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HOMOGENEOUS CONTINUA AND NON-BLOCKERS

SERGIO MACÍAS

Abstract. We prove that the decomposition given by C. Piceno in Nonblockers in homogeneous continua, Topology Appl., 249 (2018), 127-134, coincides with already known decompositions, for decomposable homogeneous continua, for one-dimensional indecomposable homogeneous continua and homogeneous indecomposable continua admitting an essential map onto the figure eight continuum. We give the description of the set of non-blockers of singletons for a couple of classes of continua.

1. Introduction

The purpose of this note is to show that the decomposition given by C. Piceno in [12, Theorem 4.2] coincides with already known decompositions for decomposable homogeneous continua (Theorem 3.5), for one-dimensional indecomposable homogeneous continua (Theorem 3.11) and homogeneous indecomposable continua admitting an essential map onto the figure eight continuum (Theorem 3.9). We include the description of the set of non-blockers of singletons for a couple of classes of continua.

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2. Definitions

Given a metric space Z, a *decomposition* of Z is a family $\mathcal{G}$ of nonempty and mutually disjoint subsets of Z such that $\bigcup \mathcal{G} = Z$. A decomposition $\mathcal{G}$ of a metric space Z is said to be *upper semicontinuous* if the quotient map $q\colon Z \twoheadrightarrow Z/\mathcal{G}$ is closed. The decomposition $\mathcal{G}$ is *lower semicontinuous* provided that q is open. Then $\mathcal{G}$ is *continuous* if q is open and closed. A decomposition $\mathcal{G}$ is *monotone* if each element of $\mathcal{G}$ is connected.

A *continuum* is a compact, connected, metric space. A continuum is *decomposable* if it is the union of two of its proper subcontinua. A continuum is *indecomposable* if it is not decomposable. A subcontinuum K of a continuum X is *terminal* if for each subcontinuum L of X such that $L \cap K \neq \emptyset$, we have that either $L \subset K$ or $K \subset L$. A continuum X is *aposyndetic* provided that for each pair of points x_1 and x_2 of X, there exists a subcontinuum W of X such that $x_1 \in Int_X(W) \subset W \subset X \setminus \{x_2\}$. The *figure eight continuum* is any space homeomorphic to the union of two circles whose intersection is just one point.

Given a continuum X, a decomposition $\mathcal{G}$ of X is *terminal* provided that each element of $\mathcal{G}$ is a proper nondegenerate terminal subcontinuum of X. Information about terminal decompositions in homogeneous conitnua can be found in [13].

Given a continuum X, we define the set function $\mathcal{T}$ as follows: if A is a subset of X then

$$\mathcal{T}(A) = X \setminus \{x \in X \mid \text{there exists a subcontinuum } W \text{ of } X \text{ such that } x \in Int_X(W) \subset W \subset X \setminus A\}.$$

Let us observe that for any subset A of X, $\mathcal{T}(A)$ is a closed subset of X and $A \subset \mathcal{T}(A)$.

A *map* is a continuous function. A surjective map $f\colon Z \twoheadrightarrow Y$ between continua is *monotone* provided that $f^{-1}(C)$ is connected for every connected subset C of Y. The surjective map f is *atomic* if for each subcontinuum K of Z such that $f(K)$ is nondegenerate, then $K = f^{-1}(f(K))$.

Given a continuum X, we consider its hyperspace of subcontinua, denoted by $\mathcal{C}(X)$ and the space of singletons, denoted by $\mathcal{F}_1(X)$, both topologized with the Hausdorff metric [11, (0.1)]. An *order arc* in $\mathcal{C}(X)$ is a map $\alpha\colon [0,1] \to \mathcal{C}(X)$ such that if $0 \leq s < t \leq 1$, then $\alpha(s) \subsetneq \alpha(t)$. A *Whitney map* for $\mathcal{C}(X)$ is a map $\mu\colon \mathcal{C}(X) \twoheadrightarrow [0,1]$ such that $\mu(X) = 1$, $\mu(A) = 0$ if $A \in \mathcal{F}_1(X)$ and if $A \subsetneq B$, then $\mu(A) < \mu(B)$. A *Whitney level*, is a set of the form $\mu^{-1}(t)$ for some $t \in [0,1]$.

A set A *does not block the singletons of* X provided that for each $x \in X\setminus A$, there exists an order arc $\alpha\colon [0,1] \to \mathcal{C}(X)$ such that $\alpha(0) = \{x\}$, $\alpha(1) = X$ and for every $t \in [0,1)$, $\alpha(t) \cap A = \emptyset$. The family of subsets of X that do not block singletons is denoted by $\mathcal{NB}(\mathcal{F}_1(X))$.

3. Results

Lemma 3.1. *If G is a proper terminal subcontinuum of X, $A \in \mathcal{NB}(\mathcal{F}_1(X))$ and $A \cap G \neq \emptyset$, then $G \subset A$.*

Proof. Suppose that $A \in \mathcal{NB}(\mathcal{F}_1(X))$ and G is a proper terminal subcontinuum of X such that $A \cap G \neq \emptyset$ and there exists $z \in G \setminus A$. Then there exists an order arc $\alpha\colon [0,1] \to \mathcal{C}(X)$ such that $\alpha(0) = \{z\}$, $\alpha(1) = X$ and for every $t \in [0,1)$, $\alpha(t) \cap A = \emptyset$. Let $t_0 < 1$ be such that $\alpha(t_0) \setminus G \neq \emptyset$. Since G is a terminal subcontinuum of X, we obtain that $G \subset \alpha(t_0)$, a contradiction to the fact that $\alpha(t_0) \cap A = \emptyset$. Therefore, $G \subset A$. $\square$

Corollary 3.2. *If X is a continuum admitting a terminal decomposition $\mathcal{G}$ such that the elements of $\mathcal{G}$ are nondegenerate, then $\mathcal{NB}(\mathcal{F}_1(X)) \cap \mathcal{F}_1(X) = \emptyset$.*

As a consequence of [10, 3.1.21 and 5.1.18] and Lemma 3.1, we have:

Corollary 3.3. *Let X be a decomposable nonaposyndetic homogeneous continuum and let x be a point in X. If A is a proper subset of $\mathcal{T}(\{x\})$, then $A \notin \mathcal{NB}(\mathcal{F}_1(X))$.*

Given a continuum X, for each element x of X, define

$$\pi(x) = \bigcap\{A \in \mathcal{NB}(\mathcal{F}_1(X)) \mid x \in A\}.$$

From Lemma 3.1, we obtain:

Lemma 3.4. *Let X be a continuum and let x_0 be a point of X. If G_{x_0} is a terminal subcontinuum of X and $\pi(x_0) \neq \emptyset$, then $G_{x_0} \subset \pi(x_0)$.*

Theorem 3.5. *Let X be a decomposable homogeneous continuum and suppose that $\mathcal{NB}(\mathcal{F}_1(X)) \neq \emptyset$. Then $\mathcal{T}(\{x\}) = \pi(x)$ for all points x of X.*

Proof. Let X be a decomposable homogeneous continuum. Since $\mathcal{NB}(\mathcal{F}_1(X)) \neq \emptyset$, $\pi(x) \neq \emptyset$ for all points x in X [12, Propositions 3.9 and 3.1]. If X is an aposyndetic continuum, then $\mathcal{T}(\{x\}) = \{x\}$ for all x in X [10, 3.1.28]. Hence, clearly, $\mathcal{T}(\{x\}) \subset \pi(x)$ for each x in X. Suppose X is not aposyndetic, then for every x in X, $\mathcal{T}(\{x\})$ is a terminal subcontinuum of X [10, 3.1.21 and 5.1.18]. Hence, by Lemma 3.4, $\mathcal{T}(\{x\}) \subset \pi(x)$ for all points x of X. By [12, Theorem 4.3], $\pi(x)$ is a terminal subcontinuum of X for every x in X. By [10, 5.1.19], $\mathcal{T}(\{x\})$ is a maximal terminal subcontinuum of X for every x in X. Therefore, $\mathcal{T}(\{x\}) = \pi(x)$ for all x in X. $\square$

Remark 3.6. Note that in Theorem 3.5, by Lemma 3.1 and [12, Lemma 4.1], if $\mathcal{NB}(\mathcal{F}_1(X) \cap \mathcal{F}_1(X) \neq \emptyset$, then $\pi(x) = \{x\}$ for every point x of X. In particular, X is an aposyndetic continuum [10, 3.1.28]. If $\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{F}_1(X)$, then X is a simple closed curve [2, Theorem 5.3].

Notation 3.7. Let $\mathbb{H}$ be the Poincaré model of the hyperbolic plane, let $\mathbb{F}$ be the double torus, let W be the figure eight continuum essentially embedded in $\mathbb{F}$ [10, p. 221] and let $\mathcal{Q}$ be the Hilbert cube. Let $\sigma \times 1_{\mathcal{Q}} \colon \mathbb{H} \times \mathcal{Q} \twoheadrightarrow \mathbb{F} \times \mathcal{Q}$ be the universal covering map.

The following is a special case of the Terminal Decomposition Theorem by James T. Rogers, Jr. [10, 5.3.28].

Theorem 3.8. *Let X be an indecomposable homogeneous continuum admitting an essential map onto the figure eight continuum. Hence, we may consider X essentially embedded in $\mathbb{F} \times \mathcal{Q}$. With Notation 3.7, Let $\widetilde{X} = (\sigma \times 1_{\mathcal{Q}})^{-1}(X)$. If $\mathcal{G} = \{(\sigma \times 1_{\mathcal{Q}})(\mathcal{T}_{\mathbb{Y}}(\tilde{x})) \mid \tilde{x} \in \mathbb{K}$, where $\mathbb{K}$ is a component of $\widetilde{X}$, and $\mathbb{Y} = Cl_{(\mathbb{H} \times \mathcal{Q}) \cup \mathcal{S}^1}(\mathbb{K})\}$, then $\mathcal{G}$ is a continuous decomposition such that the following hold:*

(1) *$\mathcal{G}$ is a monotone terminal decomposition of X.*

(2) *The elements of $\mathcal{G}$ are mutually homeomorphic, indecomposable, cell-like homogeneous continua.*

(3) *The quotient space, $X/\mathcal{G}$, is an homogeneous continuum.*

(4) *$X/\mathcal{G}$ does not contain any proper nondegenerate terminal subcontinuum.*

(5) *If the elements of $\mathcal{G}$ are nondegenerate, then they have the same dimension as X and $X/\mathcal{G}$ is one dimensional.*

Theorem 3.9. *Let X be an indecomposable homogeneous continuum admitting an essential map onto the figure eight continuum and let $\mathcal{G} = \{G_x \mid x \in X\}$ be the terminal decomposition given in Theorem 3.8. If $\mathcal{NB}(\mathcal{F}_1(X)) \neq \emptyset$, then $G_x = \pi(x)$ for all points x of X.*

Proof. Since $\mathcal{NB}(\mathcal{F}_1(X)) \neq \emptyset$, $\pi(x) \neq \emptyset$ for all points x in X [12, Propositions 3.9 and 3.1]. By [12, Theorem 4.3], $\pi(x)$ is a terminal subcontinuum of X for all x in X. By Theorem 3.8, G_x is a terminal subcontinuum of X for every x in X. Hence, by Lemma 3.1, $G_x \subset \pi(x)$ for each x in X.

Let $q \colon X \twoheadrightarrow X/\mathcal{G}$ be the quotient map. Let x be a point of X and suppose $G_x \neq \pi(x)$. By [10, 5.1.11], $q(\pi(x))$ is a terminal subcontinuum of $X/\mathcal{G}$. Since $X/\mathcal{G}$ does not contain nondegenerate terminal proper subcontinua (Theorem 3.8), $q(\pi(x)) = \{q(x)\}$. Thus, $\pi(x) = q^{-1}(q(x)) = G_x$, a contradiction. Hence, $G_x = \pi(x)$ for all points x of X. $\square$

The following is Krupski and Prajs Decomposition Theorem [7, Theorem 6.6].

Theorem 3.10. *Let X be a one-dimensional homogeneous continuum, then there exists a unique continuous decomposition $\mathcal{G}$ of X such that*

(1) *each $G \in \mathcal{G}$ is an hereditarily indecomposable tree-like, homogeneous continuum which is terminal in X;*

(2) *$h(G) \in \mathcal{G}$ for each $G \in \mathcal{G}$ and every homeomorphism $h\colon X \to X$;*

(3) *$X/\mathcal{G}$ is an homogeneous one-dimensional continuum with no proper, nondegenerate, terminal subcontinua.*

By Theorem 3.10, for a one-dimensional indecomposable homogeneous continuum X, we have a similar result to Theorem 3.8. The common part is the existence of a terminal decomposition $\mathcal{G}$ of X such that the quotient space does not contain nondegenerate proper terminal subcontinua (the requirement of the existence of an essential map onto the figure eight continuum is not needed). Hence, we have:

Theorem 3.11. *If X is a one-dimensional indecomposable homogeneous continuum and $\mathcal{G} = \{G_x \mid x \in X\}$ is a terminal decomposition such that the quotient space $X/\mathcal{G}$ does not contain a terminal subcontinua and $\mathcal{NB}(\mathcal{F}_1(X)) \neq \emptyset$, then $G_x = \pi(x)$ for all points x of X.*

Remark 3.12. Observe that by Theorem 3.5 the decomposition $\Lambda_X = \{\pi(x) \mid x \in X\}$, constructed in [12, Theorem 4.2], with the assumption that $\mathcal{NB}(\mathcal{F}_1(X)) \neq \emptyset$, coincides with F. Burton Jones Aposyndetic Decomposition, for the class of decomposable homogeneous continua. Also, by Theorem 3.9, for many indecomposable homogeneous continua, such decomposition coincides with James T. Rogers, Jr. Terminal Decomposition. In addition, Theorems 3.5 and 3.11 tell us that for all one-dimensional homogeneous continua the decomposition Λ_X coincides either with Jones Aposyndetic Decomposition or with P. Krupski and J. R. Prajs Decomposition.

4. Families of non-blockers of singletons

Next, for two classes of continua, we give an explicit description of the set of non-blockers of singletons.

Given a continuum X and a point x in X, the *composant of x in X*, denoted by κ_x, is the union of all proper subcontinua of X containing x. It is known that for an indecomposable continuum X, the composants are pairwise disjoint [6, Theorem 3–47] and X has uncountably many of them [6, Theorem 3–46].

Let $\mathbf{n} = \{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers and let $\mathcal{S}^1$ be the unit circle in the complex plane. The $\mathbf{n}$-*solenoid*, $\Sigma_{\mathbf{n}}$, is the space $\{(z_k)_{k=1}^{\infty} \in (\mathcal{S}^1)^{\infty} \mid z_k = z_{k+1}^{n_{k+1}}\}$, with the product topology. It is known that solenoids, different from the simple closed curve, are indecomposable continua [5]. For a classification of solenoids see [1].

Theorem 4.1. *Let $\mathbf{n} = \{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers. If $\Sigma_{\mathbf{n}}$ is the $\mathbf{n}$-solenoid, different from a simple closed curve, then $\mathcal{NB}(\mathcal{F}_1(\Sigma_{\mathbf{n}})) \cap \mathcal{C}(X) = \mathcal{C}(\Sigma_{\mathbf{n}}) \setminus \{\Sigma_{\mathbf{n}}\}$.*

Proof. Let A be a proper subcontinuum of $\Sigma_{\mathbf{n}}$ and let $x \in \Sigma_{\mathbf{n}} \setminus A$. If $A \cap \kappa_x = \emptyset$, then let $\alpha\colon [0,1] \to \mathcal{C}(\Sigma_{\mathbf{n}})$ be an order arc such that $\alpha(0) = \{x\}$ and $\alpha(1) = \Sigma_{\mathbf{n}}$. By indecomposablilty, $A \cap \alpha(t) = \emptyset$ for all $t \in [0,1)$.

Suppose $A \subset \kappa_x$. By [3, p. 181], the composants of solenoids coincide with their arc components. By [14, Theorem 7], there exists a one-to-one onto map $f\colon \mathbb{R} \twoheadrightarrow \kappa_x$ such that $f(0) = x$ and $f((-\infty, 0])$ and $f([0,\infty))$ are both dense in $\Sigma_{\mathbf{n}}$. Without loss of generality, we assume that $A \subset f((-\infty, 0])$. Then $\beta\colon [0,1] \to \mathcal{C}(\Sigma_{\mathbf{n}})$ given by

$$\beta(t) = \begin{cases} f([0, \frac{t}{1-t}]), & \text{if } t < 1; \\ \Sigma_{\mathbf{n}}, & \text{if } t = 1; \end{cases}$$

is an order arc such that $\beta(0) = \{x\}$, $\beta(1) = \Sigma_{\mathbf{n}}$ and $A \cap \beta(t) = \emptyset$ for each $t \in [0,1)$.

In both cases, we have that $A \in \mathcal{NB}(\mathcal{F}_1(\Sigma_{\mathbf{n}}))$. □

By [9], for each solenoid Σ, there exists a solenoid of pseudo-arcs $P\Sigma$ which is a homogeneous one-dimensional continuum that admits a continuous terminal decomposition into pseudo-arcs such that the quotient space is homeomorphic to Σ. By [8, Theorem 3], $P\Sigma$ is unique.

Note that the next theorem tells us that non-blocker subcontinua of the singletons of $P\Sigma$ are the proper subcontinua containing maximal pseudo-arcs of $P\Sigma$.

Theorem 4.2. *For each solenoid of pseudo-arcs $P\Sigma$, there exists a Whitney map $\mu\colon \mathcal{C}(P\Sigma) \twoheadrightarrow [0,1]$ such that $\mathcal{NB}(\mathcal{F}_1(P\Sigma)) \cap \mathcal{C}(X) = \mu^{-1}(\frac{1}{2}, 1)$.*

Proof. Let $q\colon P\Sigma \twoheadrightarrow \Sigma$ be the quotient map. By [4, Theorem 5.2], $\mathcal{NB}(\mathcal{F}_1(P\Sigma)) = \{q^{-1}(B) \mid B \in \mathcal{NB}(\mathcal{F}_1(\Sigma))\}$. By Theorem 4.1, $\mathcal{NB}(\mathcal{F}_1(\Sigma)) \cap \mathcal{C}(X) = \mathcal{C}(X) \setminus \{X\}$. Note that for each x in X, $q^{-1}(x)$ is equal to one of pseudo-arcs forming the continuous decomposition whose quotient space is Σ. These form a closed subset of $\mathcal{C}(P\Sigma)$. Since these pseudo-arcs are nondegenerate, by [10, 6.11.12], there exists a Whitney map $\mu\colon \mathcal{C}(P\Sigma) \twoheadrightarrow [0,1]$ such that if P is a pseudo-arc of the decomposition, then $\mu(P) = \frac{1}{2}$. In fact, by [10, 6.11.21], these pseudo-arcs form the

Whitney level $\mu^{-1}(\frac{1}{2})$. Since the quotient map q is atomic [10, 8.1.25], for each subcontinuum K of $P\Sigma$ such that K interesects at least two elements of the decomposition, $K = q^{-1}(q(K))$. Thus, if K is a subcontinuum of $P\Sigma$ and $\mu(K) > \frac{1}{2}$, we obtain that $K = q^{-1}(q(K))$. Therefore, $\mathcal{NB}(\mathcal{F}_1(P\Sigma)) \cap \mathcal{C}(X) = \mu^{-1}(\frac{1}{2}, 1)$. □

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