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Joan E. Hart and Kenneth Kunen

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Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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SPACES WITH NO S OR L SUBSPACES

JOAN E. HART AND KENNETH KUNEN

Abstract. We consider spaces that contain neither an S-space nor an L-space. We call such a space ESLC, and show that it is consistent with ZFC for a product of two ESLC spaces to contain both an S-space and an L-space.

1. Introduction

All topological spaces considered in this paper are T_3 (Hausdorff and regular).

The following terminology is standard: A space X is *hereditarily separable* (*HS*) iff all subspaces of X are separable, and *hereditarily Lindelöf* (*HL*) iff all subspaces of X are Lindelöf. Then, X is an *S-space* iff X is HS but not HL, and X is an *L-space* iff X is HL but not HS.

S-spaces are consistent with $\mathsf{MA}(\aleph_1)$ [14], but are refuted by PFA [16], while L-spaces exist in ZFC [10]. Under CH, a large variety of S-spaces and L-spaces have been constructed, and still more have been built under $\diamondsuit$.

In this paper, we shall study the following notion:

Definition 1.1. The space X is *ESLC* iff every subspace of X is either both HS and HL or neither HS nor HL.

This is the same as saying that no subspace of X is either an S-space or an L-space. The reason for the "C" in "ESLC" is the following slightly non-standard terminology for a standard concept:

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Definition 1.2. The space X is *HC* (hereditarily ccc) iff every subspace of X has the ccc (countable chain condition).

Translating to more standard terminology, X is HC iff X has countable spread iff X has no uncountable discrete subspaces. The "no uncountable" version parallels the well-known ([12], Theorem 3.1) characterizations of HS and HL: X is HS iff X has no left separated ω_1 sequences and X is HL iff X has no right separated ω_1 sequences. A sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ is *left separated* provided that each $x_\alpha \notin \mathrm{cl}(\{x_\xi : \xi < \alpha\})$ and *right separated* provided that each $x_\alpha \notin \mathrm{cl}(\{x_\xi : \xi > \alpha\})$.

Observe that HS$\to$HC and HL$\to$HC hold trivially, and the disjoint union of an S-space and an L-space is HC but is neither HS nor HL. As the "C" suggests, X is ESLC iff for every subspace of X HS $\leftrightarrow$ HL $\leftrightarrow$ HC. To prove the $\Rightarrow$ direction of this "iff", say X is ESLC and $Y \subseteq X$ is not HL. Shrinking Y, WLOG Y is right separated in type ω_1. Applying ESLC, Y is also not HS, so there is a $D \in [Y]^{\aleph_1}$ that is also left separated by the same ordering. But then, D is discrete, so Y is not HC. So X is **E**SLC iff these three properties are **E**quivalent for X.

Two classes of ESLC spaces are the spaces that are both HS and HL (the spaces we call HS+HL), and the metrizable spaces. The HS+HL spaces, such as the Sorgenfrey line, are HS+HL+HC. Metrizable spaces are trivially ESLC, because every metric space is either second countable or has an uncountable discrete subspace.

In Section 4 we see that the *semi*-metrizable spaces provide a much broader class of ESLC spaces. We also show that countable products of generalized butterfly spaces form a large class of ESLC spaces. A special case of this is: Let X_n for $n \in \omega$ be a subspace of the Sorgenfrey line; then each X_n is HS and HL, and hence ESLC. There are many examples, in various models of set theory, of $\prod_n X_n$ being HS and HL, or being neither HS nor HL. But $\prod_n X_n$ is always ESLC by Lemmas 4.6 and 3.2, so it can never contain an S-space or an L-space.

Even though ESLC holds in products of our butterfly spaces, ESLC productivity is not provable in ZFC; in Section 7 we shall show that it is consistent to have both X, Y satisfy HS+HL (and hence ESLC) with $X \times Y$ containing an S-space. This follows from a stronger result:

Theorem 1.3. *It is consistent with* ZFC *to have* X, Y *both strongly HG* (*and hence strongly HS+HL*), *with* $X \times Y$ *containing a strong S-space and a strong L-space.*

The property HG, defined in Section 2, is a natural strengthening of HS+HL. Recall that X is a strong S-space iff X is not HL and X^n is HS for all $n \in \omega$ (and hence also X^ω is HS by Lemma 3.2). Finite powers determine strong L-spaces and other *strong* properties similarly:

Definition 1.4. If $\mathcal{P}$ is some property of spaces, then we say that X is *strongly* $\mathcal{P}$, or X is *st*$\mathcal{P}$, iff all finite powers of X have $\mathcal{P}$.

Strong S-spaces and strong L-spaces are refuted by $\mathsf{MA}(\aleph_1)$ [8], but exist under CH. Furthermore, just in ZFC, there exists a strong S-space iff there exists a strong L-space [17]. For more details about these topics, see [12].

The ideas behind the proof of Theorem 1.3 are discussed in Section 6, where we introduce a new axiom, which we call the COMA (Chain of Models Axiom). This axiom follows from CH and is also true in every forcing extension $V[G]$ obtained by the addition of one Cohen real or one random real. The COMA unifies features of Cohen and random real forcing and of CH constructions of elementary submodel chains. These three classes of models were already known to provide examples of strong S-spaces and L-spaces, as well as of two ccc posets whose product is not ccc. In Section 6 we show how the COMA produces these old results, and in Section 7 we show how it yields the strongly HG spaces of Theorem 1.3.

2. A Fourth Property

We describe here the property HG, which is called the *pointed ccc* in Gruenhage [5]. This property is a natural strengthening of HS+HL:

$$\begin{tikzcd} & \mathrm{HS} \arrow[dr] & \\ \mathrm{HG} \arrow[ur] \arrow[dr] & & \mathrm{HC} \\ & \mathrm{HL} \arrow[ur] & \end{tikzcd}$$

As pointed out in the Introduction, the two arrows on the right do not reverse assuming CH because the direct sum of an S-space and an L-space is HC, but neither HS nor HL. Although restricting the diagram to ESLC spaces gives us HS $\leftrightarrow$ HL $\leftrightarrow$ HC, it does *not* make HG an equivalent property too; for example, the Sorgenfrey line is ESLC because it is HS+HL, but it is not HG. This, and the two new implications will be obvious from the definition of HG:

Definition 2.1. A *weakly separated sequence* in X is a sequence $\langle x_\alpha : \alpha < \omega_1\rangle$ together with open $U_\alpha \ni x_\alpha$ for $\alpha < \omega_1$, such that for all $\{\alpha, \beta\} \in [\omega_1]^2$, either $x_\alpha \notin U_\beta$ or $x_\beta \notin U_\alpha$ (or both). Then X is *HG (hereditarily good)* iff there are no weakly separated sequences in X.

The connection between weakly separated subsets and HS and HL is also discussed in [16] (see page 29). In [15], Tkačenko calls a subspace слабо *растянутым* if it satisfies the condition on sequences in our Definition 2.1, and he defines the associated cardinal function $R(X)$.

Hajnal and Juhász, in [7], loosely translate the term to weakly *separated* and note that $R(X) = \aleph_0$ implies that X is HS and HL, because $R(X)$ is the supremum of the sizes of weakly separated subspaces of X. Now, a space X is HG iff $R(X) \leq \aleph_0$.

To summarize: Let $\mathcal{U}$ denote an arbitrary sequence $\langle (x_\alpha, U_\alpha) : \alpha < \omega_1 \rangle$, where each U_α is open in X and $x_\alpha \in U_\alpha$. Then X is:

HS iff $\forall \mathcal{U}\, \exists \alpha < \beta\, [x_\alpha \in U_\beta]$ iff X has no left separated sequence;
HL iff $\forall \mathcal{U}\, \exists \alpha < \beta\, [x_\beta \in U_\alpha]$ iff X has no right separated sequence;
HC iff $\forall \mathcal{U}\, \exists \alpha \neq \beta\, [x_\beta \in U_\alpha]$ iff X has no discrete sequence;
HG iff $\forall \mathcal{U}\, \exists \alpha \neq \beta\, [x_\beta \in U_\alpha \;\&\; x_\alpha \in U_\beta]$ iff X has no weakly separated sequence.

Definition 2.2. If $\mathcal{U} = \langle (x_\alpha, U_\alpha) : \alpha < \omega_1 \rangle$ witnesses the failure of X to be HS, HL, HC, or HG, call $\mathcal{U}$ a *separating sequence* for X.

The related strong properties (see Definition 1.4) give us essentially the same diagram, but with the prefix "st" in each node. Assuming CH, the existence of strong S-spaces and strong L-spaces shows that none of the four arrows in the "strong" picture reverse. The Sorgenfrey line is HS+HL, but its square is not even HC. Assuming CH, however, we can let X be a Sorgenfrey line built on an uncountable set of reals that is n-entangled for all $n < \omega$ (see [1, 2]); then X is both stHS and stHL but not HG.

In contrast to the *non*-strong diagram, the rightmost arrows in the strong version *almost* reverse: stHC $\leftrightarrow$ (stHS *or* stHL). If X is neither stHS nor stHL, then we have $m, n \in \omega$, along with a sequence $\langle \vec{x}_\alpha : \alpha < \omega_1 \rangle$ that is left separated in X^m and a sequence $\langle \vec{y}_\alpha : \alpha < \omega_1 \rangle$ that is right separated in X^n. But then the sequence $\langle (\vec{x}_\alpha, \vec{y}_\alpha) : \alpha < \omega_1 \rangle$ is discrete in X^{m+n}.

If $\mathcal{P}$ is one of our properties, consider the following four closure meta-properties (properties of properties):

1. $\mathcal{P}$ is *hereditary* iff it is closed under subspaces. In general, if $\mathcal{P}$ is hereditary then so is st$\mathcal{P}$. The properties HS, HL, HC, HG are clearly hereditary.

2. $\mathcal{P}$ is closed under *countable unions* iff whenever $X = \bigcup_{n \in \omega} Z_n$ and each Z_n has $\mathcal{P}$, then X has $\mathcal{P}$. The properties HS, HL, HC, HG are closed under countable unions; but assuming CH, the corresponding strong properties are not. For stHL, stHS, stHC, there is a well-known counter-example: Let $X = Z_0 \,\dot\cup\, Z_1$ be the disjoint union of the Sorgenfrey line Z_0 built on an uncountable set of reals E that is n-entangled for all $n < \omega$, and the Sorgenfrey line Z_1 built on the set of reals $\{-z : z \in E\}$. Then both Z_0 and Z_1 are stHL, stHS, and stHC, but $\{(z, -z) : z \in E\}$

is discrete in X^2 so that X is not even stHC. For stHG: Section 7 will produce two stHG spaces Z_0, Z_1 such that $Z_0 \times Z_1$ is not HG; then their disjoint union $Z_0 \dot{\cup} Z_1$ is not stHG.

3. $\mathcal{P}$ is closed under *continuous images* iff whenever f maps X continuously *onto* Y and X has $\mathcal{P}$, then Y has $\mathcal{P}$. In general, if $\mathcal{P}$ is closed under continuous images then so is st$\mathcal{P}$.

Lemma 2.3. *Each of the properties HS, HL, HC, HG is closed under continuous images.*

Proof. If Y is not H$\mathcal{P}$, for one of these properties $\mathcal{P}$, let $\mathcal{U} = \langle (y_\alpha, U_\alpha) : \alpha < \omega_1 \rangle$ be a separating sequence, as in Definition 2.2. Choose $x_\alpha \in f^{-1}\{y_\alpha\}$ for each α. Then in X, $\langle (x_\alpha, f^{-1}(U_\alpha)) : \alpha < \omega_1 \rangle$ is also a separating sequence. $\square$

4. $\mathcal{P}$ is *productive* iff whenever X and Y both have $\mathcal{P}$, then $X \times Y$ has $\mathcal{P}$. See Section 3 for an overview of productivity properties of HS, HL, HC, and HG. Section 4 looks at two more classes of spaces that are of interest when examining products: the productive semi-metrizable spaces, and the generalized butterfly spaces, whose products need not be butterflies, but are ESLC.

3. Quasi-Productive Properties

As usual, we have called a property $\mathcal{P}$ of topological spaces *productive* iff whenever X and Y both have $\mathcal{P}$, $X \times Y$ also has $\mathcal{P}$. It follows that whenever X_n have $\mathcal{P}$ for $n < k \in \omega$, then $\prod_{n<k} X_n$ also has $\mathcal{P}$. But the same may fail for ω spaces. So we introduce the following:

Definition 3.1. A property $\mathcal{P}$ of topological spaces is *quasi-productive* iff whenever X_n are spaces for $n < \omega$ and $\prod_{n<k} X_n$ has $\mathcal{P}$ for all finite k, then $\prod_{n<\omega} X_n$ also has $\mathcal{P}$.

For $k = 0$, let $\prod_{n<0} X_n$ be the trivial one-point space $\{\emptyset\}$.

Lemma 3.2. *The following properties are quasi-productive: HS, HL, HC, HG, ESLC, compactness, ccc.*

Proof. The proof for compactness is clear from the Tychonov Theorem, and the proof for ccc is well-known.

The proofs for HS, HL, HC, HG are all similar. For HS: Suppose that $\prod_{n<\omega} X_n$ is not HS, so it contains a left separated ω_1 sequence $\langle \vec{x}_\alpha : \alpha < \omega_1 \rangle$. Then we have open U_α for $\alpha < \omega_1$ such that $\vec{x}_\alpha \in U_\alpha$ but $\alpha < \beta \to \vec{x}_\alpha \notin U_\beta$. Each $\vec{x}_\alpha = \langle x^n_\alpha : n < \omega \rangle$, with $x^n_\alpha \in X_n$. Shrinking the U_α if necessary, WLOG each $U_\alpha = \prod_{n<\omega} U^n_\alpha$ where U^n_α is open in X_n and $U^n_\alpha = X_n$ for all $n \geq k_\alpha$ for some $k_\alpha \in \omega$. Then, thinning the sequence, WLOG all $k_\alpha = k$ for some fixed $k \in \omega$. But then $\langle \vec{x}_\alpha \restriction k : \alpha < \omega_1 \rangle$ is left separated in $\prod_{n<k} X_n$, so $\prod_{n<k} X_n$ is not HS.

For ESLC: Suppose that $\prod_{n<\omega} X_n$ is not ESLC, so it contains an S-space or an L-space. Say it contains an L-space. Then it contains a left separated ω_1 sequence $\langle \vec{x}_\alpha : \alpha < \omega_1 \rangle$ with the additional property that $\{\vec{x}_\alpha : \alpha < \omega_1\}$ is HL. By the above argument, we may thin the sequence and say that WLOG, there is a $k \in \omega$ such that $\langle \vec{x}_\alpha \restriction k : \alpha < \omega_1 \rangle$ is left separated in $\prod_{n<k} X_n$. But $\{\vec{x}_\alpha \restriction k : \alpha < \omega_1\}$ is also HL because it is the continuous image of an HL space (Lemma 2.3), so $\prod_{n<k} X_n$ is not ESLC. □

A property may be both productive and quasi-productive (such as compactness), or productive but not quasi-productive (such as "finite" or "scattered" or "discrete"), or quasi-productive but not productive (such as HS, HL, HC), or neither productive nor quasi-productive (such as "homeomorphic to some ordinal $\leq \omega + 1$"). For HS, HL, HC, the Sorgenfrey line is a simple counter-example to productivity. The productivity of the ccc is independent of ZFC. We shall see in Section 7 that HG and ESLC are consistently non-productive.

4. Butterflies and Semi-metrics

These two standard classes do *not* provide examples in which ESLC is non-productive.

The first class is a generalization of metrizable spaces, discussed in detail in the chapter by Gruenhage [6]. He discusses two generalizations:

$$\text{metrizable} \Rightarrow \text{semi-metrizable} \Rightarrow \text{symmetrizable} \quad .$$

But symmetrizable spaces need not be ESLC, since Shakhmatov [13] shows that it is consistent to have a symmetrizable L-space. Semi-metrizable spaces *are* ESLC, so we give the definition in detail:

Definition 4.1. Given any (T_3) space $(X, \mathcal{T})$, the function $d : X \times X \to \mathbb{R}$ is a *semi-metric* for X iff d satisfies the following four conditions. As with metrics, define $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$.

1. For each $x, y \in X$: $d(x, y) = d(y, x)$.
2. For each $x, y \in X$: $d(x, y) \geq 0$, and $d(x, y) = 0 \leftrightarrow x = y$.
3. For each $x \in X$ and $\varepsilon > 0$: $x \in \text{int}(B(x, \varepsilon))$.
4. Each $U \subseteq X$ is open iff for each $x \in U$ there exists $\varepsilon > 0$ with $x \in B(x, \varepsilon) \subseteq U$.

Then X is *semi-metrizable* iff there exists a semi-metric for X.

Unlike with metrics, $B(x, \varepsilon)$ need not be open. But semi-metrizability is hereditary. We use this to get the following:

Lemma 4.2. *If X is semi-metrizable, then the properties HC, HS, HL, HG are all equivalent for X. In particular, X is ESLC.*

Proof. It is sufficient to show that if X is HC, then it is also HG. The "in particular" then follows from the fact that semi-metrizability is hereditary.

Assume that X is not HG, and let $\mathcal{U} = \langle (x_\alpha, U_\alpha) : \alpha < \omega_1 \rangle$ be a separating sequence. Then for all $\{\alpha, \beta\} \in [\omega_1]^2$, either $x_\alpha \notin U_\beta$ or $x_\beta \notin U_\alpha$. Using (4), each $U_\alpha \supseteq B(x_\alpha, 2^{-n_\alpha})$ for some $n_\alpha \in \omega$. Thinning the sequence, WLOG $n_\alpha = n \in \omega$ for all α. But now, we have $d(x_\alpha, x_\beta) \geq 2^{-n}$ for all $\{\alpha, \beta\} \in [\omega_1]^2$, so by (3) the sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ is discrete. $\square$

In contrast, the Sorgenfrey line is HS+HL but not HG.

It is easy to see that products of semi-metrizable spaces will still be ESLC:

Lemma 4.3. *The product of two semi-metrizable spaces is semi-metrizable.*

Proof. Assume that d_1, d_2 are semi-metrics on spaces X_1, X_2, respectively. The function d on $X_1 \times X_2$ defined by: $d((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2))$ yields $B((x_1, x_2), \varepsilon) = B(x_1, \varepsilon) \times B(x_2, \varepsilon)$, and satisfies the conditions of Definition 4.1. $\square$

The second class here, that of generalized butterflies, is not closed under its own products, but its countable products are ESLC. For example, the Sorgenfrey line is a generalized butterfly space, but its square is not, even though it is still ESLC.

The butterfly space (or bowtie space) is a standard example in topology. The details vary with the exposition. The next definition presents a generalization of this concept. For a somewhat different generalization, see Burke and van Douwen [3].

Definition 4.4. Let (X, d) be a separable metric space with topology $\mathcal{T}$. A *butterfly refinement* $\widehat{\mathcal{T}}$ of $\mathcal{T}$ is obtained as follows. For each $x \in X$ choose sets U_x^n for $n \in \omega$ such that $x \in U_x^n$, and $U_x^n \backslash \{x\}$ is $\mathcal{T}$ open, and $\mathrm{diam}(U_x^n) \searrow_n 0$, and each $\mathrm{cl}(U_x^{n+1}, \mathcal{T}) \subseteq U_x^n$. Then $\widehat{\mathcal{T}}$ is the topology on X with base $\{U_x^n : x \in X \ \& \ n \in \omega\}$. A *generalized butterfly space* is a butterfly refinement of some separable metric space.

Note that for each x, $\{U_x^n : n \in \omega\}$ is a local base at x in $\widehat{\mathcal{T}}$.

A simple example: If $X \subseteq \mathbb{R}$ with the usual metric and $U_x^n = X \cap (x - 2^{-n}, x]$, then $\widehat{\mathcal{T}}$ is the set X with the Sorgenfrey topology. So, unlike semi-metrizable spaces, generalized butterfly spaces can be HS and HL without being HG.

Some remarks on the definition: The assumption that $U_x^n \backslash \{x\}$ is $\mathcal{T}$ open and $\operatorname{diam}(U_x^n) \searrow_n 0$ guarantees that $\{U_x^n : x \in X \ \& \ n \in \omega\}$ is indeed a base for some topology that refines $\mathcal{T}$, and the assumption that each $\operatorname{cl}(U_x^{n+1}, \mathcal{T}) \subseteq U_x^n$ guarantees that $\widehat{\mathcal{T}}$ is T_3. If each U_x^n is $\mathcal{T}$ open, then $\widehat{\mathcal{T}} = \mathcal{T}$. At the other extreme, $\forall x\, [U_x^n = \{x\}]$ satisfies the definition, making $\widehat{\mathcal{T}}$ discrete.

The $\widehat{\mathcal{T}} = \mathcal{T}$ and discrete examples show that $\widehat{\mathcal{T}}$ might be both HS and HL or neither HS nor HL. In fact, a butterfly refinement $(X, \widehat{\mathcal{T}})$ is always ESLC. More generally, Lemmas 4.6 and 3.2 show that every countable product of generalized butterfly spaces is ESLC. On the other hand, Lemma 4.5 shows that a product of even two generalized butterflies need not be a generalized butterfly.

The Sorgenfrey square is not a butterfly refinement of *any* separable metric topology on $\mathbb{R}^2$. Actually, a much stronger statement is true:

Lemma 4.5. *Give $\mathbb{R}$ the Sorgenfrey topology, and let Y be any non-discrete T_3 space. Then $\mathbb{R} \times Y$ is not a butterfly refinement of* any *separable metric topology $\mathcal{T}$ on $\mathbb{R} \times Y$.*

Proof. If $\mathbb{R} \times Y$ were a butterfly refinement, then since all such butterflies are first countable, Y would be first countable. Then Y would contain a copy of $\omega+1$ with its usual topology; that is, a simple convergent sequence. Then, since butterfly is hereditary, we may assume that $Y = \omega + 1$. Let $\widehat{\mathcal{T}}$ denote the product topology on the set $Z := \mathbb{R} \times (\omega + 1)$, using the Sorgenfrey topology on $\mathbb{R}$. Let $\mathcal{T}$ be an *arbitrary* separable metric topology on the set Z. Now, assume that $\widehat{\mathcal{T}}$ is a butterfly refinement of $\mathcal{T}$, and we shall derive a contradiction.

In the following, $\operatorname{diam}(E)$ is the diameter of E using some fixed metric for $\mathcal{T}$. Using the terminology of Definition 4.4, we are assuming that: In $\widehat{\mathcal{T}}$, each $z \in Z$ has a local open neighborhood base $\{U_z^n : n \in \omega\}$ such that $z \in U_z^n$, and $U_z^n \backslash \{z\}$ is $\mathcal{T}$ open, and $\operatorname{diam}(U_z^n) \searrow_n 0$, and each $\operatorname{cl}(U_z^{n+1}, \mathcal{T}) \subseteq U_z^n$.

For each $x \in \mathbb{R}$: $(-\infty, x] \times (\omega+1)$ is a $\widehat{\mathcal{T}}$ open neighborhood of the point (x, ω), so we can fix $n_x \in \omega$ such that $U_{(x,\omega)}^{n_x} \subseteq (-\infty, x] \times (\omega + 1)$. Then, since $U_{(x,\omega)}^{n_x}$ is also a $\widehat{\mathcal{T}}$ open neighborhood of (x, ω), we can fix $m_x \in \omega$ and $\varepsilon_x > 0$ such that $(x - \varepsilon_x, x] \times [m_x, \omega] \subseteq U_{(x,\omega)}^{n_x}$. Let $W_x = U_{(x,\omega)}^{n_x} \backslash \{(x, \omega)\}$, which is $\mathcal{T}$ open.

Let $\mathcal{B} \subseteq \mathcal{P}(Z)$ be some countable open base for the topology $\mathcal{T}$. For each $x \in \mathbb{R}$, choose $\mathcal{D}_x \subseteq \mathcal{B}$ such that $W_x = \bigcup \mathcal{D}_x$. Let $\mathcal{E}_x = \{B \in \mathcal{D}_x : B \cap (\{x\} \times [m_x, \omega)) \neq \emptyset\}$.

Note that $\mathcal{E}_x \neq \emptyset$ because $\{x\} \times [m_x, \omega) \subseteq W_x = \bigcup \mathcal{D}_x$. Also, $x_1 < x_2 \to \mathcal{E}_{x_1} \cap \mathcal{E}_{x_2} = \emptyset$ because $B \in \mathcal{E}_{x_1} \to B \subseteq W_{x_1} \subseteq (-\infty, x_1] \times (\omega+1)$, while $B \in \mathcal{E}_{x_2} \to B \cap (\{x_2\} \times (\omega+1)) \neq \emptyset$. But this is a contradiction because $|\mathcal{B}| = \aleph_0 < |\mathbb{R}|$. $\square$

Whether or not it is a butterfly, a countable product of butterflies is still ESLC, by Lemmas 3.2 and 4.6.

Lemma 4.6. *Every finite product of generalized butterfly spaces is ESLC.*

Proof. Our product space is $\prod_{i<n} Z_i$, where each Z_i has the butterfly topology $\widehat{\mathcal{T}_i}$. All that we use about $\widehat{\mathcal{T}_i}$ is that it refines some separable metric topology $\mathcal{T}_i$ on Z_i, and for each $z \in Z_i$, $\{V \in \widehat{\mathcal{T}_i} : z \in V \ \&\ (V\backslash\{z\}) \in \mathcal{T}_i\}$ is a local base at z in $\widehat{\mathcal{T}_i}$. On $\prod_{i<n} Z_i$, let $\widehat{\mathcal{T}}$ denote the product topology $\prod_{i<n} \widehat{\mathcal{T}_i}$, and let $\mathcal{T}$ denote the separable metric topology $\prod_{i<n} \mathcal{T}_i$. So, we need to show that $(\prod_{i<n} Z_i, \widehat{\mathcal{T}})$ is ESLC.

We proceed by induction on n, with $n = 0$ being the trivial case for the one-element space. So, fix $n > 0$, assume that the lemma holds for $n-1$, and we shall prove that in $\widehat{\mathcal{T}}$, every left separated ω_1 sequence in $\prod_{i<n} Z_i$ has an uncountable discrete subsequence. Essentially the same argument works starting from a right separated ω_1 sequence.

Let our left separated sequence be $\langle \vec{z}_\alpha : \alpha < \omega_1 \rangle$, where each $\vec{z}_\alpha = \langle z^i_\alpha : i < n \rangle$. Let V_α for $\alpha < \omega_1$ denote the left separating neighborhoods, which we may assume are of the form $V_\alpha = \prod_{i<n} V^i_\alpha$. So $\vec{z}_\alpha \in V_\alpha$, and each V^i_α is $\widehat{\mathcal{T}_i}$ open, and $z^i_\alpha \in V^i_\alpha$, and we may furthermore assume that each $V^i_\alpha \backslash \{z^i_\alpha\}$ is $\mathcal{T}_i$ open. By "left separating" we mean that whenever $\alpha < \beta$: $\vec{z}_\alpha \notin V_\beta$, so $\exists i < n\ [z^i_\alpha \notin V^i_\beta]$.

Thinning the sequence, we may assume that for each i, either (1) $\forall \alpha, \beta\ [z^i_\alpha = z^i_\beta]$ or (2) $\forall \alpha, \beta\ [\alpha \neq \beta \to z^i_\alpha \neq z^i_\beta]$. But if (1) holds for some i, then we can delete that coordinate and apply inductively the $n-1$ case of the lemma. So, WLOG (2) holds for all i.

Thinning some more, WLOG the set $\{\vec{z}_\alpha : \alpha < \omega_1\}$ is $\aleph_1$–dense in itself in the separable metric topology $\mathcal{T}$. Then, let $C \subset \omega_1$ be a club such that for all $\xi \in C$, the set $\{\vec{z}_\alpha : \xi < \alpha < \xi^{+C}\}$ is $\mathcal{T}$ dense in $\{\vec{z}_\alpha : \alpha < \omega_1\}$, where $\xi^{+C} = \min(C \backslash (\xi+1))$.

We shall now show that whenever $\xi, \eta \in C$ with $\min(C) < \xi < \eta$ and $\xi < \alpha < \xi^{+C} \leq \eta < \beta < \eta^{+C}$, we have $\vec{z}_\beta \notin V_\alpha$. Since we already know that $\vec{z}_\alpha \notin V_\beta$, it follows that whenever $E \subset \omega_1$ contains exactly one element from the interval (ξ, ξ^{+C}) for each $\xi \in C$, the set $\{\vec{z}_\alpha : \alpha \in E\}$ is discrete in $\widehat{\mathcal{T}}$, so we are done.

To prove $\vec{z}_\beta \notin V_\alpha$: Fix $\zeta \in C$ with $\zeta < \xi < \eta$. Then $\zeta^{+C} \leq \xi < \alpha$. Since V_α is left separating, $\emptyset = \{\vec{z}_\delta : \zeta < \delta < \zeta^{+C}\} \cap V_\alpha \supseteq \{\vec{z}_\delta : \zeta < \delta < \zeta^{+C}\} \cap \prod_{i<n}(V^i_\alpha \backslash \{z^i_\alpha\})$. Thus, since $\vec{z}_\beta$ is a $\mathcal{T}$ limit point of $\{\vec{z}_\delta : \zeta < \delta < \zeta^{+C}\}$

and $\prod_{i<n}(V_\alpha^i\backslash\{z_\alpha^i\})$ is $\mathcal{T}$ open, we have $\vec{z}_\beta \notin \prod_{i<n}(V_\alpha^i\backslash\{z_\alpha^i\})$. Finally, $\vec{z}_\beta \notin \prod_{i<n} V_\alpha^i = V_\alpha$, because $\alpha \neq \beta \to z_\alpha^i \neq z_\beta^i$. $\square$

5. Duality and Normal Forms

The results that we shall prove in Sections 6 and 7 will use the following general method for constructing spaces, suggested by the article of Roitman [12].

Start with a function $\Psi : I \times J \to \omega$, where the sets I, J will often (but not always) be ω_1. We may view Ψ as a matrix encoding either an I-indexed subfamily of ω^J (the vertical functions) or a J-indexed subfamily of ω^I (the horizontal functions). The basic notation is given by:

Definition 5.1. Let I, J be any non-empty sets and fix $\Psi : I \times J \to \omega$. For $\alpha \in I$ and $\beta \in J$, define $f_\beta^\Psi(\alpha) = g_\alpha^\Psi(\beta) = \Psi(\alpha, \beta)$. Then let $\mathcal{F}^\Psi = \{f_\beta^\Psi : \beta \in J\} \subseteq \omega^I$ and $\mathcal{G}^\Psi = \{g_\alpha^\Psi : \alpha \in I\} \subseteq \omega^J$. We drop the superscript Ψ when Ψ is clear from context.

Our examples in Sections 6 and 7 will all be spaces of the form $\mathcal{F}^\Psi$ or $\mathcal{G}^\Psi$, where Ψ is constructed by using the COMA. We remark that if X is any zero dimensional space, then $X \cong \mathcal{F}^\Psi$ for an appropriate Ψ (with $|J| = |X|$ and $|I| = w(X)$), and $X \cong \mathcal{G}^\Psi$ for an appropriate Ψ (with $|I| = |X|$ and $|J| = w(X)$).

In this section, we shall show that if $\mathcal{F}$ or $\mathcal{G}$ *fails* to have one of the properties stHL, stHS, stHC, stHG, then there is a simple *normal form* for that failure. There are two reasons for doing this:

1. In Sections 6 and 7, these normal forms for failure will help us prove that a space that we construct in fact *has* one of the properties stHL, stHS, stHC, stHG.

2. In this section, we shall use these normal forms to augment the known *duality results* for stHL/stHS to get also duality results for stHG/stHG and stHC/stHC:

Theorem 5.2. *For $\mathcal{F}$ and $\mathcal{G}$ as in Definition 5.1, the following duality results hold:*
1. $\mathcal{F}$ is stHS iff $\mathcal{G}$ is stHL. *2. $\mathcal{F}$ is stHL iff $\mathcal{G}$ is stHS.*
3. $\mathcal{F}$ is stHG iff $\mathcal{G}$ is stHG. *4. $\mathcal{F}$ is stHC iff $\mathcal{G}$ is stHC.*

We shall see that if $\mathcal{F}$ or $\mathcal{G}$ is *not* st$\mathcal{P}$ for $\mathcal{P}$ one of HS, HL, HC, HG, then a *block pattern* (defined below) codes neighborhoods witnessing the corresponding failure; then the normal forms are defined in terms of the block patterns. Proposition 3.9 of [12] essentially proves (1) and (2). The same method will also prove (3) and (4).

We shall begin by describing the block patterns and normal forms for general I, J, and we shall show that every failure of one of stHL, stHS, stHC, stHG can be transcribed into normal form. We shall then describe a simplification of the block patterns and normal forms in the case that $I = J = \omega_1$.

If I or J is countable, then $\mathcal{F}$ and $\mathcal{G}$ are trivially stHG (since they are either countable or second countable), so assume always that $|I| \geq \aleph_1$ and $|J| \geq \aleph_1$.

Definition 5.3. Assume that $|I| \geq \aleph_1$ and $|J| \geq \aleph_1$ and $m, n \in \omega\backslash\{0\}$. Assume that I and J are totally ordered by a relation $\lhd$. Then $(\mathcal{A}, \mathcal{B})$ is an (m, n) *block pattern* for $I \times J$ iff

$\mathcal{A} = \langle A_\xi : \xi < \omega_1\rangle$, where all $A_\xi \in [I]^m$ and the A_ξ are pairwise disjoint, and

$\mathcal{B} = \langle B_\xi : \xi < \omega_1\rangle$, where all $B_\xi \in [J]^n$ and the B_ξ are pairwise disjoint.

Given $\mathcal{A}, \mathcal{B}$, the A_ξ and B_ξ are listed in $\lhd$–increasing order as $A_\xi = \{\alpha_\xi^i : i < m\}$ and $B_\xi = \{\beta_\xi^j : j < n\}$ respectively.

The order $\lhd$ is completely unrelated to the topology; it just gives us a convenient way of listing each A_ξ and B_ξ. Here, I need not contain a $\lhd$-increasing ω_1-sequence. In the case that $I = J = \omega_1$, as the discussion preceding Definition 5.6 clarifies, we may thin a block pattern so that $\xi < \eta$ implies $\max(A_\xi \cup B_\xi) < \min(A_\eta \cup B_\eta)$.

Given $\Psi : I \times J \to \omega$, the duality results are based on the observation that there are two views of the block pattern: In the *horizontal view*, the sets B_ξ code $\aleph_1$ n–tuples from $\mathcal{F}^\Psi$, and each n–tuple is assigned a neighborhood coded by an A_ξ. In the *vertical view*, the sets A_ξ code $\aleph_1$ m–tuples from $\mathcal{G}^\Psi$, and each m–tuple is assigned a neighborhood coded by a B_ξ.

Horizontal view: For each $\xi < \omega_1$, we have the n–tuple $\vec{f_\xi} = (f_{\beta_\xi^0}, \ldots, f_{\beta_\xi^{n-1}}) \in \mathcal{F}^n$; each $\vec{f_\xi}$ is in its neighborhood $U_\xi := \{\vec{h} \in \mathcal{F}^n : \forall i < m\ \forall j < n\ h_j(\alpha_\xi^i) = c_{i,j,\xi}\} \subseteq (\omega^I)^n$. Here, $c_{i,j,\xi} = f_{\beta_\xi^j}(\alpha_\xi^i) = \Psi(\alpha_\xi^i, \beta_\xi^j)$.

Vertical view: For each $\xi < \omega_1$, we have the m–tuple $\vec{g}_\xi = (g_{\alpha_\xi^0}, \ldots, g_{\alpha_\xi^{m-1}}) \in \mathcal{G}^m$; each $\vec{g}_\xi$ is in its neighborhood $V_\xi := \{\vec{k} \in \mathcal{G}^m : \forall i < m\ \forall j < n\ k_i(\beta_\xi^j) = c_{i,j,\xi}\} \subseteq (\omega^J)^m$. Here, $c_{i,j,\xi} = g_{\alpha_\xi^i}(\beta_\xi^j) = \Psi(\alpha_\xi^i, \beta_\xi^j)$.

These views let us express topological properties of $\mathcal{F}^\Psi$ and $\mathcal{G}^\Psi$ in terms of *block pattern properties*, defined as follows:

Definition 5.4. Fix $\Psi : I \times J \to \omega$. For any (m, n) block pattern $(\mathcal{A}, \mathcal{B})$ and $c_{i,j} \in \omega$, say that $(\mathcal{A}, \mathcal{B}, \langle c_{i,j} : i < m \ \&\ j < n\rangle)$ satisfies $(\star)$ with respect to Ψ iff

$$(\star) \qquad \forall \xi < \omega_1\ \forall i < m\ \forall j < n\ \Psi(\alpha^i_\xi, \beta^j_\xi) = c_{i,j}\ .$$

Then Ψ satisfies the *weak block pattern property WLR, WUL, WCD*, or *WSD*, respectively, iff for *all* m, n and $(\mathcal{A}, \mathcal{B}, \langle c_{i,j} : i < m \ \&\ j < n\rangle)$ that satisfy $(\star)$, we have, respectively:

WLR : $\exists \xi < \eta < \omega_1\ \forall i, j\ [\Psi(\alpha^i_\eta, \beta^j_\xi) = c_{i,j}]$
WUL : $\exists \xi < \eta < \omega_1\ \forall i, j\ [\Psi(\alpha^i_\xi, \beta^j_\eta) = c_{i,j}]$
WCD : $\exists \{\xi, \eta\} \in [\omega_1]^2\ \forall i, j\ [\Psi(\alpha^i_\xi, \beta^j_\eta) = c_{i,j}]$
WSD : $\exists \{\xi, \eta\} \in [\omega_1]^2\ \big[\forall i, j\ [\Psi(\alpha^i_\xi, \beta^j_\eta) = c_{i,j}]$ AND $\forall i, j\ [\Psi(\alpha^i_\eta, \beta^j_\xi) = c_{i,j}]\big]$.

The natural construction of Ψ satisfying a weak block pattern property yields Ψ with the Stronger properties, beginning with "S" instead of "W", of Definition 6.8. The acronyms of Definition 5.4 may be taken to refer to the diagonal of the domain of the sequence $\langle A_\xi \times B_\eta : (\xi, \eta) \in \omega_1 \times \omega_1\rangle$, with the last two acronyms representing the Complement of the Diagonal, and Symmetric about the Diagonal. The first two acronyms are explained in Definition 5.8; they make sense for a *normalized* block pattern, which we define below.

Now, Theorem 5.2 is obvious from the following:

Lemma 5.5. *For $\Psi : I \times J \to \omega$ and $\mathcal{F}$ and $\mathcal{G}$ as in Definition 5.1:*
$\mathcal{F}^\Psi$ is stHS iff Ψ satisfies WLR iff $\mathcal{G}^\Psi$ is stHL
$\mathcal{F}^\Psi$ is stHL iff Ψ satisfies WUL iff $\mathcal{G}^\Psi$ is stHS
$\mathcal{F}^\Psi$ is stHC iff Ψ satisfies WCD iff $\mathcal{G}^\Psi$ is stHC
$\mathcal{F}^\Psi$ is stHG iff Ψ satisfies WSD iff $\mathcal{G}^\Psi$ is stHG

Remark. We know that stHC $\leftrightarrow$ (stHS *or* stHL) (see Section 2). It follows that WCD $\leftrightarrow$ (WLR *or* WUL). Here, the $\leftarrow$ direction is obvious, and the $\to$ direction can also be proved directly by showing that a counter-example to the WLR and a counter-example to the WUL can be merged to yield a counter-example to the WCD.

Proof. The Lemma has eight iffs, but they all involve the same concepts, so we shall only write out the proof of the first iff:

For the $\to$ direction, we show that the failure of WLR yields a left separated ω_1 sequence in some finite power of $\mathcal{F}^\Psi$. Specifically, we have an (m, n) block pattern $(\mathcal{A}, \mathcal{B})$ along with $c_{i,j} \in \omega$ for $i < m$ and $j < n$ such that $(\mathcal{A}, \mathcal{B}, \langle c_{i,j} : i < m \ \&\ j < n\rangle)$ satisfies $(\star)$ with respect to Ψ but $\forall \xi < \eta < \omega_1\ \exists i, j\ [\Psi(\alpha^i_\eta, \beta^j_\xi) \neq c_{i,j}]$; that is, the WLR statement fails. But this shows that $\langle \vec{f}_\xi : \xi < \omega_1\rangle$ is left separated, where $\vec{f}_\xi = (f_{\beta^0_\xi}, \ldots, f_{\beta^{n-1}_\xi}) \in \mathcal{F}^n$.

For the $\leftarrow$ direction, assume that $\mathcal{F}^\Psi$ is not stHS, and we shall construct a block pattern that contradicts the WLR. We proceed in two steps:

Step 1. For this step, we temporarily let $X = \mathcal{F}^\Psi$ to emphasize that this argument works with any non stHS space. Assume that X is not stHS, and let $n \in \omega\backslash\{0\}$ be *least* such that X^n is $\neg$HS. Then there is a left separated sequence $\langle \vec{x}_\xi : \xi < \omega_1 \rangle$ in X^n, where $\vec{x}_\xi = (x_\xi^0, \ldots, x_\xi^{n-1})$. Thinning the sequence, we may assume that whenever $i < j < n$, either (1) $\forall \xi\ [x_\xi^i = x_\xi^j]$ or (2) $\forall \xi\ [x_\xi^i \neq x_\xi^j]$. Then by minimality of n, for each pair $i < j < n$, we have (2). Thinning again, WLOG the n–element sets $\{x_\xi^0, \ldots, x_\xi^{n-1}\}$ for $\xi < \omega_1$ form a delta system with some root R. But then $X^{n-|R|}$ is $\neg$HS, so $|R| = 0$. So, WLOG we may assume that the n–element sets are disjoint, with no repetitions; that is, we have $x_\xi^i = x_\eta^j \rightarrow \xi = \eta\ \&\ i = j$. Finally, assume that X is totally ordered by some $\lhd$ (which is unrelated to the topology). Thinning once more, WLOG all the n–tuples are ordered the same way, and then, permuting them, WLOG the order is increasing.

Step 2. Now, $X = \mathcal{F}^\Psi$. So, we now have $x_\xi^j = f_{\beta_\xi^j} \in \omega^I$, where $\beta_\xi^j \in J$ and $\beta_\xi^0 \lhd \cdots \lhd \beta_\xi^{n-1}$. If $B_\xi = \{\beta_\xi^j : j < n\}$ then $B_\xi \in [J]^n$ and the B_ξ are pairwise disjoint by Step 1. Let $\vec{f_\xi}$ denote $\vec{x}_\xi = (f_{\beta_\xi^0}, \ldots, f_{\beta_\xi^{n-1}})$.

Use the following notation for neighborhoods: If $\vec{k} = \langle k_r : r < n \rangle \in (\omega^I)^n$, then basic neighborhoods of $\vec{k}$ are the sets $W_A(\vec{k}) := \{\vec{h} \in (\omega^I)^n : \forall r < n\ [h_r \restriction A = k_r \restriction A]\}$, where $A \in [I]^{<\aleph_0}$. Thinning again, we may assume now that the separating neighborhoods are $U_\xi = W_{A_\xi}(\vec{f_\xi})$, where $A_\xi \in [I]^m$ for some fixed $m \in \omega\backslash\{0\}$. Then $\vec{f_\xi} \in W_{A_\xi}(\vec{f_\xi})$, and for all $\xi < \eta$ $\vec{f_\xi} \notin U_\eta = W_{A_\eta}(\vec{f_\eta})$, and hence $\exists \alpha \in A_\eta\ \exists j < n\ [f_{\beta_\xi^j}(\alpha) \neq f_{\beta_\eta^j}(\alpha)]$.

Thinning, WLOG the A_ξ form a delta system, and thinning again, WLOG all the $f_{\beta_\eta^j}(\alpha)$ are the same for α in the root. Discarding the root, we may assume that the A_ξ are pairwise disjoint. List A_ξ in increasing order as $A_\xi = \{\alpha_\xi^i : i < m\}$. Finally, thinning again, WLOG for each pair i, j, there is $c_{i,j}$ so that $f_{\beta_\xi^j}(\alpha_\xi^i) = c_{i,j}$ (equivalently, $\Psi(\alpha_\xi^i, \beta_\xi^j) = c_{i,j}$) for all ξ. Now we have our block pattern that refutes the WLR because it satisfies $\forall \xi < \eta\ \exists i, j\ [\Psi(\alpha_\eta^i, \beta_\xi^j) = f_{\beta_\xi^j}(\alpha_\eta^i) \neq c_{i,j}]$. $\square$

Step 2 of the construction of a block pattern in the Lemma 5.5 proof may be augmented when $I = J = \omega_1$ to produce a *normalized* block pattern $\mathcal{A}$. This will simplify the notation in the proofs in Sections 6 and 7. Given a block pattern $(\mathcal{A}, \mathcal{B})$, before "finally" thinning to get the sequence $\langle c_{i,j} : i < m \ \& \ j < n\rangle$, we reduce the pairs of sets A_ξ, B_ξ to a single block A_ξ as follows. For each $\xi < \omega_1$, the sets A_ξ and B_ξ may intersect, and their elements can be shuffled together in various ways. But we may always thin our sequence to have the block pattern satisfy

$$\xi < \eta \to \max(A_\xi \cup B_\xi) < \min(A_\eta \cup B_\eta) \quad .$$

Thinning the sequence once more we get $|A_\xi \cup B_\xi| = |A_\eta \cup B_\eta|$ for all $\xi < \eta < \omega_1$; we may also assume that each pair A_ξ, B_ξ is shuffled together in the same way so that $\langle \gamma_\xi^i : i < |A_\xi \cup B_\xi|\rangle$ lists its elements in increasing order. Finally, thin again for each pair $i, j < |A_\xi \cup B_\xi|$ so that $\Psi(\gamma_\xi^i, \gamma_\xi^j) = c_{i,j}$ for all $\gamma_\xi^i, \gamma_\xi^j \in A_\xi \cup B_\xi$. Then each union $A_\xi \cup B_\xi$, now called A_ξ, replaces the pair A_ξ, B_ξ to get a normalized block pattern $\mathcal{A}$.

Definition 5.6. Assume that $I = J = \omega_1$ and $n \in \omega \backslash \{0\}$. Then $\mathcal{A}$ is a *normalized block pattern* of block size n iff $\mathcal{A} = \langle A_\xi : \xi < \omega_1\rangle$, where all $A_\xi \in [\omega_1]^n$ and the A_ξ are pairwise disjoint and $\xi < \eta \to \max(A_\xi) < \min(A_\eta)$. Then, we shall list each A_ξ in increasing order as $A_\xi = \{\alpha_\xi^i : i < n\}$.

To verify that a normalized block pattern satisfies a property of Definition 5.4, we replace each β by α. This leads to the normalized block pattern properties:

Definition 5.7. Fix $\Psi : \omega_1 \times \omega_1 \to \omega$, normalized block pattern $\mathcal{A}$ of block size n, and $\langle c_{i,j} : i, j < n\rangle$ with each $c_{i,j} \in \omega$. The requirements for the *normalized block pattern properties LR, UL, CD, or SD* are:

LR : $\exists \xi < \eta < \omega_1 \ \forall i, j \ [\Psi(\alpha_\eta^i, \alpha_\xi^j) = c_{i,j}]$
UL : $\exists \xi < \eta < \omega_1 \ \forall i, j \ [\Psi(\alpha_\xi^i, \alpha_\eta^j) = c_{i,j}]$
CD : $\exists \{\xi, \eta\} \in [\omega_1]^2 \ \forall i, j \ [\Psi(\alpha_\xi^i, \alpha_\eta^j) = c_{i,j}]$
SD : $\exists \{\xi, \eta\} \in [\omega_1]^2 \ \big[\forall i, j \ [\Psi(\alpha_\xi^i, \alpha_\eta^j) = c_{i,j}] \text{ AND } \forall i, j \ [\Psi(\alpha_\eta^i, \alpha_\xi^j) = c_{i,j}]\big]$.

Now the following explains the acronyms "LR" and "UL".

Definition 5.8. Partition $\omega_1 \times \omega_1$ into $D \ \dot\cup \ D_{LR} \ \dot\cup \ D_{UL}$, with $D = \{(\alpha, \alpha) : \alpha < \omega_1\}$ the *diagonal*, and $D_{UL} = \{(\alpha, \beta) : \alpha < \beta < \omega_1\}$ the *upper left* of the diagonal, and $D_{LR} = \{(\alpha, \beta) : \beta < \alpha < \omega_1\}$ the *lower right* of the diagonal.

If $\xi < \eta$, then $A_\xi \times A_\eta \subseteq D_{UL}$ and $A_\eta \times A_\xi \subseteq D_{LR}$. We shall see that the strong block pattern property SLR depends only on $\Psi \restriction D_{LR}$, and the SUL depends only on $\Psi \restriction D_{UL}$. Each weak block pattern property, however, depends on all three of $\Psi \restriction D_{LR}$, $\Psi \restriction D_{UL}$ and $\Psi \restriction D$; in the construction of a normalized block pattern that proves Lemma 5.5, the values of the $c_{i,j}$ depend on all pairs $(\alpha_\xi^i, \alpha_\xi^j) \in A_\xi \times A_\xi$.

We end this section with one more application of duality: it provides us with some ZFC examples of strongly HG spaces beyond the obvious ones (such as countable spaces and second countable spaces and countable sums and products of these). It also illustrates a case where we do *not* want to have $I = J = \omega_1$.

Lemma 5.9. *If X is strongly HG and zero dimensional, then so is $C(X, \{0,1\})$ with the topology of pointwise convergence.*

Proof. Recall that the topology of pointwise convergence on $C(X, \{0,1\})$ is the Tychonov topology on $\{0,1\}^X$ restricted to the subset $C(X, \{0,1\})$.

Now, let $I = X$ and $J = C(X, \{0,1\})$, and define $\Psi(i,j) = j(i)$, and use this Ψ to define $\mathcal{F}$ and $\mathcal{G}$ as in Definition 5.1. But $\mathcal{G} \cong X$ because the map $i \mapsto g_i \in \{0,1\}^J$ is just the Tychonov embedding of the zero dimensional space X into the cube $\{0,1\}^{C(X,\{0,1\})}$. Also, $\mathcal{F} \subseteq \{0,1\}^I = \{0,1\}^X$ and it is easily seen that $\mathcal{F} \cong C(X, \{0,1\})$. So, the lemma follows from the fact that $\mathcal{F}$ is stHG iff $\mathcal{G}$ is stHG. $\square$

Even when X is separable metric, $C(X, \{0,1\})$ can be "big" in the sense that it has size and weight $\mathfrak{c}$. This is as large as possible, since any space Y that is HS and HL satisfies $w(Y) \le \mathfrak{c}$ (since Y is separable) and $|Y| \le \mathfrak{c}$ (since every point is a G_δ). To get $C(X, \{0,1\})$ to be "big", apply the following lemma with, for example, any $X \subset \mathbb{R}$ such that both X and $\mathbb{R} \backslash X$ are $\mathfrak{c}$ dense in $\mathbb{R}$:

Lemma 5.10. *Let X be separable metric and zero dimensional, and assume that $|X| = \mathfrak{c}$ and X has $\mathfrak{c}$ different clopen sets. Then $|C(X, \{0,1\})| = \mathfrak{c}$ and $w(C(X, \{0,1\})) = \mathfrak{c}$.*

Proof. Both $|C(X, \{0,1\})| = \mathfrak{c}$ and $w(C(X, \{0,1\})) \le \mathfrak{c}$ are obvious. Now assume that $\kappa := w(C(X, \{0,1\})) < \mathfrak{c}$ and we shall derive a contradiction.

The *standard base* for $C(X, \{0,1\})$, inherited from $\{0,1\}^X$, is $\{N_\sigma : \sigma \in \mathrm{Fn}(X, 2)\}$, where $N_\sigma = \{\varphi \in C(X, \{0,1\}) : \varphi \supset \sigma\}$. So the standard base has size $\mathfrak{c}$.

Let $\mathcal{B}$ be an open base for $C(X, \{0,1\})$ with $|\mathcal{B}| = \kappa$. Since $C(X, \{0,1\})$ is HL (by Lemma 5.9), every element of $\mathcal{B}$ is a countable union of sets from the standard base, so there is an $E \in [X]^\kappa$ such that $\{N_\sigma : \sigma \in \mathrm{Fn}(E, 2)\}$ is a base for $C(X, \{0,1\})$.

Fix any $a \in X \backslash E$. Then fix $\sigma \in \mathrm{Fn}(E, 2)$ such that $N_\sigma \subseteq N_{\{(a,1)\}}$. But now we have a contradiction, since $a \notin \mathrm{dom}(\sigma)$ implies that there is an $f \in C(X, \{0,1\})$ such that $f \supset \sigma$ and $f(a) = 0$. $\square$

We remark that our $C(X, \{0,1\})$ does have countable net weight, which is another way to see that it is stHG. Hajnal and Juhász [7] show that it's consistent with ZFC to have an HG space with uncountable net weight. Such a space also exists under the COMA by Theorem 1.3 or Lemma 7.4.

6. The COMA

This axiom does not mention forcing explicitly, although it is inspired by forcing and it is true in $V[1\,\text{Cohen real}\,]$ and in $V[1\,\text{random real}\,]$. It is also a consequence of CH. This COMA implies all the new consistency results in this paper (see Section 7), along with the existence of strong S-spaces and L-spaces, and the non-productivity of the ccc, as we shall show in this section. We begin by defining a notion of *pseudo-genericity*, which is inspired by forcing.

Definition 6.1. Let M be a transitive model of $\mathsf{ZFC} - \mathsf{P}$. Fix $I, \Theta \in M$ such that $(|I| = \omega)^M$ and $1 \le |\Theta| < \omega$ (which is absolute). Call $\mathcal{D}$ an (M, I, Θ) *demand* iff $\mathcal{D} \in M$ and $\mathcal{D} \subseteq \mathrm{Fn}(I, \Theta)$ and $\forall \{p, q\} \in [\mathcal{D}]^2\ [\mathrm{dom}(p) \cap \mathrm{dom}(q) = \emptyset]$ and $(|\mathcal{D}| = \omega)^M$ and $\sup\{|p| : p \in \mathcal{D}\} < \omega$.

If $g : I \to \Theta$, we say that g *meets* the demand $\mathcal{D}$ iff $\exists p \in \mathcal{D}\ g \supset p$; call g (I, Θ) *pseudo-generic* over M iff $g : I \to \Theta$ and g meets *every* (M, I, Θ) demand $\mathcal{D} \in M$.

Remarks. It is trivial to find a $g \in M$ that meets a given demand $\mathcal{D} \in M$, but when $|\Theta| > 1$, a pseudo-generic g cannot be in M. Of course, the case $|\Theta| = 1$ is trivial. Note that the "$\exists p \in \mathcal{D}\ g \supset p$" in the definition can be strengthened to:

Lemma 6.2. *Let $M, I, \Theta, \mathcal{D}$ be as in Definition 6.1, with g (I, Θ) pseudo-generic over M. Then there are infinitely many $p \in \mathcal{D}\ g \supset p$.*

Proof. If not, the demand $\mathcal{D} \setminus \{p \in \mathcal{D} : g \supset p\}$ would yield a contradiction. □

Two examples of pseudo-generic g are "Cohen reals" and "random reals". To prove this, we note that a demand $\mathcal{D}$ itself is not a dense set, but for Cohen and random forcing, it determines a dense set in a natural way:

The Cohen real g is obtained by forcing with $\mathbb{P} = \mathrm{Fn}(I, \Theta)$. It is pseudo-generic, because for each demand $\mathcal{D} \in M$, $\widehat{\mathcal{D}} := \{q \in \mathbb{P} : \exists p \in \mathcal{D}\ q \le p\}$ is dense and in M.

The random real g is obtained by forcing with the measure algebra of Θ^I, using the counting probability measure on Θ. It is pseudo-generic, because for each demand $\mathcal{D} \in M$, $\widehat{\mathcal{D}} := \{\varphi \in \Theta^I : \exists p \in \mathcal{D}\ \varphi \supset p\}$ is a Borel set of measure 1. To prove that $\widehat{\mathcal{D}}$ has measure 1, we use the assumption that $n := \sup\{|p| : p \in \mathcal{D}\} < \omega$, so that $\widehat{\mathcal{D}}$ is a union of an infinite collection of independent sets, each of measure $\ge |\Theta|^{-n}$.

One can develop the basics of random real forcing in a model of $\mathsf{ZFC} - \mathsf{P}$, even though the definition of the measure algebra itself requires the Power Set Axiom.

In the case that M is countable, the Cohen and random reals exist; but also, a pseudo-generic g is easily produced directly, without any mention of forcing.

The COMA involves a chain of ω_1 models, but we first study a pair of models in the chain, so we define the notion of a *pseudo-generic extension*:

Definition 6.3. Let M, N be transitive models of $\mathsf{ZFC} - \mathsf{P}$. Then $\sharp(g, I, \Theta, M)$ abbreviates the statement that $I, \Theta \in M$ and $(|I| = \omega)^M$ and $1 \le |\Theta| < \omega$ and g is (I, Θ) pseudo-generic over M. Also, $\flat(I, \Theta, M, N)$ says that there exists a $g \in N$ such that $\sharp(g, I, \Theta, M)$. Then $M \subset_{PG} N$ holds iff $M \subset N$ and $\flat(I, \Theta, M, N)$ holds for all $I, \Theta \in M$ such that $(|I| = \omega)^M$ and $1 \le |\Theta| < \omega$.

We only assume that $M \subset N$. We do not assume that $M \prec N$ (although this will hold in some applications of the COMA), nor do we assume that M is definable in any way in N (although this will hold in some other applications).

Our applications of $M \subset_{PG} N$ will use $\flat(I, \Theta, M, N)$ for various values of I, Θ, but in constructing such pairs, M, N, it will be useful to note that:

Lemma 6.4. *Assume that M, N are transitive models of $\mathsf{ZFC} - \mathsf{P}$ and $M \subset N$ and $\flat(\omega, 2, M, N)$. Then $M \subset_{PG} N$.*

Proof. $\flat(I, \Theta, M, N)$ follows from $\flat(\omega, 2, M, N)$ by the following argument, in which it is always assumed that $I, \Theta \in M$ and $(|I| = \omega)^M$ and $1 \le |\Theta| < \omega$:

First, note that $\flat(I, \Theta, M, N) \leftrightarrow \flat(\omega, k, M, N)$, where $k = |\Theta|$ (which is absolute). To prove this, note that M contains bijections from I onto ω and from Θ onto k.

Next, note that $\flat(I, \Theta, M, N) \to \flat(I, \Theta', M, N)$ whenever $|\Theta'| \le |\Theta|$.

Finally, note that $\flat(\omega, k, M, N) \to \flat(\omega \times \{0,1\}, k, M, N) \to \flat(\omega, k \times k, M, N)$, and from this we are done. To prove the second $\to$: Observe that each $p \in \mathrm{Fn}(\omega, k \times k)$ may be coded by a $\hat{p} \in \mathrm{Fn}(\omega \times 2, k)$, where $p(\ell) = (\hat{p}(\ell, 0), \hat{p}(\ell, 1))$. $\square$

Definition 6.5. The Chain of Models Axiom (COMA) is the assertion that statements (1)(2)(3)(4) below hold:

1. We have transitive sets $M_\alpha \models \mathsf{ZFC} - \mathsf{P}$ for $\alpha < \omega_1$, with $\alpha < \beta \to M_\alpha \subset M_\beta$.
2. M_α contains α and an injection from α into ω.
3. $M_\alpha \subset_{PG} M_{\alpha+1}$.
4. Whenever $X \in [\omega_1]^{\aleph_1}$ and $F : X \to \bigcup_\alpha M_\alpha$ and $\forall \alpha\, [F(\alpha) \in [M_\alpha \cap ON]^{<\aleph_0}]$: there is a $Y \in [X]^{\aleph_0}$ and a $\rho < \omega_1$ such that $F \restriction Y \in M_\rho$.

Remarks. Note that $\alpha < \beta \to M_\alpha \subset_{PG} M_\beta$. We do *not* assume continuity at limits ($M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$), although this will hold in some cases. We also do *not* assume that $\alpha < \beta \to M_\alpha \prec M_\beta$.

The COMA cannot be obtained just in ZFC, since it contradicts $\mathsf{MA}(\aleph_1)$. We do not know an elementary proof of this, but we shall see that the COMA yields strong S-spaces and strong L-spaces, and the non-productivity of the ccc, all of which contradict $\mathsf{MA}(\aleph_1)$.

We can prove the consistency of the COMA in $2\frac{1}{2}$ different ways:

Lemma 6.6. *A chain satisfying the COMA exists in any model of* $\mathsf{ZFC}+\mathsf{CH}$ *or in any Cohen or random real extension of a model of* ZFC.

Proof. Assuming CH, we obtain $\langle M_\alpha : \alpha < \omega_1\rangle$ as a chain of countable elementary submodels of $H(\aleph_1)$, built by recursion on α in the standard way, except that in view of (2), we do not just take unions at limit steps. The chain is easily built so that (1),(2),(3) hold. The natural construction also gives us $\alpha < \beta$ implies $M_\alpha \prec M_\beta$. Using CH, we can build the chain so that $\bigcup_{\alpha<\omega_1} M_\alpha = H(\aleph_1)$, which implies (4).

Now assume that V is a model of ZFC.

In V, fix a regular $\kappa > 2^{\aleph_0}$, and fix $E_\alpha \in [\omega]^{\aleph_0}$ for $\alpha < \omega_1$, with $\alpha < \beta \to E_\alpha \subset^* E_\beta$; also, fix bijections $\psi_\alpha : \omega \twoheadrightarrow E_{\alpha+1}\backslash E_\alpha$. Still in V, let $M = H(\kappa)$.

First, consider a Cohen real extension $V[1\,\text{Cohen real}\,] = V[G]$, where G is $\mathrm{Fn}(\omega, 2)$ generic over V. Let $\Gamma = \bigcup G : \omega \to 2$. Let $M_\alpha = M[\Gamma \restriction E_\alpha]$. Here, (1) and (2) are trivial because $M_\alpha \supseteq M \supset \omega_1$.

To verify (3): $M_{\alpha+1}$ contains $h_\alpha := \Gamma \restriction (E_{\alpha+1}\backslash E_\alpha)$, so let $f_\alpha = h_\alpha \circ \psi_\alpha : \omega \to 2$. Then f_α is $(\omega, 2)$ pseudo-generic over M_α, so $\flat(\omega, 2, M_\alpha, M_{\alpha+1})$. Then (3) follows by Lemma 6.4.

To verify (4): Note that each $M_\alpha \cap ON = \kappa$, so in $V[G]$, we have $X \in [\omega_1]^{\aleph_1}$ and $F : X \to [\kappa]^{<\aleph_0}$. Then, since $\mathrm{Fn}(\omega, 2)$ is countable, there is $Z \in [X]^{\aleph_1}$ such that $Z, F\restriction Z \in V$. Then, Y can be any countably infinite subset of Z such that $Y \in V$, and ρ can be 0.

Finally, a random real extension $V[1\,\text{random real}\,]$ also satisfies the COMA, by essentially the same argument, replacing $\mathrm{Fn}(\omega, 2)$ with the measure algebra of 2^ω. To verify (4), we again have, in $V[G]$, $X \in [\omega_1]^{\aleph_1}$ and $F : X \to [\kappa]^{<\aleph_0}$. We obtain our Y using the fact that the measure algebra is separable. $\square$

COMA clauses (2) and (4) make possible standard CH-like constructions of chains of models. For example, suppose that we are given a disjoint collection $\{A_\xi : \xi \in \omega_1\} \subseteq [\omega_1]^{<\aleph_0}$ with $\xi < \eta \to \max(A_\xi) < \min(A_\eta)$, as we might get from a block pattern. Let $\alpha_\xi = \max(A_\xi)$ and

$X = \{\alpha_\xi : \xi \in \omega_1\}$ and $F(\alpha_\xi) = A_\xi$. By (2), we have $\forall\alpha\ [F(\alpha) \in [M_\alpha \cap ON]^{<\aleph_0}]$. Then (4) gives us a $Y \in [X]^{\aleph_0}$ and a $\rho < \omega_1$ such that $F{\restriction}Y \in M_\rho$. So, there is an $I \in [\omega_1]^{\aleph_0}$ such that $\langle A_\xi : \xi \in I\rangle \in M_\rho$. The typical CH elementary submodel argument would get this just by assuming that $\bigcup_{\alpha<\omega_1} M_\alpha = H(\aleph_1)$.

We shall use the COMA to construct functions Ψ on $\omega_1 \times \omega_1$ with various block pattern properties. The natural construction will lead not to the Weak properties of Definition 5.4, but rather to the corresponding Stronger properties defined below. The Strong properties require that Ψ obey a *restriction* on its images of the partition $D \mathbin{\dot\cup} D_{LR} \mathbin{\dot\cup} D_{UL}$.

Definition 6.7. A *restriction* is a triple $\mathfrak{T} = (\Theta_D, \Theta_{LR}, \Theta_{UL})$, with $\Theta_D, \Theta_{LR}, \Theta_{UL} \in [\omega]^{<\aleph_0} \setminus \{\emptyset\}$. Then $\Psi : \omega_1 \times \omega_1 \to \omega$ *obeys* this restriction $\mathfrak{T}$ iff $\Psi(D) \subseteq \Theta_D$ and $\Psi(D_{LR}) \subseteq \Theta_{LR}$ and $\Psi(D_{UL}) \subseteq \Theta_{UL}$.

The Strong properties replace the requirement $(\star)$ of the definition of the Weak properties with requirement $\dagger$, which ties the $c_{i,j}$ to the restriction $\mathfrak{T}$ on Ψ:

Definition 6.8. Fix a restriction $\mathfrak{T} = (\Theta_D, \Theta_{LR}, \Theta_{UL})$. For any $c_{i,j} \in \omega$, say that $\langle c_{i,j} : i, j < n\rangle$ satisfies ($\dagger$) with respect to $\mathfrak{T}$ iff

$$(\dagger) \quad \forall(i,j) \in n \times n \left[c_{i,i} \in \Theta_D \mathbin{\&} \big(i < j \to [c_{i,j} \in \Theta_{UL} \mathbin{\&} c_{j,i} \in \Theta_{LR}]\big)\right]$$

If $\Psi : \omega_1 \times \omega_1 \to \omega$ obeys restriction $\mathfrak{T}$, then $(\Psi, \mathfrak{T})$ satisfies the *strong block pattern property SLR, SUL, SCD,* or *SSD*, respectively, iff for *all* $n > 0$ and *all* normalized block patterns $\mathcal{A}$ of block size n and *all* $\langle c_{i,j} : i, j < n\rangle$ that satisfy ($\dagger$), Ψ satisfies the corresponding normalized block pattern property LR, UL, CD, or SD of Definition 5.7.

The values of Ψ on D are irrelevant for these Strong properties, because $\alpha^i_\eta \neq \alpha^j_\xi$ in each normalized block pattern property.

When $I = J = \omega_1$ and Ψ obeys restriction $\mathfrak{T}$, each Strong property implies the corresponding Weak property. Suppose $(\mathcal{A}, \mathcal{B}, \langle c_{i,j} : i < m \mathbin{\&} j < n\rangle)$ satisfies $(\star)$ with respect to Ψ, and $\mathcal{A}$ is the corresponding normalized block pattern. Then each $c_{i,j} = \Psi(\alpha^i_\xi, \alpha^j_\xi)$ by $(\star)$. The normalized block pattern lists each A_ξ in increasing order; thus, since Ψ obeys restriction $\mathfrak{T}$, $\langle c_{i,j} : i, j < n\rangle$ satisfies ($\dagger$). The weak properties do not, however, imply the strong; for a trivial counter-example, let Ψ be the identically zero function.

The strong block pattern properties use $\mathfrak{T}$ to restrict the range of Ψ. The SUL requires $\Psi(\omega_1 \times \omega_1) \subseteq \Theta_{UL}$, the SLR requires $\Psi(\omega_1 \times \omega_1) \subseteq \Theta_{LR}$, and the SSD requires $\Psi(\omega_1 \times \omega_1) \subseteq \Theta_{LR} = \Theta_{UL}$. Satisfying a block pattern property implies that Ψ maps onto any $\{c_{i,j} : i, j < n\}$, so these restrictions on the range of Ψ are easy to see by considering $c_{i,j} \in \Psi(\omega_1 \times \omega_1)$. Moreover, letting $\langle c_{i,j} : i, j < n\rangle$ list $\Theta_D \cup \Theta_{LR} \cup \Theta_{UL}$, we see that $\Psi(\omega_1 \times \omega_1) = \Theta_D \cup \Theta_{LR} \cup \Theta_{UL}$. The SCD is uninteresting, since (as with the Weak properties) it is equivalent to SLR or SUL; a counter-example to SLR and a counter-example to SUL can be combined to yield a counter-example to SCD.

The values on D_{LR} determine whether Ψ satisfies the SLR property. That is, for a given $(\Theta_{UL}, \Theta_{LR}, \Theta_D)$, whenever $\Psi_1{\restriction}D_{LR} = \Psi_2{\restriction}D_{LR}$, Ψ_1 has the SLR property iff Ψ_2 has the SLR property. This "iff" is false for the WLR. It is easy to make $\mathcal{F}^{\Psi_1}$ countable (and hence trivially stHS), while $\mathcal{F}^{\Psi_2}$ has an uncountable discrete subset. Specifically, we can have $\Psi_1(\alpha, \beta) = 0$ whenever $\beta \geq \omega$ and Ψ_2 the same as Ψ_1 with the exception that $\Psi_2(\gamma, \gamma + 1) = 1$ whenever $\omega < \gamma < \omega_1$.

In contrast to the Weak versions, some Strong properties *alone* contradict $\mathsf{MA}(\aleph_1)$. For example, we get the following "*combinatorial refutation*" of the productivity of the ccc, as described in Lemma 3.1 of Galvin [4]. Galvin [4] showed that CH implies that there are two ccc posets whose product is not ccc. His proof can be simplified by using a chain of countable elementary submodels, as described in the text [9] (see Theorem III.8.19). This elementary chain proof is similar to the proof in Lemma 6.6 that CH implies the COMA. The fact that there are such posets in $V[1\,\text{Cohen real}\,]$ and $V[1\,\text{random real}\,]$ is due to Roitman [11].

Lemma 6.9. *Fix $\Psi : \omega_1 \times \omega_1 \to \omega$ and restriction $\mathfrak{T} = (\Theta_D, \Theta_{LR}, \Theta_{UL})$ with $0, 1 \in \Theta_D \cap \Theta_{LR} \cap \Theta_{UL}$. If $(\Psi, \mathfrak{T})$ satisfies either the SUL or the SLR, then there are two ccc posets whose product is not ccc.*

Proof. Start with any partition $\Pi : [\omega_1]^2 \to \omega$. This yields posets $\mathbb{R}_0, \mathbb{R}_1$ obtained by forcing with homogeneous sets. That is, for $\tau \in \{0, 1\}$, define the poset $\mathbb{R}_\tau = \{r \in [\omega_1]^{<\aleph_0} : \forall\{\alpha, \beta\} \in [r]^2\ [\Pi(\{\alpha, \beta\}) = \tau]\}$. Each $\mathbb{R}_\tau$ is ordered by $\supseteq$, so $\mathbb{1} = \emptyset$. Then $\mathbb{R}_0 \times \mathbb{R}_1$ is not ccc because it has the antichain $\{(\{\alpha\}, \{\alpha\}) : \alpha < \omega_1\}$. Of course, now we need to define Π so that $\mathbb{R}_0$ and $\mathbb{R}_1$ *are* ccc.

Assume that $(\Psi, \mathfrak{T})$ satisfies the SUL. Then define $\Pi : [\omega_1]^2 \to \omega$ by: $\Pi(\{\alpha, \beta\}) = \Psi(\alpha, \beta)$ whenever $\alpha < \beta < \omega_1$.

Finally, fix $\tau \in \{0, 1\}$ and assume that $\mathbb{R}_\tau$ is not ccc. Let $\{A_\xi : \xi < \omega_1\} \subset \mathbb{R}_\tau$ be an antichain; so $\xi < \eta \to A_\xi \cup A_\eta \notin \mathbb{R}_\tau$. Thinning the sequence, WLOG the A_ξ form a delta system with root R. Then note that $\xi < \eta \to (A_\xi \backslash R) \cup (A_\eta \backslash R) \notin \mathbb{R}_\tau$. Replacing A_ξ by $A_\xi \backslash R$, WLOG the A_ξ are pairwise disjoint. Thinning some more, WLOG all $|A_\xi| = n$ and $\xi < \eta \to \max(A_\xi) < \min(A_\eta)$; that is, the A_ξ form a normalized block pattern of block size n. Then $A_\xi = \{\alpha_\xi^i : i < n\}$.

Now, in Definition 6.8, let $c_{i,j} = \tau$ for all i, j. Then (†) holds. Applying the SUL, we get $\xi < \eta < \omega_1$ such that $\forall i, j\ [\Psi(\alpha^i_\xi, \alpha^j_\eta) = c_{i,j} = \tau]$. But this, along with $A_\xi, A_\eta \in \mathbb{R}_\tau$ implies that $A_\xi \cup A_\eta \in \mathbb{R}_\tau$, so we have a contradiction. □

Note that we get no such result with the Weak properties. In ZFC, we can get such a Ψ satisfying the WUL or the WLR or even the WSD just by letting $\mathcal{F}^\Psi$ be one of the stHG spaces described in Section 5.

A strong S-space is also easily read off of Strong block pattern properties:

Lemma 6.10. *Fix $\Psi : \omega_1 \times \omega_1 \to \omega$ and restriction $\mathfrak{T} = (\Theta_D, \Theta_{LR}, \Theta_{UL})$ with $0, 1 \in \Theta_D \cap \Theta_{LR} \cap \Theta_{UL}$. If $(\Psi, \mathfrak{T})$ satisfies the SLR, then there is a strong S-space.*

Proof. Just by the WLR, $\mathcal{F}^\Psi$ is stHS (see Lemma 5.5). But now, WLOG $\Psi \equiv 1$ on D and $\Psi \equiv 0$ on D_{UL}, since changing Ψ on $D \cup D_{UL}$ does not change the SLR. But then $\mathcal{F}^\Psi$ is not Lindelöf because for each β, $\{h : h(\beta) = 1\}$ is a neighborhood of f_β that does not contain f_α for any $\alpha > \beta$. □

The COMA implies some non-trivial instances of the SLR, including the ones used in Lemmas 6.9 and 6.10 (and hence it contradicts $\mathsf{MA}(\aleph_1)$). The following uses the COMA to construct a canonical map Ψ.

Theorem 6.11. *Assume the COMA, and let $\mathfrak{T} = (\Theta_D, \Theta_{LR}, \Theta_{UL})$ be a restriction. Then there is a $\Psi : \omega_1 \times \omega_1 \to \omega$ such that:*

A. $(\Psi, \mathfrak{T})$ satisfies SLR iff $\Theta_{UL} \cup \Theta_D \subseteq \Theta_{LR}$.

B. $(\Psi, \mathfrak{T})$ satisfies SUL iff $\Theta_{LR} \cup \Theta_D \subseteq \Theta_{UL}$.

C. $(\Psi, \mathfrak{T})$ satisfies SSD iff $\Theta_D \subseteq \Theta_{LR} = \Theta_{UL}$.

As mentioned after Definition 6.8, the “=” and “⊆” hypotheses about $\Theta_D, \Theta_{LR}, \Theta_{UL}$ in this theorem are required by the strong block pattern property definitions.

The following introduces the structure of the proof for all three parts, but the conclusion of part (C) uses some additional lemmas.

Proof of Theorem 6.11. Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be as in the statement of the COMA (Definition 6.5). Since $\Theta_D \neq \emptyset$ (by definition of “restriction”), WLOG $0 \in \Theta_D$; this will simplify our notation. We obtain the *“canonical map”* $\Psi : \omega_1 \times \omega_1 \to \omega$ as follows:

1. $\Psi(\alpha, \alpha) = 0$.

2. $\Psi \restriction (\{\alpha\} \times \alpha)$ is in $M_{\alpha+1}$, and $\sharp(\Psi \restriction (\{\alpha\} \times \alpha), \{\alpha\} \times \alpha, \Theta_{LR}, M_\alpha)$ holds.

3. $\Psi \restriction (\alpha \times \{\alpha\})$ is in $M_{\alpha+1}$, and $\sharp(\Psi \restriction (\alpha \times \{\alpha\}), \alpha \times \{\alpha\}, \Theta_{UL}, M_\alpha)$ holds.

Assume that $\langle c_{i,j} : i,j < n \rangle$ satisfies (†) from Definition 6.8. Let $\mathcal{A}$ be a normalized block pattern of block size n.

Part (B): Assume that $\Theta_{LR} \cup \Theta_D \subseteq \Theta_{UL}$. Then by (†) each $c_{i,j} \in \Theta_{UL}$.

To verify SUL, we find $\xi < \eta < \omega_1$ so that $\Psi(\alpha^i_\xi, \alpha^j_\eta) = c_{i,j}$ for all i,j.

First, apply COMA(4) to get $\rho < \omega_1$ and $E \subset \omega_1$ such that type$(E) = \omega$ and $\langle A_\xi : \xi \in E \rangle \in M_\rho$. In this application of COMA(4), $X = \{\max(A_\xi) : \xi < \omega_1\}$ and $F(\max(A_\xi)) = A_\xi$.

Then $E \in M_\rho$, and, enlarging ρ if necessary, we may assume that $\sup(E) < \rho$ and $A_\xi \subset \rho$ for all $\xi \in E$.

Then, *fix* η with $\rho < \eta < \omega_1$. We now use successively pseudo-genericity of $\Psi(\cdot, \alpha^0_\eta), \Psi(\cdot, \alpha^1_\eta), \ldots, \Psi(\cdot, \alpha^{m-1}_\eta)$. Here, $\Psi(\cdot, \alpha)$ abbreviates $\Psi{\restriction}(\alpha \times \{\alpha\})$.

Let $E_1 = \{\xi \in E : \forall i < m\ [\Psi(\alpha^i_\xi, \alpha^0_\eta) = c_{i,0}]\}$. Then E_1 is infinite by pseudo-genericity of $\Psi(\cdot, \alpha^0_\eta)$ (see Lemma 6.2). Also, $E_1 \in M_{\alpha^1_\eta}$ so by pseudo-genericity of $\Psi(\cdot, \alpha^1_\eta)$, $E_2 := \{\xi \in E_1 : \forall i < m\ [\Psi(\alpha^i_\xi, \alpha^1_\eta) = c_{i,1}]\}$ is also infinite.

If $m = 2$, we are done.

If $m > 2$, proceed to get infinite sets $E_0 = E \supseteq E_1 \supseteq E_2 \supseteq \cdots \supseteq E_m$, and for all $\xi \in E_m$, we have $\forall i,j < m\ [\Psi(\alpha^i_\xi, \alpha^j_\eta) = c_{i,j}]$. Then, since $E_m \neq \emptyset$, we are done.

The proof of (A) is very similar. □

We have not yet proved (C). If $\Theta_D \subseteq \Theta_{LR} = \Theta_{UL}$, then we have proved that Ψ satisfies the SLR and the SUL, but this does not give us SSD, for which we need the *same* ξ, η to work for both the SLR and the SUL.

For this we introduce a new concept, which is motivated by the forcing fact that adding two Cohen reals is the same as adding one:

Definition 6.12. Let M be a transitive model for $\mathsf{ZFC} - \mathsf{P}$ and fix $I, J, \Theta, \Lambda \in M$ such that $(|I| = |J| = \omega)^M$ and $1 \leq |\Theta| < \omega$ and $1 \leq |\Lambda| < \omega$. A *pair demand* is a $\mathcal{D} = \langle (p_n, q_n) : n \in \omega \rangle$, where $\{p_n : n \in \omega\}$ is an (M, I, Θ) demand and $\{q_n : n \in \omega\}$ is an (M, J, Λ) demand.

For $g : I \to \Theta$ and $h : J \to \Lambda$, say (g,h) *meets* the pair demand $\mathcal{D}$ iff there is $m \in \omega$ so that $g \supset p_m$ and $h \supset q_m$. The pair (g,h) is *pair pseudo-generic* over M iff (g,h) *meets* every *pair demand* $\mathcal{D} \in M$.

The important point here is that *the same* m produces the pair extended by g and h, so this pair pseudo-genericity is stronger than the assertion that g and h are each separately pseudo-generic. To be pedantic, we should say that (g,h) is (I, Θ, J, Λ) pair pseudo-generic if I, Θ, J, Λ are not clear from context.

Lemma 6.13. *Suppose (g,h) is pair pseudo-generic over M. Then, using the notation of Definition 6.12, there are infinitely many m such that $[g \supset p_m \ \&\ h \supset q_m]$.*

We also need to verify that such g,h exist:

Lemma 6.14. *Fix $I, J, \Theta, \Lambda \in M$ as in Definition 6.12 and assume that $M \subset_{PG} N$. Then there exist functions $g : I \to \Theta$ and $h : J \to \Lambda$ such that (g,h) is pair pseudo-generic over M and $g, h \in N$.*

Proof. WLOG, $I \cap J = \emptyset$. Then, fix $k \in N$ such that $k : I \cup J \to \Theta \cup \Lambda$ is $(I \cup J, \Theta \cup \Lambda)$ pseudo-generic over M. In N, choose $g : I \to \Theta$ such that $g(i) = k(i)$ whenever $k(i) \in \Theta$; and $h : J \to \Lambda$ such that $h(j) = k(j)$ whenever $k(j) \in \Lambda$.

Given the pair demand $\mathcal{D} = \langle (p_n, q_n) : n \in \omega\rangle$, let $\widehat{\mathcal{D}} = \{p_n \cup q_n : n \in \omega\}$; note that $p_n \cup q_n \in \mathrm{Fn}(I \cup J, \Theta \cup \Lambda)$. Since k is pseudo-generic, we may fix m such that $k \supset p_m \cup q_m$, which implies that $g \supset p_m$ and $h \supset q_m$. □

Proof of Theorem 6.11 (C). We build the *"canonical map"* $\Psi : \omega_1 \times \omega_1 \to \omega$ as in the beginning of the proof of Theorem 6.11, so that it also satisfies:

4. $\Psi \restriction \big((\{\alpha\}\times\alpha)\cup(\alpha\times\{\alpha\})\big)$ is in $M_{\alpha+1}$ and the pair $\big(\Psi\restriction(\{\alpha\}\times\alpha), \Psi\restriction(\alpha \times \{\alpha\})\big)$ is pair pseudo-generic over M_α.

Actually, (4) implies (2) and (3).

As in the proof of parts (A) and (B), assume that $\langle c_{i,j} : i,j < n\rangle$ satisfies (†) from Definition 6.8, and let $\mathcal{A}$ be a normalized block pattern of block size n.

Now we are assuming that $\Theta_D \subseteq \Theta_{LR} = \Theta_{UL}$, and we must show that $(\Psi, \mathfrak{T})$ satisfies SSD. That is, we must find $\xi < \eta < \omega_1$ so that both $\forall i,j\ [\Psi(\alpha^i_\xi, \alpha^j_\eta) = c_{i,j}]$ and $\forall i,j\ [\Psi(\alpha^i_\eta, \alpha^j_\xi) = c_{i,j}]$ hold.

Note that, by our assumption $\Theta_D \subseteq \Theta_{LR} = \Theta_{UL}$, each $c_{i,j} \in \Theta_{LR} = \Theta_{UL}$, so that we may simultaneously use the arguments for parts (A) and (B).

First, apply COMA(4) to get $\rho < \omega_1$ and $E \subset \omega_1$ such that $\mathrm{type}(E) = \omega$ and $\langle A_\xi : \xi \in E\rangle \in M_\rho$. As before, we can get $\sup(E) < \rho$ and $E \in M_\rho$ and $A_\xi \subset \rho$ for all $\xi \in E$. Then, *fix* η with $\rho < \eta < \omega_1$. We now use successively *pair* pseudo-genericity of $(\Psi(\cdot, \alpha^\eta_0), \Psi(\alpha^\eta_0, \cdot))$, $(\Psi(\cdot, \alpha^\eta_1), \Psi(\alpha^\eta_1, \cdot))$, $\dots$, $(\Psi(\cdot, \alpha^\eta_{m-1}), \Psi(\alpha^\eta_{m-1}, \cdot))$.

Let $E_1 = \{\xi \in E : \forall i < m\ [\Psi(\alpha^i_\xi, \alpha^0_\eta) = c_{i,0} \ \&\ \Psi(\alpha^0_\eta, \alpha^i_\xi) = c_{0,i}]\}$. Then E_1 is infinite by Lemma 6.13 and pair pseudo-genericity of $(\Psi(\cdot, \alpha^\eta_0), \Psi(\alpha^\eta_0, \cdot))$. Also, $E_1 \in M_{\alpha^1_\eta}$ so by pair pseudo-genericity of $(\Psi(\cdot, \alpha^\eta_1), \Psi(\alpha^\eta_1, \cdot))$, $E_2 := \{\xi \in E_1 : \forall i < m\ [\Psi(\alpha^i_\xi, \alpha^1_\eta) = c_{i,1} \ \&\ \Psi(\alpha^1_\eta, \alpha^i_\xi) = c_{1,i}]\}$ is also infinite.

If $m = 2$, we are done. If $m > 2$, proceed to get infinite sets $E_0 = E \supseteq E_1 \supseteq E_2 \supseteq \cdots \supseteq E_m$ such that for all $\xi \in E_m$: $\forall i, j < m$ $[\Psi(\alpha^i_\xi, \alpha^j_\eta) = c_{i,j}$ & $\Psi(\alpha^j_\eta, \alpha^i_\xi) = c_{j,i}]$. Then, since $E_m \neq \emptyset$, we are done. $\square$

Properties of $\mathcal{F}^\Psi$ and $\mathcal{G}^\Psi$ under possible combinations of $\Theta_D, \Theta_{UL}, \Theta_{LR}$ are given by the following.

Lemma 6.15. *Assume the COMA, and let $\mathfrak{T} = (\Theta_D, \Theta_{LR}, \Theta_{UL})$ be a restriction, with $\Theta_D \subseteq \Theta_{LR} \cap \Theta_{UL}$ and $|\Theta_D| = 1$. There are four possible inclusion relations that can hold between Θ_{UL} and Θ_{LR}, and the following table summarizes the properties of $\mathcal{F}^\Psi$ and $\mathcal{G}^\Psi$ under each of the four, where $\Psi : \omega_1 \times \omega_1 \to \omega$ is built using the "canonical map" of the proof of Theorem 6.11 (replacing the "$\Psi(\alpha, \alpha) = 0$" by "$\Psi(\alpha, \alpha) \in \Theta_D$").*

Inclusion Relation	$\mathcal{F}^\Psi$	$\mathcal{G}^\Psi$
$\Theta_{UL} = \Theta_{LR}$	*stHG*	*stHG*
$\Theta_{UL} \subsetneqq \Theta_{LR}$	*stHS* + $\neg$*HL*	*stHL* + $\neg$*HS*
$\Theta_{UL} \supsetneqq \Theta_{LR}$	*stHL* + $\neg$*HS*	*stHS* + $\neg$*HL*
$\Theta_{UL} \not\subseteq \Theta_{LR}$ *and* $\Theta_{UL} \not\supseteq \Theta_{LR}$	$\neg$*HC*	$\neg$*HC*

Proof. The positive results are clear from Theorem 6.11 and Lemma 5.5.

As a typical negative result, consider the last line in the table. Assume that $\Theta_{UL} \not\subseteq \Theta_{LR}$ and that $\Theta_{UL} \not\supseteq \Theta_{LR}$, and fix $m \in \Theta_{UL} \backslash \Theta_{LR}$ and $n \in \Theta_{LR} \backslash \Theta_{UL}$. Then, use pseudo-genericity to choose ordinals $\xi_\mu, \alpha_\mu, \eta_\mu \in \omega_1$ for $\mu < \omega_1$ such that $\mu < \nu \to \xi_\mu < \alpha_\mu < \eta_\mu < \xi_\nu < \alpha_\nu < \eta_\nu$ and $\Psi(\xi_\mu, \alpha_\mu) = m$ and $\Psi(\eta_\mu, \alpha_\mu) = n$. Note that $(\xi_\mu, \alpha_\mu) \in D_{UL}$ and $(\eta_\mu, \alpha_\mu) \in D_{LR}$, so these values m, n are consistent with our restriction $\mathfrak{T}$. Now, let $U_\mu = \{h \in \omega^{\omega_1} : h(\xi_\mu) = m \;\&\; h(\eta_\mu) = n\}$. Then $f_{\alpha_\mu} \in U_\mu$, but $\nu \neq \mu \to f_{\alpha_\nu} \notin U_\mu$ by the choices of $n \notin \Theta_{UL}$ and $m \notin \Theta_{LR}$. Thus $\{f_{\alpha_\mu} : \mu < \omega_1\}$ is a discrete subset of $\mathcal{F}$, and so $\mathcal{F}$ is not HC.

To choose the $\xi_\mu, \alpha_\mu, \eta_\mu \in \omega_1$: Proceed by recursion on μ. Given the $\xi_\mu, \alpha_\mu, \eta_\mu$ for $\mu < \nu$: Let $\tau = \sup\{\eta_\mu : \mu < \nu\}$. Then let $\eta_\nu = \tau + \omega + \omega$. Then, using pseudo-genericity of $\Psi\restriction(\{\eta_\nu\} \times \eta_\nu)$, fix α_ν with $\tau + \omega < \alpha_\nu < \eta_\nu$ and $\Psi(\eta_\nu, \alpha_\nu) = n$. Then, using pseudo-genericity of $\Psi\restriction(\alpha_\nu \times \{\alpha_\nu\})$, fix ξ_ν with $\tau < \xi_\nu < \tau + \omega$ such that $\Psi(\xi_\nu, \alpha_\nu) = m$. $\square$

7. Products of HG Spaces

Lemma 6.15 points out that using the COMA, one can construct $\mathcal{F}^\Psi$ to be stHG. This may seem uninteresting, since non-trivial examples of stHG spaces exist in ZFC (see Section 5). Using the COMA, however, lets us construct spaces with features not possible in ZFC. In particular, we can build a pair of stHG spaces whose product contains a strong S-space, or a strong L-space, or both; thereby proving Theorem 1.3.

The proof of Theorem 6.11(C) required us to introduce the concept of pair pseudo-genericity. The combinations of stHG spaces we build here will require the notion of *s–fold pseudo-genericity*, which naturally generalizes Definition 6.12 and Lemmas 6.13 and 6.14:

Definition 7.1. Let M be a transitive model of $\mathsf{ZFC} - \mathsf{P}$ and fix $s \in \omega \backslash \{0\}$. Assume that we have $I_\mu, \Theta_\mu \in M$ for $\mu < s$ such that $(|I_\mu| = \omega)^M$ and $1 \le |\Theta_\mu| < \omega$ for each $\mu < s$. An *s–fold demand* $\mathcal{D}$ is of the form $\mathcal{D} = \langle \langle p_n^\mu : \mu < s \rangle : n \in \omega \rangle$, where for each $\mu < s$ $\{p_n^\mu : n \in \omega\}$ is an (M, I_μ, Θ_μ) demand. We say that $\langle g^\mu : \mu < s \rangle$ *meets* this demand iff each $g^\mu : I_\mu \to \Theta_\mu$ and $\exists m \in \omega\ \forall \mu < s\ [g^\mu \supset p_m^\mu]$. The sequence $\langle g^\mu : \mu < s \rangle$ is *s–fold pseudo-generic* over M iff it meets every s-fold demand in M.

Note that as for pair pseudo-genericity, the same m must work for each μ in the definition of *meets*, so it is not sufficient that each g^μ be separately pseudo-generic.

Lemma 7.2. *Suppose $\langle g^\mu : \mu < s \rangle$ is s–fold pseudo-generic over M. Then using the notation of Definition 7.1, there are infinitely many $m \in \omega\ \forall \mu < s\ [g^\mu \supset p_m^\mu]$.*

Lemma 7.3. *Fix $s \in \omega$ and $\langle I_\mu : \mu < s \rangle, \langle \Theta_\mu : \mu < s \rangle \in M$ as in Definition 7.1, and assume that $M \subset_{PG} N$. Then there exist a sequence $\langle g^\mu : \mu < s \rangle \in N$ that is s–*fold pseudo-generic *over M.*

This machinery helps us build the spaces that prove Theorem 1.3, which we restate:

Theorem 1.3. *Assuming the COMA, there are spaces X, Y that are strongly HG (and hence strongly ESLC) such that $X \times Y$ contains a strong S-space and a strong L-space.*

We first prove a weaker version:

Lemma 7.4. *Assuming the COMA, there are spaces X, Y that are strongly HG such that $X \times Y$ contains a strong S-space or a strong L-space.*

Proof. We shall get *two* functions $\Psi^1, \Psi^2 : \omega_1 \times \omega_1 \to \{0, 1\}$. Each one *separately* will be constructed using the COMA as above, using the restriction $\mathfrak{T} = (\Theta_D, \Theta_{LR}, \Theta_{UL})$, where $\Theta_D = \{0\} \subset \Theta_{UL} = \Theta_{LR} = \{0, 1\}$.

We now have $f_\beta^1(\alpha) = \Psi^1(\alpha, \beta)$ and $f_\beta^2(\alpha) = \Psi^2(\alpha, \beta)$ and $\mathcal{F}^1 = \{f_\beta^1 : \beta < \omega_1\}$ and $\mathcal{F}^2 = \{f_\beta^2 : \beta < \omega_1\}$. Now $\mathcal{F}^1, \mathcal{F}^2 \subset \{0, 1\}^{\omega_1}$.

Using $\Theta_{LR} = \Theta_{UL}$ for $\mathcal{F}^1$ and $\mathcal{F}^2$ *separately*, each one will be strongly HG. We shall arrange for the diagonal Δ of $\mathcal{F}^1 \times \mathcal{F}^2$ to be a strong S-space; so $X = \mathcal{F}^1$ and $Y = \mathcal{F}^2$.

Note that $\Delta = \{(f_\beta^1, f_\beta^2) : \beta < \omega_1\}$. We may think of Δ as arising from a single function $\Psi : \omega_1 \times \omega_1 \to \{0, 1\} \times \{0, 1\}$, where $\Psi(\alpha, \beta) = (\Psi^1(\alpha, \beta), \Psi^2(\alpha, \beta))$. Although Ψ maps into $\{0, 1\} \times \{0, 1\}$ instead of into $4 = \{0, 1, 2, 3\}$, all the previous results will apply to Ψ if we arrange for Ψ to be constructed properly from our pseudo-generic functions.

Furthermore Ψ^2 will be related to Ψ^1 in the following way: For $(\alpha, \beta) \in D_{UL}$, we shall have $\Psi^2(\alpha, \beta) = \Psi^1(\alpha, \beta)$, so the only values of Ψ on D_{UL} will be $(0,0)$ and $(1,1)$. On D_{LR}, Ψ will take all four values $(0,0), (0,1), (1,0), (1,1)$. Of course, on D, Ψ will take only the value $(0,0)$.

So, the combined function Ψ corresponds to $\mathfrak{T}^* = (\Theta^*_D, \Theta^*_{LR}, \Theta^*_{UL})$, where $\Theta^*_D = \{(0,0)\}$ (so $(0,0)$ acts like 0) and $\Theta^*_{LR} = \{(0,0), (0,1), (1,0), (1,1)\}$, and $\Theta^*_{UL} = \{(0,0), (1,1)\}$; so that for $\Delta = \mathcal{F}^\Psi$, we have $\Theta^*_{UL} \subsetneqq \Theta^*_{LR}$, so that by Lemma 6.15, Δ will be stHS and $\neg$HL; that is, a strong S-space.

This argument requires the three functions Ψ^1, Ψ^2, and Ψ to be obtained by the *canonical construction* with appropriate restrictions. To get this, we modify the proof of Theorem 6.11 and obtain the following for $i = 1, 2$:

1. $\Psi^i(\alpha, \alpha) = 0$.
2. $\Psi^i\restriction(\{\alpha\} \times \alpha) \in M_{\alpha+1}$, and $\sharp(\Psi^i\restriction(\{\alpha\} \times \alpha), \{\alpha\} \times \alpha, \Theta_{LR}, M_\alpha)$ holds.
3. $\Psi^i\restriction(\alpha \times \{\alpha\}) \in M_{\alpha+1}$, and $\sharp(\Psi^i\restriction(\alpha \times \{\alpha\}), \alpha \times \{\alpha\}, \Theta_{UL}, M_\alpha)$ holds.
4. The pair $\big(\Psi^i\restriction(\{\alpha\} \times \alpha), \Psi^i\restriction(\alpha \times \{\alpha\})\big)$ is pair pseudo-generic over M_α.
5. $\Psi^1\restriction(\alpha \times \{\alpha\}) = \Psi^2\restriction(\alpha \times \{\alpha\})$.
6. The triple $\big(\Psi^1\restriction(\{\alpha\} \times \alpha), \Psi^1\restriction(\alpha \times \{\alpha\}), \Psi^2\restriction(\{\alpha\} \times \alpha)\big)$ is in $M_{\alpha+1}$ and is three-fold pseudo-generic over M_α.
7. The pair $\big(\Psi^1\restriction(\{\alpha\} \times \alpha), \Psi^2\restriction(\{\alpha\} \times \alpha)\big)$ is pair pseudo-generic over M_α.

If we instead want the diagonal to be a strong L-space, then we would replace (5) by $\Psi^1\restriction(\{\alpha\} \times \alpha) = \Psi^2\restriction(\{\alpha\} \times \alpha)$.

Actually, (2)(3)(4)(7) are consequences of (5) + (6). We list them because for each i, items (1)(2)(3)(4) construct $(\Psi^i, \mathfrak{T})$ satisfying SSD so that each $\mathcal{F}^i$ is strongly HG.

To verify that this also produces a canonical construction for Ψ, the following table lists the canonical requirement number (1)(2)(3)(4) and the relevant condition from the list $(1-7)$ above. In the table "ppg" abbreviates "pair pseudo-generic"

requirement		condition
(1)	$\Psi(\alpha, \alpha) = (0,0)$	(1)
(2)	$\sharp(\Psi \restriction (\{\alpha\} \times \alpha), \{\alpha\} \times \alpha, \Theta^*_{LR}, M_\alpha)$	(6)
(3)	$\sharp(\Psi \restriction (\alpha \times \{\alpha\}), \alpha \times \{\alpha\}, \Theta^*_{UL}, M_\alpha)$	(3)(5)
(4)	$(\Psi \restriction (\{\alpha\} \times \alpha), \Psi \restriction (\alpha \times \{\alpha\}))$ is ppg over M_α	(5)(6)

□

The preceding proof builds a sequence of approximations to our spaces X, Y, and the diagonal Δ. At stage α, we construct sequences of type α: the horizontal ones, $\Psi^1{\restriction}(\alpha \times \{\alpha\})$ and $\Psi^2{\restriction}(\alpha \times \{\alpha\})$, along with the vertical ones, $\Psi^1{\restriction}(\{\alpha\} \times \alpha)$ and $\Psi^2{\restriction}(\{\alpha\} \times \alpha)$. Each one separately is pseudo-generic over M_α. The condition that $\Psi^1{\restriction}(\alpha \times \{\alpha\}) = \Psi^2{\restriction}(\alpha \times \{\alpha\})$ reduces this to three different sequences, leaving us with the triple $\big(\Psi^1{\restriction}(\{\alpha\} \times \alpha), \Psi^1{\restriction}(\alpha \times \{\alpha\}), \Psi^2{\restriction}(\{\alpha\} \times \alpha)\big)$, which we make three-fold pseudo-generic over M_α. Together, these conditions build the diagonal of $\mathcal{F}^1 \times \mathcal{F}^2$ to be a strong S-space. If we instead want the diagonal to be a strong L-space, using the revised condition (5), we make the triple of functions $\big(\Psi^1{\restriction}(\{\alpha\} \times \alpha), \Psi^1{\restriction}(\alpha \times \{\alpha\}), \Psi^2{\restriction}(\alpha \times \{\alpha\})\big)$ three-fold pseudo-generic over M_α.

Proof of Theorem 1.3. We build two examples as in Lemma 7.4: Ψ^{1S}, Ψ^{2S}, where the diagonal is a strong S-space, and Ψ^{1L}, Ψ^{2L}, where the diagonal is a strong L-space. That is, we build two pairs of stHG spaces: One pair is $\mathcal{F}^{1S}$ and $\mathcal{F}^{2S}$, with $\mathcal{F}^{1S} \times \mathcal{F}^{2S}$ containing a strong S-space. The other pair is $\mathcal{F}^{1L}$ and $\mathcal{F}^{2L}$, with $\mathcal{F}^{1L} \times \mathcal{F}^{2L}$ containing a strong L-space. Then $X = \mathcal{F}^{1S} \dot\cup \mathcal{F}^{1L}$ and $Y = \mathcal{F}^{2S} \dot\cup \mathcal{F}^{2L}$. We build in enough s–fold pseudo-genericity so that X and Y are both strongly HG.

At stage α we construct the following sequences of type α. There are horizontal ones:

$$\Psi^{1S}{\restriction}(\alpha \times \{\alpha\})\ ,\ \Psi^{2S}{\restriction}(\alpha \times \{\alpha\})\ ,\ \Psi^{1L}{\restriction}(\alpha \times \{\alpha\})\ ,\ \Psi^{2L}{\restriction}(\alpha \times \{\alpha\})\ ,$$

along with vertical ones:

$$\Psi^{1S}{\restriction}(\{\alpha\} \times \alpha)\ ,\ \Psi^{2S}{\restriction}(\{\alpha\} \times \alpha)\ ,\ \Psi^{1L}{\restriction}(\{\alpha\} \times \alpha)\ ,\ \Psi^{2L}{\restriction}(\{\alpha\} \times \alpha)\ .$$

Each of these separately is pseudo-generic over M_α, but we also have two equations

$$\Psi^{1S}{\restriction}(\alpha \times \{\alpha\}) = \Psi^{2S}{\restriction}(\alpha \times \{\alpha\}) \quad \text{and} \quad \Psi^{1L}{\restriction}(\{\alpha\} \times \alpha) = \Psi^{2L}{\restriction}(\{\alpha\} \times \alpha)$$

relating them, leaving us with a sextuple of different functions, which we make six-fold pseudo-generic over M_α. $\square$

Thus, by Theorem 1.3, productivity of ESLC does not follow from CH; our example builds a strong S-space and a strong L-space into a product of *strongly* HG spaces. In contrast, strong S-spaces and strong L-spaces are refuted by $\mathsf{MA}(\aleph_1)$ [8]. Assuming PFA, there are no S-spaces [16], and hence HS implies ESLC. That is, under PFA, a space is ESLC iff none of its subspaces is an L-space. Just in ZFC, [10] constructs an L-space. We do not know whether it is possible to build an L-space into a product of ESLC spaces.

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(Hart) University of Wisconsin, Oshkosh, WI 54901, U.S.A.
Email address: `hartj@uwosh.edu`

(Kunen) University of Wisconsin, Madison, WI 53706, U.S.A.
Email address: `kunen@math.wisc.edu`