h)$, is a proper ideal. It follows that $Z(X) = \mathcal{R}_A(h) \subseteq \mathcal{R}_A((h))$, i.e., $\mathcal{R}_A((h))$ is not a $z$-filter on $X$. Consequently, if $A(X)$ is not a $C$-ring, then $\mathcal{R}_A$ does not map ideals of $A(X)$ to $z$-filters on $X$. Thus, the answer of the second mentioned question is negative, i.e., there is no ring $A(X)$ for which $\mathcal{R}_A$ maps ideals of it to the $z$-filters and $\mathcal{R}_A \neq \Im_A$. Indeed, if $A(X)$ is a $C$-ring, then, by Theorem 2.6, $\Im_A = \mathcal{R}_A$ and if $A(X)$ is not a $C$-ring, then $\mathcal{R}_A$ does not map ideals of $A(X)$ to $z$-filters.

The following problem has been raised in [20].

**Problem** [20, Problem 5.3]**.** Characterize those intermediate rings $A(X)$ for which $\Im_A(M)$ is a $z$-ultrafilter on $X$ whenever $M$ is a maximal ideal in $A(X)$.

In [16, Theorem 4.7] it is shown that $\Im_A^{-1}(\mathcal{U})$ is a maximal ideal in $A(X)$ for each $z$-ultrafilter $\mathcal{U}$ on $X$. The next lemma investigates a more precise version for this fact. Recall that $z$-ultrafilters on $X$ are precisely the $z$-filters $\mathcal{U}^p = \{Z \in Z(X) : p \in \mathrm{cl}_{\beta X} Z\}$ for $p \in \beta X$. Remark that we use $M_A^p$ to designate the set $\{f \in A(X) : p \in S_A(f)\}$ for each $p \in \beta X$.

Evidently, $M_C^p = M^p$ and $M_{C^*}^p = M^{*p}$. From [15, Theorem 2.8 and Theorem 2.9] it follows that maximal ideals of an intermediate ring $A(X)$ are precisely the ideals $M_A^p$ for $p \in \beta X$.

**Lemma 2.8.** *Let $A(X)$ be an intermediate ring of $C(X)$. Then, $\Im_A^{-1}(\mathcal{U}^p) = M_A^p$ for each $p \in \beta X$.*

*Proof.* It suffices to show that $M_A^p \subseteq \Im_A^{-1}(\mathcal{U}^p)$ for each $p \in \beta X$. Let $f \in M_A^p$ and $E \in \Im_A(f)$. If $E \notin \mathcal{U}^p$, then $p \notin \mathrm{cl}_{\beta X} E$. Thus, there exists $Z \in Z(X)$ such that $p \in \mathrm{cl}_{\beta X} Z$ and $\mathrm{cl}_{\beta X} Z \cap \mathrm{cl}_{\beta X} E = \emptyset$. It follows that $Z \subseteq X \setminus E$ and hence, as $E \in \Im_A(f)$, by Theorem 2.1, $S_A(f) \subseteq \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X}(X \setminus Z)$ which contradicts the fact that $p \in \mathrm{cl}_{\beta X} Z$. Therefore, $\Im_A(f) \subseteq \mathcal{U}^p$; i.e., $f \in \Im_A^{-1}(\mathcal{U}^p)$. □

The following statement could be considered as an answer to the mentioned problem. Be reminded that a $z$-filter $\mathcal{F}$ is called a $\Im_A$-filter if it satisfies the equality $\Im_A^{-1}\Im_A(\mathcal{F}) = \mathcal{F}$. A maximal $\Im_A$-filter is a maximal $\Im_A$-ultrafilter.

**Theorem 2.9.** *Let $A(X)$ be an intermediate ring of $C(X)$. Then $\Im_A(M)$ is a $z$-ultrafilter on $X$ for each maximal ideal $M$ in $A(X)$ if and only if every $z$-ultrafilter on $X$ is a $\Im_A$-ultrafilter.*

*Proof.* ($\Rightarrow$) We first show that $\Im_A$-ultrafilters on $X$ are precisely the $\Im_A$-filters $\Im_A(M)$ where $M$ is a maximal ideal in $A(X)$. Let $M$ be a maximal ideal in $A(X)$ and $\mathcal{F}$ be a $\Im_A$-filter on $X$ containing $\Im_A(M)$. It follows that $M = \Im_A^{-1}\Im_A(M) \subseteq \Im^{-1}(\mathcal{F})$. By [13, Answer to Question 3], $\Im_A^{-1}(\mathcal{F})$ is an ideal in $A(X)$ which implies that $M = \Im_A^{-1}(\mathcal{F})$ and hence $\Im_A(M) = \Im_A\Im_A^{-1}(\mathcal{F}) = \mathcal{F}$. Now, let $\mathcal{U}^p$ be a $z$-ultrafilter on $X$ for some $p \in \beta X$. Then, by Lemma 2.8, $\Im_A^{-1}(\mathcal{U}^p)$ is the maximal ideal $M_A^p$ in $A(X)$. By our hypothesis, $\Im_A(M_A^p)$ is a $z$-ultrafilter on $X$. Thus, we clearly have $\Im_A(M_A^p) = \mathcal{U}^p$ which means that $\mathcal{U}^p$ is a $\Im_A$-ultrafilter.

($\Leftarrow$) This is evident by the fact that every $\Im_A$-ultrafilter is of the form $\Im_A(M)$ for some maximal ideal $M$ in $A(X)$. □

*Remark* 2.10. In [19, Theorem 5.1] it is stated that whenever $A(X)$ is an intermediate $C$-ring for which every $z$-filter is a $\Im_A$-filter, then $\Im_A(M)$ is a $z$-ultrafilter on $X$ for each maximal ideal $M$ in $A(X)$. However, it easily follows from Theorem 2.9 that whenever $A(X)$ is an intermediate ring for which every $z$-filter is a $\Im_A$-filter, then $\Im_A(M)$ is a $z$-ultrafilter for each maximal ideal $M$.

It is shown in [20, Theorem 5.2] that whenever $X$ is a $P$-space and $A(X)$ is an intermediate $C$-ring of $C(X)$, then $\Im_A(M_A^p)$ is a $z$-ultrafilter for each $p \in \beta X$. We provide an extended version of this fact by the next theorem which would be a partial answer to the problem.

**Theorem 2.11.** *Let $X$ be a $P$-space and $A(X)$ be an intermediate ring of $C(X)$. Then $\Im_A(M_A^p)$ is a $z$-ultrafilter on $X$ for each $p \in \beta X$.*

*Proof.* Let $p \in \beta X$ be given. It is clear that $\mathcal{Z}_A(M_A^p) \subseteq \Im_A(M_A^p)$. Also, if $E \in \Im_A(M_A^p)$, then as $X$ is a $P$-space, $X \setminus E$ is a zero-set. Hence, Theorem 2.1 implies that $E \in \mathcal{Z}_A(M_A^p)$. Thus, $\Im_A(M_A^p) = \mathcal{Z}_A(M_A^p)$. Moreover, by Theorem 2.1, $\mathcal{Z}_A(M_A^p) = \{E \in Z(X) : p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} E\} = \{E \in Z(X) : p \in \mathrm{cl}_{\beta X} E\} = Z(M^p)$ which is clearly a $z$-ultrafilter on $X$. Therefore, $\Im_A(M_A^p)$ is a $z$-ultrafilter on $X$. □

**Acknowledgment.** The author would like to thank to the well-informed referee for reading this paper carefully and giving useful suggestions.

## References

[1] S.K. Acharyya and D. De, *An interesting class of ideals in subalgebras of $C(X)$ containing $C^*(X)$*, Comment. Math. Univ. Carolin., **48** (2007), 273–280.

[2] A.R. Aliabad and M. Parsinia, *Remarks on subrings of $C(X)$ of the form $I + C^*(X)$*, Quaest. Math., **40** (1) (2017), 63–73.

[3] F. Azarpanah and M. Parsinia, *On the sum of $z$-ideals in subrings of $C(X)$*, J. Commut. Algebra., to appear.

[4] H.L. Byun and S. Watson, *Prime and maximal ideals in subrings of $C(X)$*, Topology Appl., **40** (1991), 45–62.

[5] D. De and S.K. Acharyya, *Characterization of function rings between $C^*(X)$ and $C(X)$*, Kyungpook Math. J., **46** (2006), 503–507.

[6] J.M. Dominguez, J. Gomez, *Intersections of maximal ideals in algebras between $C^*(X)$ and $C(X)$*, Topology Appl., **98** (1999), 149–165.

[7] J.M. Dominguez, J. Gomez and M.A. Mulero, *Intermediate algebras between $C^*(X)$ and $C(X)$ as ring of fractions of $C^*(X)$*, Topology Appl., **77** (1997), 115–130.

[8] L. Gillman and M. Jerison, *Rings of Continuous Functions.* Springer-Verlag, New York, 1978.

[9] W. Murray, J. Sack and S. Watson, *$P$-spaces and intermediate rings of continuous functions*, Rocky Mountain J. Math., **47** (8) (2017), 2757–2775.

[10] M. Parsinia, *Remarks on $LBI$-subalgebras of $C(X)$*, Comment. Math. Univ. Carolin., **57** (2) (2016), 261–270.

[11] ———, *Remarks on intermediate $C$-rings of $C(X)$*, Quaest. Math., **41** (5) 2018, 675–682.

[12] ———, *$R$-$P$-spaces and subrings of $C(X)$*, FILOMAT, **32** (1) (2018), 319–328.

[13] ———, *On the mappings $\mathcal{Z}_A$ and $\Im_A$ in intermediate rings of $C(X)$*, Comment. Math. Univ. Carolin., **59** (3) (2018), 383–390.

[14] ———, *Mappings to realcompactifications*, Categ. General Alg. Struct. Appl., **10** (1) (2019), 107–116.

[15] D. Plank, *On a class of subalgebras of $C(X)$ with applications to $\beta X - X$*, Fund. Math., **64** (1969), 41–54.

[16] P. Panman, J. Sack and S.Watson, *Correspondence between ideals and $z$-filters for rings of continuous functions between $C^*$ and $C$*, Commentationes Mathematicae., **52** (1) (2012), 11–20.

[17] H. Redlin and S. Watson, *Maximal ideals in subalgebras of $C(X)$*, Proc. Amer. Math. Soc., **100** (1987), 763–766.

[18] J. Sack and S. Watson, *Characterizations of ideals in intermediate $C$-rings $A(X)$ via $A$-compactifications of $X$*, Internat. J. Math. Math. Sci, Volume 2013, 1–6.

[19] ———, *$C$ and $C^*$ among intermediate rings*, Topology Proc., **43** (2014), 69–82.

[20] ———, *Characterizing $C(X)$ among intermediate $C$-rings on $X$*, Topology Proc., **45** (2015), 301–313.

Shahid Chamran University of Ahvaz, Ahvaz, Iran.
*Email address*: parsiniamehdi@gmail.com