http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 55, 2020

Pages 127–145

http://topology.nipissingu.ca/tp/

Spaces with regular nonabelian self covers II

by

Mathew Timm

Electronically published on June 6, 2019

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 55 (2020)
Pages 127-145

E-Published on June 6, 2019

SPACES WITH REGULAR NONABELIAN SELF COVERS II

MATHEW TIMM

Abstract. We produce a disk with countably many holes which has, for each finite group G, a regular self cover with group of deck transformations isomorphic to G. This disk with holes is then used to produce other examples of continua with the same self covering property.

This is the second of two papers motivated by the problem of finding continua X which have regular self covers $f : X \to X$ with a nonabelian group of deck transformations. In the first paper, joint work with Alberto Delgado [7], the question about existence of such spaces is answered very strongly in the affirmative. It is shown that there are two metric continua, one infinite dimensional and one 1-dimensional, which have the property that given any finite group G, each of the continua has a regular $|G|$-fold self cover with the associated group of deck transformations isomorphic to G. Thus, each of these continua satisfies the *universal regular finite-sheeted (URF) self cover property*: they have regular self covers with group of deck transformations isomorphic to G for *every* finite group G. Consequently, these continua have regular self covers with deck transformations isomorphic to every finite nonabelian G. The one dimensional continuum turns out to be homeomorphic to the Sierpinski curve (aka, the Sierpinski carpet) $\mathcal{S}$.

2010 *Mathematics Subject Classification.* Primary 54F15, 54C10; Secondary 20B25, 57M10.

Key words and phrases. Regular covering space, disk with holes, control set, patterned space, Sierpinski curve, non-cohopfian group.

Thanks to the participants in the Dynamical Systems and Continuum Theory session of the 33rd Summer Conference on Topology and its Applications and the referee for their interest in these results.

It is clear that one may find other low dimensional continua which have regular nonabelian self covers by taking products of the Sierpinski curve with other spaces of various dimensions. Specifically, $\mathcal{S} \times [0,1]$ is such a 2-dimensional space, $\mathcal{S} \times [0,1]^2$ is a 3-dimensional example, and more generally, $\mathcal{S} \times M$ where M is any finite connected $(n-1)$-dimensional complex produces n-dimensional examples. However, these examples are both somewhat uninteresting in that they are products, yet at the same time, they are complicated in the sense that the local topology around every point in $\mathcal{S} \times M$ is complicated. The infinite dimensional example of [7] is also a product, an infinite direct product, and as a consequence it also has complicated local topology at each of its points. These observations, Daverman [10], and remarks in [7] indicate that the problem of finding low dimensional non-product spaces with simple local topology and regular self covers with nonabelian group of deck transformations is of interest.

In the current paper we show that there is a non-product 2-dimensional continuum with simple local topology around most of its points which has finite sheeted regular self covers with nonabelian groups of deck transformations. We prove the stronger result, that

Theorem 1 *There is a disk with holes K which satisfies the URF self cover property. Furthermore, there is a Cantor set $C \subset int(K)$ such that $K \setminus C$ is a connected 2-dimensional (planar) manifold with boundary.*

By [7] and Theorem 1, we now have two low dimensional non-product continua, the Sierpinski curve and the disk with holes of Theorem 1, which satisfy the *URF* self cover property.

The method used to prove Theorem 1 is also of interest. First, it is an abbreviation of the process used in [7] to show that the Sierpinski curve satisfies the *URF* self cover property. This process is based on Whyburn's characterisation [23] of the Sierpinski curve as the complement in a 2-disk β of the interiors of a dense null sequence of 2-disks contained in β. Theorem 1 follows from the recognition that the requirement that the null sequence of disks be dense can be relaxed and the observation that the Sierpinski curve exercises a sort of "control" over how this null sequence accumulates.

Second, the proof of Theorem 1 is of interest because it can be used as the basis for a process that allows one to build many non-homeomorphic low (and high) dimension non-product spaces which satisfy the *URF* self cover property. These constructions are captured in Corollaries 4.1 and 4.2 in Section 4.

Finally, note that because of the usual correspondence between covering spaces and subgroups of their fundamental groups, there is the related group theoretic problem of determining groups G which have proper finite index subgroups H isomorphic to G. A proper subgroup H of G which is isomorphic to G is called a *clone* and groups which possess clones are said to be *non-cohopfian.* Thus, the topological problem of determining spaces with finite index regular nonabelian self covers corresponds to the group theoretic problem of determining which non-cohopfian groups G have finite index normal clones H such that G/H is nonabelian. The fundamental groups of the continua constructed in Theorem 1, Corollaries 4.1 and 4.2, and [7] provide concrete examples of such non-cohopfian groups. In fact, the fundamental group of the infinite dimensional continuum of [7] produces a direct product which is dramatically non-cohopfian and which has quotients by appropriate clones which are nonabelian.

References for some of the continua theoretic aspects of self covers include [2], [7], [8], [12], [14], [15], [16], [17] and [21]. Techniques similar to those used in this paper are exploited to produce interesting wild knots in [19]. Related group theoretic concerns are addressed in [13] and [20]. Results at the interface of the topology and the group theory include [1], [3], [4], [5], [6], [9], [10], [11], [19], and [22]. See [7] for a more detailed summary of these sorts of results.

This paper is organized as follows. Section 1 contains background material including the definition of a controlled patterned space and a proof that the Cantor set satisfies the universal regular finite sheeted self cover property. Section 2 contains the proof that patterned disks controlled by the Cantor set are unique up to homeomorphism. Section 3 contains the proof of the Theorem 1. Section 4 shows how to use controlled patterned disks to build many spaces which satisfy the universal regular finite-sheeted self cover property. Section 5 contains a question suggested by the results in this paper.

1. Background material

All *continua* in this paper are compact connected metric spaces. A *manifold* may have nonempty boundary. A *disk with holes* is a space homeomorphic to $\beta \setminus \big(\cup \{int(B) : B \in \mathcal{B}\}\big)$ where β is a closed 2-dimensional disk and $\mathcal{B}$ is a collection of pairwise disjoint closed 2-dimensional disks each of which is contained in $int(\beta)$. If $|\mathcal{B}| = m$, $m = |\mathbb{N}|$ allowed, we say β is *a disk with m holes.* The closed disk of center p and radius ϵ is denoted $B(p, \epsilon)$. A *neighborhood* of a set S is an open set containing S. Our fundamental groups have base points but we suppress the base point in the notation. Let $\mathbb{N}^* = \{0\} \cup \mathbb{N}$.

Let X and Y be continua and $f : X \to Y$ a covering space map. The group $Aut_Y(X, f)$ of *deck transformations of* (X, f) *over* Y is the set of all homeomorphisms α of X under function composition such that $f \circ \alpha = f$.

The covering space $f : X \to Y$ is a *regular* cover, if given any $y \in Y$ and $x_1, x_2 \in f^{-1}(y)$, there is an $\alpha \in Aut_Y(X, f)$ such that $\alpha(x_1) = x_2$. Given a group G, an $|G|$-fold cover $f : X \to Y$ is a *G regular cover*, if $Aut_Y(X, f) \cong G$. A regular cover is a *cyclic cover* provided its group of deck transformation is a cyclic group. Given $n \in \mathbb{N}$, a regular n-fold cover is an *n-fold cyclic cover* if $Aut_Y(X, f) = \mathbb{Z}_n$. A cyclic cover is a *finite cyclic cover* if it is an n-fold cyclic cover for some $n \in \mathbb{N}$. A G regular cover is a *nonabelian* regular cover if G is a nonabelian group.

If $f : X \to Y$ is an n-fold cover and a subset $S \subset Y$ is contained in a fundamental domain of f, the *full lift of S by f* is $f^{-1}(S)$. When S is contained in a fundamental domain of f, the full lift of S by f will be written $f^{-1}(S) = S^{(1)} \cup \ldots \cup S^{(n)}$ where each $S^{(i)}$ is mapped homeomorphically onto S by f. Each $S^{(i)}$ is called a *lift* of S.

We give a technical meaning to the notion that a continuum is "simple" as follows. A continuum X of topological dimension n is *simple* if there is a subset $S \subset X$ such that $X \setminus S$ is a connected n-dimensional manifold which is open and dense in X. Note, $\partial(X \setminus S) \neq \emptyset$ allowed.

It follows from the classification of 2-dimensional compact manifolds that any two disks with the same finite numbers of holes are homeomorphic. However, for disks with $|\mathbb{N}|$ holes one must also pay attention to how the boundary circles of the holes accumulate. See [7, Example 2.2] and the remarks at the beginning of Section 4 in the same paper. One may also prove the following.

Proposition 1.1. *Let $\mathcal{D}_j, j = 1, \ldots, n$ be countably infinite collections of pair-wise disjoint closed disks contained in the interior of the closed 2-disk β and $\{p_{1j}, \ldots, p_{jj}\}$ finite point sets contained in $int(\beta) \setminus \cup\{D_{ji} : D_{ji} \in \mathcal{D}_j\}$. Let*

$$K_j = \beta \setminus \cup\{int(D_{ji}) : D_{ji} \in \mathcal{D}_j\},\ j = 1, \ldots, n$$

be disks with countably infinitely many holes such that

$$cl(\cup\{\partial D_{ij} : D_{ij} \in \mathcal{D}_j\}) \setminus \cup\{\partial D_{ij} : D_{ij} \in \mathcal{D}_j\} = \{p_{1j}, \ldots, p_{jj}\}.$$

Then K_j and K_k are homeomorphic if and only if $j = k$.

Consequently, there are disks with countably infinitely many holes which are not homeomorphic. Also observe that each of the K_j of Proposition 1.1 are simple: by removing the finite point sets on which the boundaries of the holes accumulate, one obtains noncompact connected 2-dimensional planar manifolds with boundary.

1.1. **Patterned spaces and control sets.** Let X be a topological space, $\mathcal{F}$ a collection of subsets of X, and S a subset of X. The pair $(X, \mathcal{F})$ is a *patterned space* and $\mathcal{F}$ is the patterned space's *pattern*. The triple $(X, \mathcal{F}, S)$ is a *controlled* patterned space with *control set* S, if $\big(\cup\{F : F \in \mathcal{F}\}\big) \subset X \setminus S$ and $S = cl\big(\big(\cup\{F : F \in \mathcal{F}\}\big)\big) \setminus (\cup\{F : F \in \mathcal{F}\})$, i.e., S is the set of accumulation points of the pattern $\mathcal{F}$.

Example 1.2. Set β equal to a 2-disk, and $\mathcal{D} = \{int(D_i) : i \in \mathbb{N}\}$, where $\{D_i : i \in \mathbb{N}\}$ is the null sequence of disks used to form the Sierpinski gasket $\mathcal{S}$ as in the introduction. Then $(\beta, \mathcal{D})$ is a patterned space with pattern $\mathcal{D}$ and $(\beta, \mathcal{D}, \mathcal{S})$ is a controlled patterned space with control set $\mathcal{S}$.

Example 1.3. The pairs $(K_j, \{\partial D_{ij} : D_{ij} \in \mathcal{D}_j\})$ of Proposition 1 are patterned spaces. The triples

$$(K_j, \{\partial D_{ij} : D_{ij} \in \mathcal{D}_j\}, \{p_{1j}\ , \dots, p_{jj}\}),$$

$j = 1, \dots, n$, are controlled patterned spaces. The control sets are, respectively, $\{p_{1j}\ , \dots, p_{jj}\}$.

1.2. **Standard patterned disks and annuli.** Choose a very small $\epsilon > 0$. Consider the closed 2-disk $\beta = [-\epsilon, 2-\epsilon] \times [-1, 1]$. Let C denote the standard middle thirds Cantor set in $[0, 1] \times 0$. Let $\mathcal{M} = \{(i, j) : (\frac{i}{3^j}, \frac{i+1}{3^j})$ is used to form $C\}$. Let $\mathcal{M}^0 = \{(0, 0)\} \cup \mathcal{M}$. For $(i, j) \in \mathcal{M}$, let $B_{ij} = B\big((\frac{2i+1}{2 \cdot 3^j}, 0), \frac{1}{4 \cdot 3^j}\big) \subset \beta$ and let $B_{00} = B\big((\frac{3}{2}, 0), \frac{1}{4}\big) \subset \beta$.

Let $\mathcal{B} = \{B_{ij} : (i, j) \in \mathcal{M}\}$ and let $\mathcal{B}^0 = \{B_{ij} : (i, j) \in \mathcal{M}^0\} = \mathcal{B} \cup \{B_{00}\}$. Let $\partial\mathcal{B} = \{\partial B : B \in \mathcal{B}\}$ and $\partial\mathcal{B}^0 = \{\partial B : B \in \mathcal{B}^0\}$.
The *standard patterned disk controlled by the Cantor set* is $(\beta, \mathcal{B}, C)$. The patterned disk $(\beta, \mathcal{B}^0, C)$ is the *augmented* patterned disk controlled by the Cantor set.

Let $K = \beta \setminus (\cup\{int(B) : B \in \mathcal{B}\})$. The *standard patterned disk with holes controlled by the Cantor set* is $(K, \partial\mathcal{B}, C)$. Let $K^0 = \beta \setminus (\cup\{int(B) : B \in \mathcal{M}^0\})$. Then $(K^0, \partial\mathcal{B}^0, C)$ is the *augmented patterned disk with holes controlled by the Cantor set.*

We sometimes say β is the standard patterned disk, β is the augmented patterned disk, K is the standard disk with holes, or K^0 is the augmented disk with holes instead of using the full triple to denote these objects. When this is done, the patterns and control sets are assumed to be those indicated above.

See Figure 1 for pictures of the patterned disks $(\beta, \mathcal{B}, C)$ and $(\beta, \mathcal{B}^0, C)$ and patterned disks K and K^0 with holes. The shaded regions in Figure 1 serve a dual purpose: they can either be thought of as the interiors of the disks in the patterns $\mathcal{B}$ and $\mathcal{B}^0$ or as the holes in K and K^0.

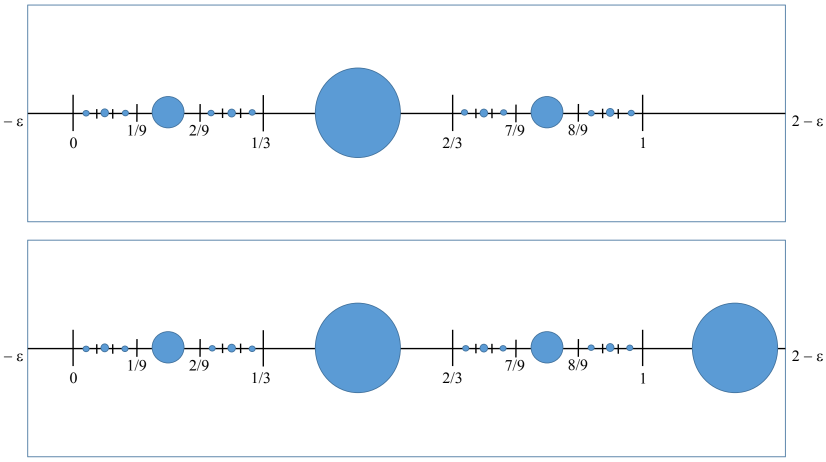

FIGURE 1. Top: The standard patterned disk or standard patterned disk with holes. Bottom: The augmented patterned disk or augmented patterned disk with holes.

Let $A = [-\epsilon, 2-\epsilon] \times [-1,1]/\sim$ denote the annulus formed by gluing $-\epsilon \times [-1,1]$ to $(2-\epsilon) \times [-1,1]$ via $(-\epsilon, t) \mapsto (2-\epsilon, t)$. Let $A^0 = A \backslash (\cup\{int(B) : B \in \mathcal{B}^0\})$. The *standard patterned annulus with holes controlled by the Cantor set* is the controlled patterned space $(A^0, \partial\mathcal{B}^0, C)$. See Figure 2.

Observe that both of K and A^0 are formed by removing open 2-disks from β which have centers at the midpoints of one of the "deleted" open intervals used to form the Cantor set C and whose diameters are only one half of the length of this interval. This observation has two consequences. First, it follows from their definitions that all of $(\beta, \mathcal{B}, C)$, $(\beta, \mathcal{B}^0, C$, $(K, \partial\mathcal{B}, C)$, $(K^0, \mathcal{B}^0, C)$, and $(A^0, \partial\mathcal{B}^0, C)$ are patterned spaces with the same Cantor set C as their control spaces. Second, the fact that the diameters of the open 2-disks removed from β to form K and from A to form A^0 are less than the total distance between the endpoints of each associated "deleted" interval used to form C allows the construction of isotopies which show K, K^0, and A^0 are just three different ways of thinking about the same disk with countably infinitely many holes. The topological equivalence of these three spaces is a special cases of the Uniqueness Theorem proved in the next section. However, it is instructive to point out that simpler proofs showing they are all isotopic are available. The isotopies giving the equivalence are shown in Figure 3.

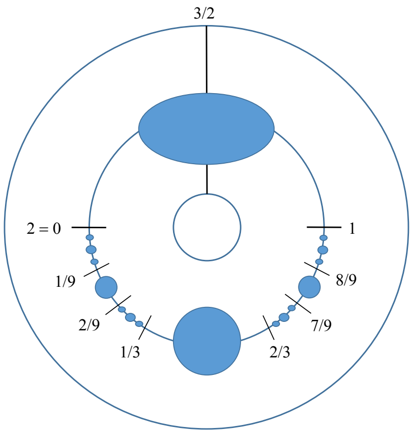

Figure 2. The standard patterned annulus with countably infinitely many holes.

Briefly, the isotopy f_t in the figure shifts $\partial B_{1,1}$ onto $\partial B_{0,0}$, shifts $\partial B_{1,2}$ onto $\partial B_{1,1}$ and shifts $\partial B_{1,n}$ onto $\partial B_{1,(n-1)}$. Consequently the standard patterned disk K is isotopic to the augmented patterned disk K^0.

The isotopy h_t shifts $\partial B_{1,1}$ onto $\partial B'_{0,0} = \partial B\big((\frac{1}{2}, \frac{1}{2}), \frac{1}{4}\big)$, and is otherwise the same as h_t. Applying first f_t, then h_t and re-coordinatizing as in Figure 4, it follows that standard patterned disk K is isotopic to the standard patterned annulus A^0 and Proposition 1.4, below, follows.

Note that f_t and h_t are the identity outside $cl(N)$ where N is as indicated in Figure 3. In particular note that N is an open set in the complement of the Cantor set and in the complement of most of the closed disks in the pattern $\mathcal{B}$. The only point in C contained in $cl(N)$ is $(0,0)$.

Proposition 1.4. *The patterned spaces K,K^0, and A^0 are isotopic.*

By inducting on the number n of holes

$$int(B_{11}), int(B_{12}), \ldots int(B_{1n})$$

that one wants to move off the x-axis to positions far from C, the proof of Proposition 1.2 generalises to produce the next result.

Proposition 1.5. *Let $\{D_1, \ldots, D_n\}$ be a collection of closed disks in $int(\beta) \setminus ([0,1] \times 0)$ such that the disks in $\mathcal{B} \cup \big(\cup_{i=1}^n D_i\big)$ are pairwise*

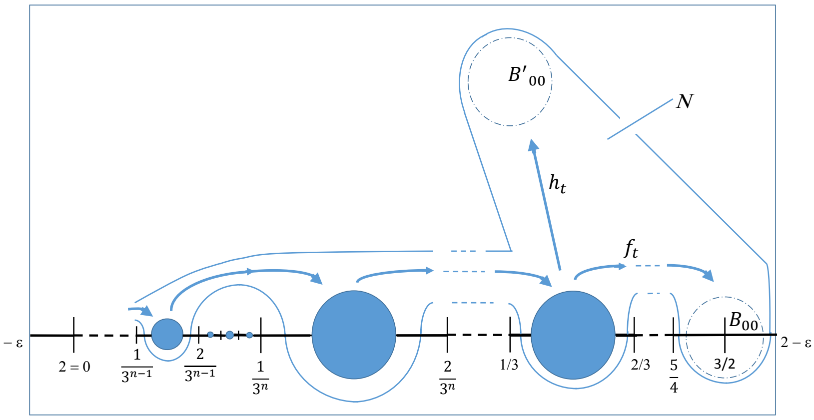

FIGURE 3. The homotopies f_t and h_t.

disjoint. Then the patterned disk with holes

$$\big(K \setminus \big(\cup_{i=1}^{n} int(D_i)\big), \partial\mathcal{B} \cup \big(\cup_{i=1}^{n} \{\partial D_i\}\big), C\big)$$

controlled by the Cantor set C is isotopic, as a triple, to the standard patterned disk with holes $(K, \partial\mathcal{B}, C)$ controlled by C.

1.3. **The Cantor set and the** *URF* **self cover property.** There are various ways, e.g, [7, 2.3], to show that the Cantor set has G regular self covers for every finite group G. The proof given here forms the basis for our approach to Theorem 1. We call the combination of this result and the additional structure provided by the remarks following its proof the *Method of Disks with Holes.*

Proposition 1.6. *Let C be the Cantor set. Then C satisfies the URF self cover property. Furthermore, setting $\beta = [-\epsilon, 2-\epsilon] \times [-1, 1]$, $\epsilon > 0$ very small, we may assume that C is the middle thirds Cantor set in $[0, 1] \times 0 \subset \beta$ and θ_G is the restriction of a G regular cover*

$$\theta_G : \beta \setminus \big(\cup_{i=1}^{n} int(V_i)\big) \to \beta \setminus \big(\cup_{j=1}^{m} int(W_j)\big)$$

where $\{V_i : i = 1, \ldots, n\}$ and $\{W_j : j = 1, \ldots m\}$ are a collections of pairwise disjoint closed disks whose unions are far from, that is, outside a neighborhood of, C.

Proof. Fix a finite group $G = \langle g_1, \ldots, g_m : r_1, \ldots, r_n\rangle$ with a fixed presentation. There are the standard algebraic results which say that

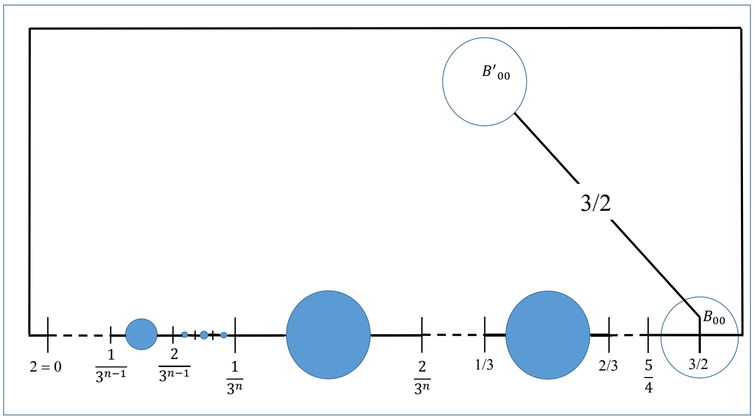

FIGURE 4. K as A^0 with B'_{00} as the inner hole in the annulus A. The inner rectangle is the circle $[0,2] \times 0/\big((0,0) \sim (2,0)\big)$ of A.

$G = F_m/F_n$ where the F_i are free groups on i generators and F_n is normal of index $|G|$ in F_m. Using the fact that the fundamental group of a disk with finitely many holes is a free group we express these results topologically in Step 1.

Step 1. Start with a disk M with m holes. Then there is another disk X with (usually a different number) n of holes and a regular $g = |G|$-fold covering space map $\theta_G : X \to M$ such that $G \cong \pi_1(M)/\theta_G^*(\pi_1(X))$ is the group of deck transformations. Let X_0, M_0 denote, respectively, the closed disks from which X and M are obtained. Also, without loss of generality, assume that θ_G maps the boundary $\partial X_0 \subset X$ onto $\partial M_0 \subset M$. Note that the boundary circles of some of the holes in X might also map onto $\partial M_0 \subset M$.

Step 2. Now choose a small closed 2-disk E of M that is evenly covered by θ_G. Identify E with a small copy of our standard patterned disk $(\beta, \mathcal{B}, C)$. Then within $int(E) \cong int(\beta)$ there are copies of the intervals $[0,1] \times 0 \subset [0, 2-\epsilon] \times 0$ and the the middle thirds Cantor set C contained in $[0,1] \times 0$. By construction, the holes in M are far from C and, in fact, far from all of $[0,1] \times 0 \cup \big(\cup\{B : B \in \mathcal{B}\}\big) \subset E$. Specifically the holes are outside of the 2-disk E.

Since the union of finitely many Cantor sets is a Cantor set, $\theta_G^{-1}(C)$ is another Cantor set contained in $\theta_G^{-1}(E)$. The restriction $\theta_G : \theta_G^{-1}(C) \to C$ is a regular G self cover. Consequently, it follows that C satisfies the *URF* self cover property. □

There is additional structure which can be imposed on θ_G. We do so in the following. More is said about $\theta_G^{-1}(C)$ in Step 3 and more about C is in Step 4.

Step 3. Let $\mathcal{B}^{(i)}$ and $\mathcal{B}^{0(i)}$ be, respectively, the lifts of the patterns $\mathcal{B}$ and $\mathcal{B}^0$ contained in E to the lift $E^{(i)}$. Within each lift $E^{(i)}$, $i = 1, \ldots, g-1$ apply a copy of the isotopy f_t of Proposition 2 and turn each of these $E^{(i)}$, thought of as copies $(\beta^{(i)}, \mathcal{B}^{(i)}, C^{(i)})$ of the standard patterned disk, into lifts which are copies $(\beta^i, \mathcal{B}^{0(i)}, C^{(i)})$ of the augmented patterned disk $(\beta, \mathcal{B}^0, C)$.

Next, for each $i = 1, \ldots, g-1$ connect the point lift $(2-\epsilon, 0)^{(i)} \in E^{(i)}$ to the point lift $(-\epsilon, 0)^{(i+1)} \in E^{(i+1)}$ by a path γ_i, an embedding of an interval where all of γ_i except its initial and terminal point are contained in $int(X) \setminus \big((\cup_{j=1}^g E^j) \cup (\cup_{k=1}^{i-1} \gamma_k)\big)$. For $i = 1, \ldots, g-1$, let $\Delta^i = \cup\{B^{0(i)} : B^{0(i)} \in \mathcal{B}^{0(i)}\}$ and let $\Delta^g = \cup\{B^{(g)} : B^{(g)} \in \mathcal{B}^{(g)}\}$. It is important to note that the Δ^i, $i = 1, \ldots, g-1$, are the union of the disks in copies of the pattern $\mathcal{B}^0$ while Δ^g is the union of the disks in a copy of the pattern $\mathcal{B}$.

Then the subspace

$$S = \big(\cup_{i=1}^{g-1} ([-\epsilon, 2-\epsilon] \times 0)^{(i)} \cup \Delta^i \cup \gamma_i\big) \cup \big(([-\epsilon, 2-\epsilon] \times 0)^{(g)} \cup \Delta^g\big)$$

of X is homeomorphic to the subspace $[0,1] \times 0 \cup \big(\cup \{B : B \in \mathcal{B}\}\big)$ of β. Furthermore X can be recoordinatized so that it becomes a copy of β with n holes punched into it. Applying an isotopy we can assume that S has been isotoped onto $[0,1] \times 0 \cup \big(\cup \{B : B \in \mathcal{B}\}\big)$. By construction, the holes in β are outside a neighborhood of $[0,1] \times 0 \cup \big(\cup \{B : B \in \mathcal{B}\}\big)$.

Step 4. Similarily, M can be recoordinatized and then isotoped so that it is a copy of the standard patterned disk $(\beta, \mathcal{B}, C)$ with m extra holes punched into it which are far from the union of $[0,1] \times 0$ and the disks in the pattern.

Step 5. Steps 1-4 produce disks with holes $X^* = \beta \setminus \cup_{i=1}^m int(W_i)$ and $M^* = \beta \setminus \cup_{j=1}^n int(V_j)$ which are, respectively, the domain and range of the G regular cover θ_G composed with the isotopies of steps 3 and 4. We abuse notation a bit and denote this composition by the same symbol. Consequently, we have a G regular covering space map of controlled patterned space triples $\theta_G : (X^*, \mathcal{B}, C) \to (M^*, \mathcal{B}, C)$.

2. The topological equivalence of patterned disks with holes controlled by the Cantor set

Theorem 2.1 (Uniqueness Theorem). *Suppose $(\beta', \mathcal{B}', C')$ is a controlled patterned disk with pattern $\mathcal{B}'$ consisting of a countable collection of pairwise disjoint closed disks contained in $int(\beta')$ controlled by a Cantor set $C' \subset int(\beta')$. Then $(\beta', \mathcal{B}', C')$ is homeomorphic, as a triple, to the standard patterned disk controlled by the Cantor set $(\beta, \mathcal{B}, C)$.*

Proof. We may assume $\beta' = \beta = [-\epsilon, 2-\epsilon] \times [-1, 1]$. The Denjoy-Riesz Theorem [18] then provides a homeomorphism of β which allows us to assume that the Cantor set C' is a middle thirds Cantor set contained in an interval I in $int(\beta)$. After an ambient isotopy of β, if necessary, we may assume that $I = [0,1] \times 0 \subset \beta$ and $C' = C$. Consequently, what remains to be shown is that if $(\beta, \mathcal{B}', C)$ is a patterned disk controlled by C, then $(\beta, \mathcal{B}', C)$ is homeomorphic as a triple to the standard patterned disk controlled by the Cantor set. We prove a somewhat stronger result: if $(\beta, \mathcal{B}', C)$ is a patterned disk controlled by the middle thirds Cantor set $C \subset [0,1] \times 0$, then $(\beta, \mathcal{B}', C)$ is ambiently isotopic as a triple to the standard patterned disk controlled by the Cantor set C via an isotopy which is the identity on C and fixes $[0,1] \times 0$ set-wise.

Let $\mathcal{B}' = \{B'_n : n \in \mathbb{N}\}$. By an ambient isotopy of β which is the identity on C, is the identity outside the closure of a small neighborhood of $\cup_{n=1}^{\infty} B'_n$, and is setwise fixed on $[0,1] \times 0$, we may assume that each B'_n is, in fact, a small round disk $B'_n = B(p_n, \epsilon_n)$ where $p_n \in int(B'_n)$ and $\epsilon_n \leq \frac{1}{4} \cdot diam(B'_n)$.

Now suppose $\left(\cup_{n=1}^{\infty} B'_n\right) \cap [0,1] \times 0 \neq \emptyset$. Let n_i be the least n such that $B'_{n_i} \cap [0,1] \times 0 \neq \emptyset$. Choose a small neighborhood $N = N(B'_{n_i})$ of B'_{n_i} whose closure is disjoint from C and the other disks in the pattern. There is an ambient isotopy which is the identity outside $cl(N)$ which shrinks ϵ_{n_i}, is the identity on $[0,1] \times 0$ and isoptopes B'_{n_i} off $[0,1] \times 0$. Continuing inductively, each B_{n_i} which intersects $[0,1] \times 0$ can be isotoped off $[0,1] \times 0$ via an ambient isotopy of β which is the identity outside the closure of a small neighborhood of B'_{n_i} and fixes $[0,1] \times 0$ and C. Hence, we may assume, after an ambient isotopy of β that $\left(\cup_{n=1}^{\infty} B'_n\right) \cap [0,1] \times 0 = \emptyset$, i.e., that the pattern in the patterned disk $(\beta, \mathcal{B}', C)$ controlled by C is disjoint from $[0,1] \times 0$.

Observe that there is some finite collection of the B'_n whose distance to $[0,1] \times 0$ is a maximum. If there are more than one disk in this collection, choose one among them that is closest to the midpoint $(\frac{1}{2}, 0)$ of the "deleted" interval of length $\frac{1}{3}$ used to form C. By renaming if necessary, we may assume that the chosen disk is B'_1. Choose a small path connected neighborhood $N_1 = N\big(B_1' \cup \{(\frac{1}{2},0)\}\big)$ whose closure is disjoint from $C \cup \big(\cup_{n=2}^{\infty} B_n'\big)$. Apply an ambient isotopy of β which is the identity outside $cl(N_1)$, fixes $[0,1]\times 0$, and isotopes B_1' onto a small round closed disk whose center is $p_1 = (\frac{1}{2},0)$, the midpoint of the "deleted" interval of length $\frac{1}{3}$, and whose radius is $\epsilon_1 \leq \frac{1}{4}\cdot\frac{1}{3}$. We denote the resulting disk by the same symbol $B_1' = B(p_1,\epsilon_1)$.

Next, in the collection $\{B_n' : n \geq 2\}$ there is a finite collection of disks whose distance to $[0,1]\times 0$ is largest and second largest. Among this finite collection, choose two, say they are B_2' and B_3', such that the distance from B_2' to $(\frac{3}{18},0)$ is minimized and the distance from B_3' to $(\frac{15}{18},0)$ is minimum, i.e., the distances from this pair of chosen disks to the pair of midpoints of the "deleted" intervals of length $\frac{1}{3^2}$ is minimized. Choose a small path connected neighborhood N_2 of $B_2' \cup \{(\frac{3}{18},0)\}$ such that $cl(N_2)$ is disjoint from C and the other disks in the pattern $\mathcal{B}'$ and choose a small path connected neighborhood of $B_3' \cup (\frac{3}{18},0)$ whose closure is disjoint from C and the other disks in the $\mathcal{B}'$. Use N_2 to ambiently isotope B_2' and N_3 to ambiently isotope B_3' onto small closed disks $B(p_2,\epsilon_2)$ and $B(p_3,\epsilon_3)$ where p_2 and p_3 are the midpoints of the "deleted" intervals of length $\frac{1}{3^2}$ contained in N_2 and N_3 respectively and $\epsilon_2,\epsilon_3 \leq \frac{1}{4}\cdot\frac{1}{3^2}$. Again, we may assume the ambient isotopies are the identity outside of $cl(N_2)\cup cl(N_3)$ and fix $[0,1]\times 0$ set wise. Also, we again denote the resulting closed disks by the original symbols $B_2' = B(p_2,\epsilon_2)$ and $B_3' = B(p_3,\epsilon_3)$.

Continuing inductively, we can assume that after an ambient isotopy which is the identity on C and fixes $[0,1]\times 0$ set-wise, that $(\beta,\mathcal{B}',C)$ is a patterned space where $\mathcal{B}' = \{B_n' = B(p_n,\epsilon_n) : n \in \mathbb{N}\}$ and for $i \in \mathbb{N}^*$ and each n, $2^i \leq n \leq 2^{i+1}-1$, p_n is the midpoint of one of the "deleted" intervals used to form C and $\epsilon_n \leq \frac{1}{4}\cdot\frac{1}{3^{i+1}}$.

Finally by an ambient isoptopy of β that is the identity on C, is the identity on the midpoints of the "deleted" intervals used to form C and fixes $[0,1]\times 0$ set-wise, those B_n' whose radii are smaller than the radii of the the disks in the standard pattern $\mathcal{B}$ centered at the same point can be enlarged until B_n' is a disk in the standard pattern $\mathcal{B}$. $\square$

3. The proof of Theorem 1

Since K is homeomorphic to the annulus A^0 with infinitely many holes, it is easy to see that it has cyclic n-fold self covers $f_n : K \cong A^0 \to K \cong A^0$ for all $n \in \mathbb{N}$. The 2- and 3-fold cyclic self covers are illustrated in Figure 5. The picture also indicates how to inductively define all of these cyclic self covers. Methods for coordinatizing these sorts of self covers are illustrated and exploited in [1], [19] and [21].

We restate Theorem 1 using the language developed in Section 1.

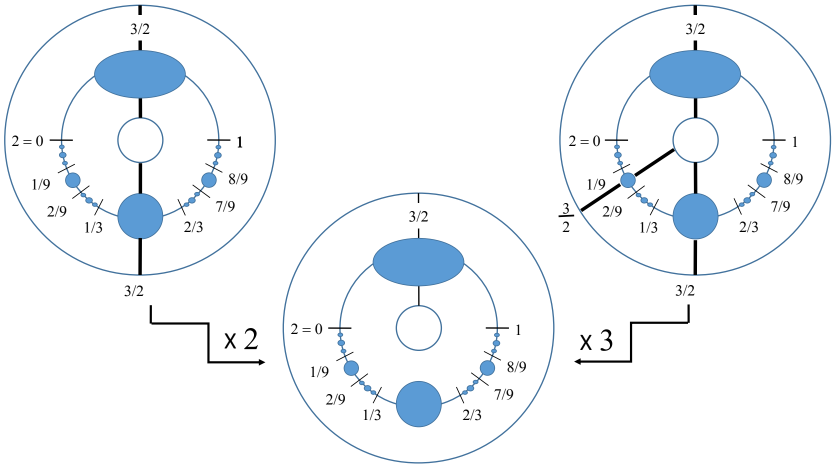

FIGURE 5. Cyclic self covers of the annulus A^0 with infinitely many holes. The lifts of $\frac{3}{2} \times [-1,1]$ from the range A^0 are shown in the domains of f_2 and f_3

Theorem 1. *There is a simple disk with countably many holes which satisfies the universal regular finite-sheeted self cover property.*

Proof. Fix a finite group G. Apply the Method of Disks with Holes to obtain the G regular cover

$$\theta_G : \big(X = \beta \setminus \big(\cup_{i=1}^{n} int(V_i)\big), \mathcal{B}, C\big) \to \big(M = \beta \setminus \big(\cup_{j=1}^{m} int(W_j)\big), \mathcal{B}, C\big)$$

of patterned spaces.

Now delete the interiors of the disks in the patterns in both X and M to obtain, respectively, $X^* = X \setminus \big(\cup \{int(B) : B \in \mathcal{B}\}\big)$ and $M^* = M \setminus \big(\cup \{int(B) : B \in \mathcal{B}\}\big)$ patterned disks with (a few extra) holes controlled by the middle thirds Cantor set C. The restriction $\theta_G : (X^*, \partial\mathcal{B}, C) \to (M^*, \partial\mathcal{B}, C)$ is a G regular cover of patterned disk with holes triples. But, applying Theorem 2 or Proposition 1.3, it follows that both the domain and range are homeomorphic to the standard disk with holes $(K, \mathcal{B}, C)$ controlled by C. The composition of these homeomorphisms with the restricted θ_G gives that K has a G regular cover. $\square$

We close this section with some remarks about $\theta_G : X^* \to M^*$. The proof above begins by specifying the finite group G of interest and shows that the X^* and M^* it determines are copies of K. The method in which X^* and M^* are developed gives primacy to certain boundary circles, namely ∂V_i and ∂W_j, of the holes in K. Specifically, these boundary circles, together with the boundary $\partial\beta \subset X^*$, are the only boundary circles on which θ_G might possibly **not** be a homeomorphism and, by construction, for any particular G there are only finitely many such boundary circles. Henceforth, we use the notation $\theta_G : X^* \to M^*$ to indicate that we are thinking of the domain copy of K as $K \setminus \left(\cup_{i=1}^{n} V_i\right)$ and the range copy as $K \setminus \left(\cup_{j=1}^{m} W_j\right)$ where the V_i are the only holes whose boundaries (together with $\partial\beta$) might not be mapped homeomorphically onto one of the boundaries of the W_j or the boundary of the disk β. The ∂V_i and ∂W_j will be called *special* boundaries.

4. Other continua satisfying the *URF* self cover property

The patterned disk with countably many holes K and the proof of Theorem 1 provide a basis for producing other continua which satisfy the *URF* self cover property. We begin with an outline of the process.

I. Begin with a controlled patterned disk with countably many holes $(K, \partial\mathcal{B}, C)$ controlled by the Cantor set C.

II. For a given finite group G the controlled patterned disk with holes K may be thought of as both the domain and range of the G regular cover $\theta_G : X^* \to M^*$. In $K = X^*$ there is a collection, ∂V_i, of special boundaries of certain holes together with $\partial\beta$ on which θ_G might not be a homeomorphism. In $K = M^*$ the images of these circles, the ∂W_j and $\partial\beta$, are also special.

III. To the special circles of II, attach a space Z_0 which has regular cyclic self covers of all orders. Z_0 is attached to each of these special circles along a circle which is mapped to itself by these self covers.

IV. Choose a countable (possibly infinite) sequence $Z_0, Z_1, Z_2, \dots$ of spaces, including Z_0, to attach to boundaries of the remaining holes in X^* and M^*.

V. Attach the spaces selected in IV to the boundaries of the nonspecial holes in the patterned disks $K = X^*$ and $K = M^*$. Extend the metrics on X^* and M^* to the copies of the Z_i attached in both III and IV. The attaching processes and extensions of the metrics are done so that every neighborhood of each point p in the control set C contains infinitely many copies of each of the Z_i. For $i \geq 1$, each copy of Z_i, in X^* is mapped homeomorphically onto its image in M^* by θ_G.

We now give the technical details. Begin by deciding whether to attach some finite number, say q, different continua or $|\mathbb{N}|$ different continua to the boundaries of the holes in the standard disk with holes $(K, \partial\mathcal{B}, C)$ controlled by C. Then let Z_0 be a continuum, e.g, the m-torus $\times_{i=1}^{m} S^1$, with n-fold cyclic self covers for all $n \in \mathbb{N}$. For specificity, choose Z_0 to be the 2-torus, $S^1 \times S^1$, choose the cyclic n-fold self cover to be that induced by the map $e^{i\theta} \mapsto e^{in\theta}$ on the first coordinate, and choose $S_0^1 = \{(e^{i\theta}, 1) : 0 \leq \theta \leq 2\pi\} \subset S^1 \times S^1$ as the circle mapped to itself by each of the self covers. Choose a collection $\mathcal{Z} = \{Z_0, Z_1, \ldots\}$ of pairwise nonhomeomorphic continua which includes the 2-torus Z_0. $\mathcal{Z}$ is called the *adjunction sequence*. Let

$$\mathcal{Z} = \begin{cases} \{Z_0, Z_1, \ldots Z_{q-1}\}, & \text{if attaching } q \text{ continua;} \\ \{Z_0, Z_1, Z_2, \ldots\}, & \text{if attaching infinitely many continua.} \end{cases}$$

Next, we attach copies of the Z_i to the boundaries ∂B_{kl} of the holes in the standard punctured disk K so that every neighborhood of each point $x \in C$ contains infinitely many copies of all the Z_i's. The case $|\mathcal{Z}| = q$ and the case $|\mathcal{Z}| = |\mathbb{N}|$ begin the same way.

Let $\partial_0 K$ denote the boundary in K of the closed disk β used to form K. Let $\varphi_0 : S_0^1 \to \partial_0 K$ be a homeomorphism. Attach a copy of Z_0 to $\partial_0 K$ via φ_0.

For each $j \geq 1$, choose a subset $\sigma_j \subset Z_j$ which is homeomorphic to a subset $\tau_j \subset S^1$ and a homeomorphism $\varphi_j : \sigma_j \to \tau_j$. The σ_j and τ_j can be as simple as finite point sets, e.g., a single point, or something more complicated.

Next, partition $\mathbb{N}$ into subset $L_n = \{2^{n-1}, 2^{n-1}+1, \ldots, 2^n - 1\}$. Note, $|L_n| = 2^{n-1}$.

For each $(i, j) \in \mathcal{M}$, define α_{ij}, the space attached to ∂B_{ij} in the pattern $\partial\mathcal{B}$, as follows:

(1) If $|\mathcal{Z}| = q$, Let $a = j \ (mod\ q),\ 0 \leq a \leq q-1$, Define $\alpha_{ij} = Z_a$.
(2) If $|\mathcal{Z}| = \infty$, observe that for each $j \in \mathbb{N}$ there is a unique $n(j) \in \mathbb{N}$ such that $j \in L_{n(j)}$. Let $a = j \ (mod\ 2^{n(j)-1})$ where $0 \leq a \leq 2^{n(j)-1}$. Define $\alpha_{ij} = Z_a$.

Finally, attach α_{ij} to ∂B_{ij} via a copy of the associated φ_a. Extend the metric on $K = \beta \setminus \big(\cup \{int(B_{ij}) : B_{ij} \in \mathcal{B}\}\big)$ to the α_{ij} so that $diameter(\alpha_{ij}) = diameter(B_{ij})$. $K_{\mathcal{Z}}$ is the resulting continuum. As a consequence of the manner in which the metric is defined on the α_{ij} it follows that $\mathcal{A}_{\mathcal{Z}} = \{\alpha_{ij} : (i, j) \in \mathcal{M}\}$ is a null sequence in $K_{\mathcal{Z}}$. Thus $(K_{\mathcal{Z}}, \mathcal{A}_{\mathcal{Z}}, C)$ is a patterned space controlled by the middle thirds Cantor set C.

To see that $K_{\mathcal{Z}}$ satisfies the *URF* self cover property, again choose a finite group G and apply the Method of Disk with Holes to obtain the G regular cover of patterned space triples

$$\theta_G : (X^*, \partial\mathcal{B}, C) \to (M^*, \partial\mathcal{B}, C).$$

In $X^* = K \setminus \big(\cup_{i=1}^{n} int(V_i)\big) \cong K$ there are the special boundaries ∂V_i of the holes $int(V_i)$. In $M^* = K \setminus \big(\cup_{j=1}^{m} int(W_j)\big) \cong K$ there are the special boundaries ∂W_j of the holes $int(W_j)$. The collection of special boundaries in the domain copy X^* of K along with $\partial_0\beta \subset X^*$ are the only boundary circles on which the restriction $\theta_G : X^* \to M^*$ might be a cyclic cover of degree greater than 1. The collection of special boundaries in the range copy M^* of K together with $\partial_0 \subset \beta M^*$ are the only circles which are possibly non-trivially covered by θ_G.

Because the 2-torus Z_0 has cyclic self covers of all orders, we attach copies of it to all of the circles $\partial_0 K \cup \big(\cup_{i=1}^{n} \partial V_i)$ in X^* and all of the circles $\partial_0 K \cup \big(\cup_{j=1}^{m} \partial W_i)$ in M^* via copies of φ_0. Because the circle and 2-torus have cyclic covers of all orders, θ_G extends to a covering space map of appropriated degree of each of the 2-tori Z_0 attached to X^* onto the appropriate copy of Z_0 attached to M^*. The degree of the extension of θ_G on each of the domain copies of Z_0 is determined by the degree of θ_G on the boundary circle ∂V_i to which it is attached.

Next, in the range copy M^* of K, there are the circles contained in the pattern $\partial\mathcal{B}$. To each of these circles $\partial B_{ij} \in \partial\mathcal{B}$ attach α_{ij} via a copy of the appropriate φ_a and extend the metric on M^* to the α_{ij} so that $\{\alpha_{ij} : (i,j) \in \mathcal{M}\}$ remains a null sequence of the resulting space as $j \to \infty$. For each ∂B_{ij} in the range M^* there are $|G|$ lifts $\big(\partial B_{ij}\big)^{(1)}, \dots \big(\partial B_{ij}\big)^{(|G|)}$ under θ_G^{-1} to circles $\partial B_{ij}^{(k)} \in \mathcal{B}$ in the domain copy X^* of K. Attach copies of α_{ij} to each of the lifts $\big(\partial B_{ij}\big)^{(1)}, \dots \big(\partial B_{ij}\big)^{(|G|)}$ via the associated φ_a and extend the metric so that each of the $|G|$ lifted collections $\{\alpha_{i,j}^{(k)} : (i,j) \in \mathcal{M}\}^{(k)}$ remains a null sequence in the resulting space. Since each lift $\big(\partial B_{ij}\big)^{(k)}$ is mapped homeomorphically onto its image ∂B_{ij} via θ_G and we have that the attaching map φ_a is a homeomorphism, it follows that θ_G extends to each of the lifted $\alpha_{ij}^{(k)}$ mapping it homeomorphically onto its image α_{ij}. Thus, the domain of the extended θ_G appears to be a copy of $K_{\mathcal{Z}}$ with some "extra" copies of Z_0 attached to the special boundaries ∂V_i. The range of the extended θ_G appears to be a copy of $K_{\mathcal{Z}}$ with "extra" copies of Z_0 attached to the special boundaries ∂W_j. Applying isotopies, e.g. Proposition 1.3, it follows that the "extra" copies of Z_0 can be absorbed into the pattern and that the domain and range of (the extended) θ_G are copies of $K_{\mathcal{Z}}$. This gives us our next result.

Corollary 4.1. *$K_{\mathcal{Z}}$ satisfies the URF self cover property.*

Our last two results are applications of Corollary 4.1 which produce $K_{\mathcal{Z}}$ with special properties. First, set $q = 2$, $Z_0 = S^1$, i.e., we attach nothing to the special boundaries or $\partial\beta$ in either of X^* or M^*, and let Z_1 be a proper subcontinuum of a 2-disk D^2 which contains at least one point of ∂D^2. Let $\mathcal{Z} = \{Z_0, Z_1\}$. The attaching maps φ_a in this case are identity maps.

Corollary 4.2. *There exist exists an adjunction sequences $\mathcal{Z}$ such that the associated $K_{\mathcal{Z}}$ is a planar continua, $K_{\mathcal{Z}}$ satisfies the URF self cover property, and $K_{\mathcal{Z}}$ is not a disk with holes.*

We can also produce $K_{\mathcal{Z}}$ which are simple. To do so, decide whether to attach finitely or infinitely many continua to the boundaries of the holes in K and choose $Z_0 = S^1$. Then, for $i \geq 1$, let Z_i be a once punctured surface, i.e., a compact 2-manifold with one boundary component $\partial Z_i = S^1$ and attach Z_i to the associated boundary circle in K via a homeomorphism. This gives our last result.

Corollary 4.3. *There exists an adjunction sequence $\mathcal{Z}$ such that the associated controlled pattern spaces $(K_{\mathcal{Z}}, \mathcal{A}_{\mathcal{Z}}, C)$ controlled by the middle thirds Cantor set C is simple and satisfies the URF self cover property.*

5. Questions

There are controlled patterned disks with holes which satisfies the universal finite-sheeted regular self cover property, but which have control sets other than the Cantor set. One such is the controlled patterned disk which yields the Sierpinski curve. We describe another.

Start with $\beta = [-\epsilon, 2-\epsilon] \times [-1, 1]$. For each $(i, j) \in \mathcal{M}$ let $D_{ij} = [\frac{2i+1}{2} - \frac{1}{4} \cdot \frac{1}{3^j}, \frac{2i+1}{2} + \frac{1}{4} \cdot \frac{1}{3^j}] \times [-\frac{1}{2}, \frac{1}{2}]$. That is, each D_{ij} is a rectangle of width $\frac{1}{2} \cdot \frac{1}{3^j}$ and height 1 centered at one of the midpoints of one of the "deleted intervals" used to form the middle thirds Cantor set $C \subset [0, 1] \times 0 \subset \beta$. Let $K^s = \beta \setminus \cup\{int(D_{ij}) : (i, j) \in \mathcal{M}\}$. Then $(K^s, \{\partial D_{ij} : (i, j) \in \mathcal{M}\}, C \times [-\frac{1}{2}, \frac{1}{2}])$ is a patterned disk with holes controlled by $C \times [-\frac{1}{2}, \frac{1}{2}]$. The proof that K^s satisfies the universal finite sheeted self cover property is an application of the Method of Disks with Holes. Observe that K^s is simple: $K^s \setminus C \times [-\frac{1}{2}, \frac{1}{2}]$ produces a disk with $|\mathbb{N}|$ holes and infinitely many punctures.

In closing, we want to pose two questions, a version of which was asked at the 2016 Summer Conference on Topology and Its applications. The disk with holes K of Theorem 1; the spaces of Corollaries 4.1, 4.2, and 4.3;

the patterned disk with square holes K^s; and even the Sierpinski curve $\mathcal{S}$ all have control sets, or in the case of S, S itself, which are fractals. In general, we ask what the relationship is between the universal regular self covering property and fractals. More specifically, we ask the following:

Question 5.1. Is it the case that if X is simple, has dimension 2, and the patterned space $(X, \mathcal{B}, Y)$ with control set Y satisfies the universal finite sheeted regular self cover property, then Y is a Cantor set or the product of a Cantor set and an interval?

Dedication. This paper is dedicated to the memory of my brother Mitch, a kind gentle man with a good practical geometric intuition.

References

[1] Bedenikovic, T., Delgado, A., Timm, M., *A classification of 2-complexes with nontrivial self-covers*, Topology and Its Applications, **153** (2006), 2073–2091.

[2] Bellamy, D. P., *Certain analytic preimages of pseudocircles are pseudocircles*, Spring Topology and Dynamical Systems Conference, Topology Proc. **29** (2005), no. 1, 19–25.

[3] Calcut, J.S., Gompf, R.E., McCarthy, J.D., *Quotient maps with connected fibers and the fundamental group*, Topology Appl. **159** (2012), 322–330. Arxiv:0904.4465v1, [math.GN], 28 April 2009.

[4] Cannon, J.W., Conner, G.R., *The combinatorial structure of the Hawaiian earring group*, Topology and its Applications **106** (2000), 225–271.

[5] ———, *On the fundamental groups of one-dimensional spaces*, Topology and its Applications **153** (2006), 2648–2672.

[6] ———, *The big fundamental group, big Hawaiian earrings, and the big free groups I*, Topology and its Applications **106** (2000), 273–291.

[7] Delgado, A., Timm, M., *Spaces with regular nonabelian self covers*, Houston J. Math, **43:4** (2017), 1323–1336.

[8] Eda, K., Mandi'cb, J., Matijevvi'cc, *Torus-like conintua which are not self-covering spaces*, Topology and its Applications, (2005), 359–369.

[9] Daverman, R.J., *Fibrator properties of manifolds*, Topology and its Applications **153** (2005), 245–260.

[10] ———, *Problem 5*, Proceedings of the 21st Annual Workshop in Geometric Topology, University of Wisconsin Milwaukee, June 10–12, 2004. www.uwm.edu/∼craigg/Annual%20Workshop%20Geometric%20Topology.html

[11] ———, *Hyperhopfian groups and approximate fibrations*, Compositio Mathematica **86** (1993), 159–176.

[12] Heath, J.W., *Weakly confluent, 2-to-1 maps on hereditarily indecomposable continua*, Proc. Amer. Math. Soc. **117** (1993), no. 2, 569–573.

[13] Hirshon, R., Misbehaved direct products., Expo. Math. **20** (2002), no. 4, 365–374.

[14] Jungck, G., Timm, M., *H-connected inverse limit spaces*, Topology Proceedings, **24** (1999), 173–187.

[15] Mardešić, S., Matijević, V., *Classifying overlay structures of topological spaces*, Topology and Appl. **113** (2001), 167–209.

[16] Matijević, V., *Classifying finite-sheeted coverings of paracompact spaces*, Revista Matematica Complutense, **16** (2003), no. 1, 311–327.

[17] ______, *Finite-sheeted covering maps over Klein bottle weak solenoid spaces*, Glasnik Matematicki, **42** (62) (2007), 19–41.

[18] Moore, R.L., Kline, J.R., *On the most general plane closed point-set through which it is possible to pass a simple continuous arc*, Annals of Mathematics, Second Series, Vol. **20**, No. 3 (Mar., 1919), 218–223.

[19] Nanyes, O., Timm, M., *On infinitely periodic knots*, Bol. Soc. Mat. Mexicana, 3 **10** (2004), 515–523.

[20] Robinson, D.J.S., Timm, M., *On groups that are isomorphic with every subgroup of finite index and their topology*, J. London Math. Soc. (2) **57** (1998), no. 1, 91–104.

[21] Timm, M., *Coverings of a 1-dimensional continuum*, Q & A, **22** (2004), 9–12.

[22] Wang, S., Wu, Y.-Q., *Covering invariants and cohopficity of 3-manifold groups*, Proc. London Math. Soc. (3) **68** (1994), 203–224.

[23] Whyburn, G., *Topological characterization of the Sierpinski curve*, Fund. Math. **45** (1958), 320—324.

Department of Mathematics; Bradley University; Peoria, Illinois 61625
Email address: `mtimm@bradley.edu`