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# THE $K(1)$ BASED BOUSFIELD-KAN SPECTRAL SEQUENCE OF CERTAIN TORIC SPACES

DAVID ALLEN AND JOSÉ LA LUZ

Abstract. In this paper we compute the homotopy groups of the unstable $K(1)$-completion of the Borel space for a class of simplicial complexes $\mathcal{L}$ that can be can be realized as the join $\mathcal{L} = \partial(\Delta^{k-1}) * \Delta^j$ for some $j$ and $k$. This decomposition allows for the differentials in a certain unstable Bousfield-Kan spectral sequence to be determined and for explicit calculations to be made.

## 1. Introduction

It is common practice to use a Bousfield-Kan spectral sequence (BKSS) to compute the homotopy groups of spaces $X$ localized at a prime, $p$. In general, computing the $E_2$-term of this spectral sequence is difficult. In [9] a Composite Functor spectral sequence (CFSS) was constructed. Its $E_2$-term is an Ext in an abelian category and converges to the $E_2$-term of the BKSS. To compute this spectral sequence the higher derived functors of the primitive element functor-$P$ of the $E$-homology of $X$ must be computed where $E$ is a suitable spectrum [9]. In addition, the comodule structure of the $R^iPE_*(X)$ in the category of unstable $U$-comodules must be computed [9]. Spaces with primitive dimension less than two allows one to obtain a long exact sequence of $E_2$-terms of the BKSS. In [16] important properties of $R^iP$ are discussed and the notion of an *injective extension sequence* is defined. In this setting, such a sequence is essentially a short exact sequence in the category of positively graded cocommutative, coalgebras. Furthermore, these sequences induce a long

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exact sequence of $R^iP(E_*(X))$ which, under certain conditions, allows one to obtain information about the higher derived functors of the primitive element functor. For spaces whose $E$-homology is nice [16], the higher derived functors of $P$ vanish when the primitive dimension exceeds one and the composite functor spectral sequence converges allowing one to tackle the $E_2$-term of the BKSS and possibly making homotopy calculations. Many toric spaces have a complicated cohomology ring which are dual to coalgebras for which this is not the case, so the spectral sequence becomes intractable, thus, making homotopy calculations very challenging. There are fibrations that relate the higher homotopy groups of these spaces to spaces which in certain cases have a less complicated cohomology ring.

The main result of this paper is the calculation of the homotopy groups of the unstable $K(1)$-completion of Borel space(s) whose Face ring is a free algebra modulo an ideal generated by one relation. Such rings have well controlled primitive dimension. The way in which the toric structure manifests is interesting in that it can be leveraged to determine the differentials without resorting to the complicated algebra that is usually part of such an endeavor. Generally speaking, we rely on the fact that "well behaved" Face rings can be decomposed into a tensor product where each term comes from less complicated objects. This follows from the fact that the underlying simplicial complex can be decomposed as a join of complexes where each term is manageable, at least from a homotopy-theoretic perspective.

The programme initiated by the second author in [25] is carried through in the Toric Topology setting for a certain class of toric spaces. We rely on such structure to simplify certain arguments, especially those focused on computing the $E_2$-term of the $K(1)$-based BKSS. The *$\pi$-primitives functor*–generalized higher derived functors of the primitive element functor– can be computed without a range restriction and since these higher derived functors vanish in degrees greater than one (in the language of [16] such colagebras are refereed to as *nice homology coalgebras*), then calculations become manageable.

This work expands the calculations of the differentials in the $K(1)$-based BKSS studied in [25]. In that paper it was proven that the class $v^k$ supports a $d_{\nu(k)+2}$ differential for the odd sphere. That particular calculation is generalized to $S^2$ (for the specific calculations in this paper), but the same argument essentially carries over to even dimensional spheres. A careful analysis of the homotopy groups of the unstable $K(1)$ completion of $\mathbb{C}P^\infty$ is determined and used as part of the input for the calculation of the homotopy groups of the unstable $K(1)$ completion of the Borel space.

Recall, for the non-connective theory $K(1)$, the coefficient ring $K(1)_* = \mathbb{Z}/p[v^{\pm}]$ where $|v| = 2(p-1)$ and $p$ is an odd prime. For a simplicial complex $\mathcal{K}$, with vertex set $[m]$, the Face ring we consider is $K(1)_*(\mathcal{K}) = K(1)_*[[v_1, ...., v_m]]/I$ where $I$ is the homogenous ideal generated by all square free monomials $v_1 \cdots v_j$ where $\{1, ...j\} \notin \mathcal{K}$ [20] pg. 35. Using results in [8] combined with results in §3 (Lemma 3.16), this ring can be realized by a simplicial complex (c.f., Definition 3.14) which allows one to consider decompositions on the simplicial level (i.e., simplicial complexes). This is especially the case when $I = \langle v_1 \cdots v_k \rangle$ for $1 < k < m$ with $m > 2$. This results in the existence of maps of spaces and certain useful tensor products. For example, the Face ring can be decomposed as a tensor product which corresponds to the Face ring of a certain simplicial complex obtained as the join of two other complexes. This join operation on the level of simplicial complexes ascends to a product on the level of Moment angle complexes, each one understood and fitting nicely into a fibration. This is one way in which the Topology informs the analysis.

The determination of certain Ext groups is made explicitly in the proof of Theorem 6.8 and the $d_2$ differential mapping one Ext to another is shown to be trivial.

The paper is organized as follows. In §2 the main results are listed along with any related terminology. Constructions from Toric Topology are discussed in §3 along with a few necessary results on the realization of certain rings as the Face rings of simplicial complexes that can be decomposed as a join. The higher derived functors of the primitive element functor and their generalization to the derived functors of the $\pi$-primitives are discussed in §4. A certain Composite Functor spectral sequence is constructed in §5 converging to the right higher derived functors of the $\pi$-primitives defined in §4. It is also shown in §5 that under suitable conditions the spectral sequence gives rise to a long exact sequence of these generalized right derived functors. In §6 we recall a crucial spectral sequence constructed in [25] where the classes in the $E_2$-term comes from those surviving to the $E_\infty$-term of a certain composite functor spectral sequence. The Ext tables for this composite functor spectral sequence are explicitly computed for a certain Borel space $B_T Z$ (c.f., §6 Notational Conventions) in a category that is very much motivated by and related to the category of *unstable U-comodules* defined in [9]. We show that the spectral sequence collapses at the $E_2$-term. Later in this section, we explicitly compute the homotopy groups of the unstable $K(1)$-completion of a certain class of toric spaces (see Theorem 6.14).

## 2. Main Result

Before summarizing the main results concerning the homotopy groups of the unstable $K(1)$ completion of certain toric spaces we provide additional commentary regarding the completion. The papers [13, 14] revolve around computing the unstable homotopy groups of the $E$-completion via the Bousfield-Kan spectral sequence–BKSS. In those papers, the authors focused on Landweber exact theories such as BP and MU. Focus on these theories assures convergence of the spectral sequence but is not sufficient to guarantee that calculations are manageable. The BKSS based on these theories have $E_2$-terms that can be computed for a certain class of spaces such as the odd-dimensional spheres. $K(1)$ is not Landweber exact so convergence is not guaranteed and the calculations of the $E_2$-term become much more involved. For this theory and the $K(1)$-based BKSS calculations centered on the odd-dimensional sphere, the reader can refer to [25].

One of the main contributions of this paper is to compute the homotopy groups of the unstable $K(1)$-completion of certain toric spaces. We obtain very specific calculations of these homotopy groups and determine the differentials explicitly in the spectral sequence.

Part of the program is to use the interplay between certain operations on the level of the underlying simplicial complex and operations on the spaces associated to these complexes to find fibrations where calculations can be made. This is made precise in §6 where explicit, detailed calculations are made.

We briefly recall the definition of the moment angle complex $Z_{\mathcal{K}}$ from [7] pgs. 94 and 137. For a fixed $\mathcal{K}$ with vertex set $[m]$ let $\mathbb{D}^m$ denote the unit polydisc in $\mathbb{C}^m$. There is an action of $T^m$ on $\mathbb{D}^m$ where the orbit space is the unit cube $I^m$ in the positive orthant $\mathbb{R}^m_{\geq}$. Using the notation found in [7], there is a map $\mu : \mathbb{D}^m \to I^m$ and the cubical subcomplex $cc(\mathcal{K})$ which is a subcomplex of $I^m$. Roughly speaking, $cc(\mathcal{K})$ is built out of certain faces of the cube that are indexed by the simplicies of $\mathcal{K}$. Details regarding this cubical complex can be found in [20] pg. 53 construction 4.9. The *moment angle complex associated to $\mathcal{K}$, $Z_{\mathcal{K}}$* is defined to be the pullback

$$\begin{tikzcd}
Z_{\mathcal{K}} \arrow[r, hook] \arrow[d] & D^2 \arrow[d, "\mu"] \\
cc(\mathcal{K}) \arrow[r, hook] & I^m
\end{tikzcd}$$

When the context is clear we write $Z = Z_{\mathcal{K}}$. There is a fibration $Z \to B_T Z \to BT^m$ where $BT_Z$ is the Borel construction. For certain simplicial complexes that can be decomposed as a join (see §3 ) one obtains a fibration where the fiber is an odd-dimensional sphere. Using results that appear in §6, we can determine the homotopy groups of the unstable $K(1)$-completion from those coming from an odd-dimensional sphere and a certain product of $\mathbb{C}P^\infty$.

We set the following notation. Let $p$ be an odd prime and $q = 2(p-1)$. If $X$ is a space, then $(X)^\wedge$ refers to the unstable $K(1)$-completion of a space $X$. The $p$-adic integers will be denoted by $\mathbb{Z}_p^\wedge$ and let $\nu(w)$ be defined by $w = ap^{\nu(w)}$ with $p \nmid a$. The main results concerning the homotopy groups of the unstable $K(1)$-completion are:

**Theorem 6.12.** *Let $p$ be an odd prime.*

$$\pi_z((\mathbb{C}P^\infty)^\wedge) = \begin{cases} \mathbb{Z}_{p^{\nu(r)+1}} & z = 1 + rq, r \in \mathbb{Z} - \{0\} \\ \mathbb{Z}_p^\wedge & z = 1, 2 \\ 0 & \textit{otherwise} \end{cases}$$

For some simplicial complex $\mathcal{K}$ on $[m] = \{1, ..., m\}$ let $T$ be the topological torus with $m$ factors. We fix $\mathcal{K}$ and write the corresponding Moment angle complex defined above as $Z = Z_{\mathcal{K}}$. There is the Borel space $B_T Z = ET \times (Z/\sim) = ET \times_T Z$ where $T$ acts on $Z$.

We now recall a few critical notions from Toric Topology. Following page 141 section 4.2 in [7] let $A_i \subseteq X_i$ be collection of subspaces of $X_i$ indexed by the vertex set $[m]$ of $\mathcal{K}$, then the *polyhedral product* $(\mathbf{X},\mathbf{A}) = \{(X_1, A_1), ...., (X_m, A_m)\}$. The authors of [7] go on to define the *polyhedral product corresponding to* $\mathcal{K}$ as

$$(\mathbf{X},\mathbf{A})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbf{X},\mathbf{A})^I = \bigcup_{I \in \mathcal{K}} \left( \prod_{i \in I} X_i \times \prod_{i \notin I} A_i \right)$$

Using this language, the Moment angle complex $Z_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$ (See page 137 and the remark on page 142 [7]). In terms of the polyhedral product, another case of particular importance is $(X, pt)^{\mathcal{K}}$ which the authors in [7] denote by $X^{\mathcal{K}}$, specifically, $(\mathbb{C}P^\infty, pt)^{\mathcal{K}} = (\mathbb{C}P^\infty)^{\mathcal{K}}$. Its cohomology ring was determined to be the Stanley-Reisner Ring of $\mathcal{K}$ and its $R$-Face ring is $H^*((\mathbb{C}P^\infty)^{\mathcal{K}}; R) \cong R[v_1, ..., v_m]/I$ for a commutative ring with unit $R$. In the case that $R$ is replaced by a graded coefficient ring $E_*$ for a generalized orientable theory that has generalized chern classes $c_i^E$, then $H^*((\mathbb{C}P^\infty)^{\mathcal{K}}; E_*) \cong E_*[[v_1, ..., v_m]]/I$. Examples might include theories corresponding to spectra such as $K(1)$, $MU$ or $BP$.

The following ring will be of interest. Let

$$Q = \begin{cases} R[v_1, .., v_m]/I & R \text{ commutative ring with unit} \\ R[[v_1, .., v_m]]/I & R \text{ graded commutative ring with unit} \end{cases}$$

where the ideal is generated by square free monomials and the indeterminants are degree two. We say a simplicial complex $\mathcal{L}$ *realizes* $Q$ if its $R$-Face ring is isomorphic to $Q$ (see §3 Definition 3.14). For a fixed $m > 2$ and a fixed $k$ such that $1 < k < m$, there is a simplicial complex $\mathcal{L}$ that realizes $K(1)_*[[v_1, ...., v_m]]/\langle v_1 \cdots v_k\rangle$ (Lemma 3.16). Let $B_TZ = (\mathbb{C}P^\infty)^{\mathcal{L}}$ and define $l(B_TZ) = k$ to be the length of the square free monomial generator of the ideal $\langle v_1 \cdots v_k\rangle$.

**Theorem 6.14.** *Let $p$ be an odd prime. For $B_TZ = (\mathbb{C}P^\infty)^{\mathcal{L}}$ with $l(B_TZ) = k$ and $(p-1) \nmid (k-2)$ the $K(1)$-Bousfield-Kan spectral sequence for $B_TZ$ converges completely in the sense of* [13] *and*

$$\pi_z((B_TZ)^\wedge) = \begin{cases} \mathbb{Z}^m_{p^{\nu(r)+1}} & z = 1 + rq, r \in \mathbb{Z} - \{0\} \\ \mathbb{Z}_{p^{\nu(r)+1}} & z = 2k - 1 + rq, r \in \mathbb{Z} - \{0\} \\ \mathbb{Z}_p^\wedge & z = 2k-2, 2k-1 \\ (\mathbb{Z}_p^\wedge)^m & z = 1, 2 \\ 0 & \textit{otherwise} \end{cases}$$

## 3. Toric Topological Preliminaries

We highlight some important constructions in Toric Topology [22],[20]. Following the exposition [7] we will use the language of polyhedral products. The background material in this section can be found in the references [7, 20]. Later in this section, a few auxiliary results needed in the sequel are proven too.

**Definition 3.1.** Let $X$ be a $T^m$-space. The Borel space $B_TX$ is the identification space

$$ET^m \times X/\sim := ET^m \times_{T^m} X$$

where the equivalence relation is defined by: $(e, x) \sim (eg, g^{-1}x)$ for any $e \in ET^n$ and $x \in X$ , $g \in T^m$.

**Notational Convention**: Throughout we will fix a simplicial complex $\mathcal{K}$ with vertex set $[m]$ and use the notation $B_TZ$ to refer to the Borel construction on the Moment angle complex $Z_{\mathcal{K}} = Z$. By $\Delta(j_1, ..., j_s)$ we mean the simplicial complex with vertex set $\{j_1, ..., j_s\}$ whose simplicies are the elements of the power set of the vertex set with no missing faces.

**Definition 3.2.** Given $\mathcal{K}$ with vertex set $[m]$ and for a fixed commutative ring $R$ with unit we have

$$R(\mathcal{K}) = R[v_1, .., v_m]/I$$

where $|v_i| = 2$ are indexed by the number of vertices and the ideal

$$I = \langle v_{i_1} \cdots v_{i_k} | \{i_1, ...., i_k\} \notin \mathcal{K} \rangle$$

is generated by square free monomials coming from those subsets of $[m]$ that are not simplicies of $\mathcal{K}$.

As mentioned above $H^*((\mathbb{C}P^\infty)^{\mathcal{K}}; R) \cong R[[v_1, ..., v_m]]/I$ when $R$ is a graded commutative ring with unit. For a complex orientable theory with generalized chern classes the $E$-Face ring of $\mathcal{K}$ is denoted by $E_*(\mathcal{K})$.

Following page 141 section 4.2 in [7] let $A_i \subseteq X_i$ be collection of subspaces of $X_i$ indexed by the vertex set $[m]$ of $\mathcal{K}$, then the *polyhedral product* $\mathbf{(X,A)} = \{(X_1, A_1), ...., (X_m, A_m)\}$. The authors of [7] go on to define the *polyhedral product corresponding to* $\mathcal{K}$ as

$$\mathbf{(X,A)}^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} \mathbf{(X,A)}^I = \bigcup_{I \in \mathcal{K}} \left( \prod_{i \in I} X_i \times \prod_{i \notin I} A_i \right)$$

We list a few examples of polyhedral products corresponding to a simplicial complex.

*Example* 3.3. The Moment angle complex $Z_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$ (See page 137 and the remark on page 142 [7]).

*Example* 3.4. Let $\mathcal{K} = \partial \Delta^{j-1}$, then the Moment angle complex $Z_{\mathcal{K}}$ is $S^{2j-1}$ [20] pg. 89 Example 6.12.

*Example* 3.5. If $\mathcal{K}$ is disjoint union of three vertices then the Moment angle complex $Z_{\mathcal{K}}$ is homotopy equivalent to $S^3 \bigvee S^3 \bigvee S^3 \bigvee S^4 \bigvee S^4$ [7] Pg. 145, 4.2.9.

In the context of a polyhedral product, a case of particular importance is $(X, pt)^{\mathcal{K}}$. This space is often denoted by $X^{\mathcal{K}}$ [7] and for calculations made in this paper we are interested in the particular example $(\mathbb{C}P^\infty, pt)^{\mathcal{K}} = (\mathbb{C}P^\infty)^{\mathcal{K}}$.

*Example* 3.6. In the case that $\mathcal{K}$ is the disjoint union of $m$ vertices $(\mathbb{C}P^\infty, pt)^{\mathcal{K}} = \mathbb{C}P_1^\infty \bigvee \cdots \bigvee \mathbb{C}P_m^\infty$ [7] pg. 144 Examples 4.2.6.

For two simplicial complexes $\mathcal{K}_1$ and $\mathcal{K}_1$ on vertex sets $[m_1]$ and $[m_2]$ the *join*, $\mathcal{K}_1 * \mathcal{K}_2$ is a simplicial complex whose simplicies are the unions of simplicies $\sigma_i \in \mathcal{K}_i$ [20] pg. 23 construction 2.9. The next result ([7] pg. 144 Proposition 4.2.5) relates products of spaces to the join of simplicial complexes.

**Theorem 3.7.** *$(\boldsymbol{X},\boldsymbol{A})^{\mathcal{K}_1 * \mathcal{K}_2} = (\boldsymbol{X}_1, \boldsymbol{A}_1)^{\mathcal{K}_1} \times (\boldsymbol{X}_2, \boldsymbol{A}_2)^{\mathcal{K}_2}$*

The next three statements and their proofs can be found in [7] (pgs. 145-147 Propositions 4.3.1, 4.3.2 and Corollary 4.3.3). They are listed here for the convenience of the reader.

**Theorem 3.8.** *The cohomology ring of $(\mathbb{C}P^\infty)^\mathcal{K}$ is isomorphic to the Face ring $\mathbb{Z}(\mathcal{K})$. The inclusion of the cellular complex $i : (\mathbb{C}P^\infty)^\mathcal{K} \hookrightarrow (\mathbb{C}P^\infty)^m = BT^m$ induces the quotient projection in cohomology $i^* : \mathbb{Z}[v_1, ..., v_m] \to \mathbb{Z}[\mathcal{K}]$.*

and

**Theorem 3.9.** *The inclusion $i : (\mathbb{C}P^\infty)^\mathcal{K} \hookrightarrow (\mathbb{C}P^\infty)^m$ is decomposed into a decomposition of a homotopy equivalence $h : (\mathbb{C}P^\infty)^\mathcal{K} \simeq ET^m \times_{T^m} Z_\mathcal{K}$ and the fibration $p : ET^m \times_{T^m} Z_\mathcal{K} \to BT^m$ with fiber $Z_\mathcal{K}$. In particular, the moment angle complex $Z_\mathcal{K}$ is the homotopy fiber of the canonical inclusion $i : (\mathbb{C}P^\infty)^\mathcal{K} \hookrightarrow (\mathbb{C}P^\infty)^m$.*

The previous statements give

**Corollary 3.10.** *The equivariant cohomology ring of the Moment angle complex $Z_\mathcal{K}$ is isomorphic to the Face ring of $\mathcal{K}$: $H^*_{T^m}(Z_\mathcal{K}) \cong \mathbb{Z}[\mathcal{K}]$. In equivariant cohomology the projection $p : ET^m \times_{T^m} Z_\mathcal{K} \to BT^m$ induces the quotient projection $p^* : \mathbb{Z}[v_1, ..., v_m] \to \mathbb{Z}[\mathcal{K}]$.*

The Borel construction $ET^m \times_{T^m} Z_\mathcal{K}$ is referred to as the *Davis Januszkiewicz space* $DJ(\mathcal{K})$ and it has homotopy type of $(\mathbb{C}P^\infty)^\mathcal{K}$ (pg. 147 [7]).

**Theorem 3.11.** *Let $k$ be a field and for $i = 1, 2$, let $\mathcal{K}_i$ be simplicial complexes on the vertex sets $[m_i]$. Suppose $k[\mathcal{K}_1]$ and $k[\mathcal{K}_2]$ are isomorphic as $k$-algebras. Then there exists a bijective map $[m_1] \to [m_2]$ which induces an isomorphism between $\mathcal{K}_1$ and $\mathcal{K}_2$.*

*Proof.* Refer to [12] and Theorem 3.1.3 [7] pg. 101. □

It is known that if $\mathcal{K}_1$ is isomorphic to $\mathcal{K}_2$ then there is an isomorphism of the Face rings. For the theory $K(1)$, the coefficient ring, $K(1)_*$ is a field, so by Theorem 3.11 it follows immediately that if $K(1)_*(\mathcal{K}_1)$ is isomorphic to $K(1)_*(\mathcal{K}_2)$ as $K(1)_*$-algebras, then the simplicial complexes are isomorphic.

*Remark* 3.12. Referring to [20] pg. 36 (3) there is a relation between the join and the tensor product of Face rings. Namely, if $k$ is a field, then $k(\mathcal{K}_1 * \mathcal{K}_2) \cong k(\mathcal{K}_1) \otimes k(\mathcal{K}_2)$.

A complex $\mathcal{K}$ is $q \geq 1$ neighborly if any $q$ element subset of $[m]$ is a simplex of $\mathcal{K}$ [20]. We state the following theorem regarding the homotopy groups of $Z_\mathcal{K}$.

**Theorem 3.13.** *Given a $q$-neighborly complex $\mathcal{K}$ and the moment angle complex $Z_{\mathcal{K}}$, then $\pi_z(Z_{\mathcal{K}}) = 0$ if $z < 2q+1$ and $\pi_{2q+1}(Z_{\mathcal{K}})$ is a free abelian group on the $q+1$-element missing faces of $\mathcal{K}$.*

*Proof.* [20] pg. 96 Theorem 6.33 9(b). □

We recall the following:

**Definition 3.14.** Given a quotient ring

$$Q = \begin{cases} R[v_1, .., v_m]/I & R \text{ commutative ring with unit} \\ R[[v_1, .., v_m]]/I & R \text{ graded commutative ring with unit} \end{cases}$$

where the ideal is generated by square free monomials and the indeterminants are degree two, then we say a simplicial complex $\mathcal{K}$ realizes $Q$ if its $R$-Face ring is isomorphic to $Q$.

In [8] the authors provide a method to realize such rings in the cases of interest in this paper. However we will need to verify that the simplicial complex that realizes the Face ring can be decomposed into the join of two simplicial complexes.

The following Lemma is obvious and we state it without proof.

**Lemma 3.15.** *For a fixed $m > 2$ and fixed $k$ such that $1 < k < m$. The module $K(1)_*[[v_1, ...., v_m]]/\langle v_1 \cdots v_k\rangle$ is isomorphic to the tensor product of $K(1)_*[[v_1, ...., v_k]]/\langle v_1 \cdots v_k\rangle$ and $K(1)_*[[v_{k+1}, .., v_m]]$.*

Given the isomorphism in the previous Lemma, we need to verify that each ring can be realized by a simplicial complex.

**Lemma 3.16.** *For a fixed $m > 2$ and a fixed $k$ such that $1 < k < m$, the rings $K(1)_*[[v_1, ...., v_k]]/\langle v_1 \cdots v_k\rangle$, $K(1)_*[[v_{k+1}, ...., v_m]]$ and $K(1)_*[[v_1, ...., v_m]]/\langle v_1 \cdots v_k\rangle$ can be realized from simplicial complexes $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{L}$.*

*Proof.* Consider $K(1)_*[[v_1, ...., v_k]]/\langle v_1 \cdots v_k\rangle$. Let $\mathcal{K}_1 = \partial\Delta^{k-1}$, then there is a space $B_T Z_{\mathcal{K}_1}$ and the $E_2$-term of the Atiyah-Hirzebruch spectral Sequence with coefficients in $K(1)_*$ converges to $K(1)^*(\mathcal{K}_1)$–the $K(1)$-Face ring and by the argument given in the proof of Proposition 6.1 [1] this is precisely $K(1)_*[[v_1, ...., v_k]]/\langle v_1 \cdots v_k\rangle$. Similar reasoning shows that the $K(1)_*$-Face ring of the complex $\Delta(k+1, ..., m)$ is $K(1)_*[[v_{k+1}, ...., v_m]]$. Let $\mathcal{L}$ be $\Delta(1, ..., m) - \{1, ..., k\}$. That is, the complex is all of the simplicies coming from the power set on the vertex set $\{1, ..., m\}$ except one; the set that gives the relation in the ring. Applying the same argument as above we obtain the result. □

From Lemma 3.16 we have $K(1)_*(\mathcal{L}) \cong K(1)_*(\mathcal{K}_1) \otimes K(1)_*(\mathcal{K}_2)$. It then follows from the definition of the join and Theorem 3.11 that $\mathcal{L} \cong \mathcal{K}_1 * \mathcal{K}_2$.

*Remark* 3.17. Since $\mathcal{L} \cong \mathcal{K}_1 * \mathcal{K}_2$, then Theorem 3.7 and the definition of the polyhedral product associated to a simplicial complex together imply $Z_{\mathcal{L}} = Z_{\mathcal{K}_1} \times Z_{\mathcal{K}_2}$. This important fact can also be found in [7] Pg 138, 4.1.3.

**Theorem 3.18.** *For a fixed $m > 2$ and a fixed $k$ such that $1 < k < m$, and the given ring $K(1)_*[[v_1, ...., v_m]]/\langle v_1 \cdots v_k \rangle$ there exists a simplicial complex $\mathcal{L}$ realzing it such that $Z_{\mathcal{L}} \simeq S^{2k-1}$.*

*Proof.* Given $K(1)_*[[v_1, ...., v_m]]/\langle v_1 \cdots v_k \rangle$ there exists a tensor product decomposition by Lemma 3.15. By Lemma 3.16 a simplicial complex realizes each ring in the isomorphism. In particular, $\mathcal{L}$ is realized by $K(1)_*[[v_1, ...., v_m]]/\langle v_1 \cdots v_k \rangle$. By Remark 3.17 the Moment angle complex can be written as the product $Z_{\mathcal{L}} = Z_{\Delta^{k-1}} \times Z_{\Delta(k+1,...,m)}$ with the second factor in the product being a disk ([7] Ex. 4.1.2 (1)) proving the result. □

*Remark* 3.19. Given $K(1)_*[[v_1, ...., v_m]]/\langle v_1, ..., v_k \rangle$ and the complex $\mathcal{L}$ that realizes it, there is a fibration $S^{2k-1} \to (\mathbb{C}P^\infty)^{\mathcal{L}} \hookrightarrow (\mathbb{C}P^\infty)^m$. In the proof of the main result (Theorem 6.14) we simply write $B_T Z$ rather than $(\mathbb{C}P^\infty)^{\mathcal{L}}$.

## 4. Higher Derived Functor Preliminaries

In order to prove our main result, we give a brief summary of certain facts concerning the higher derived functors of the primitive element functor as well as injective (or dually projective) extension sequences. The calculations can be executed for any commutative ring spectrum with unit that represents a complex orientable theory. For example, explicit $BP_*$-based calculations were made in [1, 10, 11].

Let $C$ be a coalgebra over a positively graded ring $A$ and let $(S, \mu, \eta)$ be the triple of the cofree, cocommutative, coalgebra functor over $R$. Using cosimplicial methods, there is an $S$-resolution, denoted by $\mathbf{S}(C)$. Recall, the primitive functor $P$ assigns to each coalgebra $C$, the $A$-module of primitive elements [9, 10]. Apply $P$ to the resolution $\mathbf{S}(C)$ and form the associated unaugmented chain complex; the higher derived functors of the primitive element functor, $R^iP(C)$, are obtained by taking cohomology of the chain complex [10, 16]. An *injective extension sequence* is a sequence of maps of coalgebras

$$C' \xrightarrow{f} C \xrightarrow{g} C''$$

such that

(1) $g$ is an epimorphism
(2) $f$ is an inclusion
(3) $C$ is injective as a $C^{''}$-comodule
(4) $C' = C\Box_{C''}A$

Roughly speaking, $C^{'} \to C \to C^{''}$ is a short exact sequence in the category of $A$-coalgebras. See [11, 10, 1, 2] for explicit calculations and useful reformulations. One of the key results used in computing these derived functors is given by the following [16]:

**Theorem 4.1.** *Let*

$$C^{'} \xrightarrow{f} C \xrightarrow{g} C^{''}$$

*be an injective extension sequence of $A$-coalgebras. Then there is a long exact sequence of $A$-modules:*

$$0 \to P(C^{'}) \to P(C) \to P(C^{''}) \to \cdots$$
$$\cdots \to R^iP(C^{'}) \to R^iP(C) \to R^iP(C^{''}) \to \cdots$$

**Definition 4.2.** A coalgebra $C$ is *nice* if $R^iP(C) = 0$ for $i > 1$.

The coalgebra of a Hopf Algebra, the dual of a free algebra modulo a Borel ideal and the tensor product of nice homology coalgebras are all examples of nice homology coalgebras [16]. A few calculations for coalgebras that are not nice and how the obstructions manifest in the Unstable Adams Novikov Spectral Sequence appeared in [1]. More general calculations for these higher derived functors can be found in [6].

We now give an alternative construction of certain generalized derived functors of the primitive element functor that first appeared in [25]. This approach is necessary for the computation of the derived functors of coalgebras over rings of characteristic $p > 0$. This clarification is required since we insist on working with the free $K(1)_*$-module generated by the image of the frobenius map. These functors, for a certain class of coalgebras, will serve as part of the datum for the generalized composite functor spectral sequence that is to be analyzed.

**Definition 4.3.** For any $A$-subcoalgebra $D$ of $C$ let

$$R^iP_D(C) = Coker(R^iP(D) \to R^iP(C))$$

We have the following Lemma

**Lemma 4.4.** *Let $\{C_\alpha\}$ be a collection of $A$-Coalgebras and $\{D_\alpha\}$ the corresponding coalgebras. Then*

(1) *The map $R^iP(C) \to R^iP_D(C)$ is onto for all $i$.*
(2) *Let $D = \otimes D_\alpha$. Then $R^iP_D(\otimes C_\alpha) = \oplus R^iP_{D_\alpha}(C_\alpha)$.*

*Proof.* Part (1) follows from the definitions. Since $R^iP$ takes tensor products to direct sums and the cokernel preserves direct sums part (2) follows. □

Let $A = K(1)_*$ and $H$ a Hopf algebra over $A$. Also let $A \to \mathbb{Z}_p$ be the map $u_i \mapsto 1$ and $\widetilde{H} = H \otimes_A \mathbb{Z}_p$. Essentially, the ground ring of $H$ is being changed from $A$ to $\mathbb{Z}_p$. There is a canonical map $\alpha : H \to \widetilde{H}$ given by $u_{i_1}^{k_1} \cdots u_{i_r}^{k_r} x \mapsto x$. Let $\widetilde{H}^\pi = Im(\widetilde{H} \xrightarrow{\pi} \widetilde{H})$ where $\pi$ is the Frobenious map (the $p^{th}$ power map). Let $H^\pi = \alpha^{-1}(\widetilde{H}^\pi)$. Since $char(A) = p$ it is clear that $\widetilde{H}^\pi$ is a Hopf Algebra. Since $\alpha$ is a Hopf Algebra map we have the following Lemma.

**Lemma 4.5.** *If $H$ is a Hopf algebra over $A$, then $H^\pi$ is a sub-Hopf Algebra of $H$. Moreover, if $H$ is a connected Hopf Algebra then $H^\pi$ is also a connected Hopf Algebra.*

**Notation 4.6.** If $H$ is a connected Hopf Algebra with unit map $\eta : A \to H$ then by abuse of notation we also write $\eta$ for the unit map for $H^\pi$. Also, we let $i^\pi : H^\pi \hookrightarrow H$ be the inclusion map.

Let $R^iP_\pi(H) = R^iP_{H^\pi}(H)$ and call them the *derived functors of the $\pi$-primitives.* We now calculate the $P^\pi$ derived functors of certain Hopf algebras over $A$. We now define certain Hopf algebras of interest.

We now list certain Hopf algebras that appear in [10]:

**Definition 4.7.** Let $C(x_n)$ be the Hopf Algebra with a single generator of degree $n$ such that $x_n^2 = 0$. $T(x_{2n})$ is the Hopf Algebra with generators $x_{2n}, x_{4n}, \dots$ with coalgebra structure given by: $\Delta(x_{2nm}) = \sum_{i+j=m} \binom{i+j}{i} x_{2ni} \otimes x_{2nj}$. $B(x_{2n})$ is the coalgebra of the bipolynomial algebra with generators given by $\{x_{2np^s} | s \geq 0\}$ with $x_{2n}^{p^s}$ primitive. $D$ will be a Hopf algebra such that its underlying coalgebra is cofree and cocommutative.

The coalgebra, $T(x_{2n})$, is dual to a divided power algebra and $D$ is dual to a free $A$-algebra. The $\pi$-primitive functors of the Hopf algebras $C(x_n)$, $T(x_{2n})$ and $D$ can be computed since the underlying coalgebra structure of each of these Hopf algebras coincides with the coalgebras described in [10]. We now set the following notation: $A(x_1, \dots, x_n)$ denotes the free $A$-module generated by $x_1, \dots, x_n$.

**Lemma 4.8.** *The derived functors of the $\pi$-primitives of the Hopf algebras, $C(x_n)$, $D$ and $T(x_{2n})$ are as follows:*

(1) $R^iP_\pi(C(x_{2n})) = R^iP(C(x_{2n}))$ *for* $i \geq 0$

(2) $R^iP_\pi(D) = R^iP(D)$ *for* $i > 0$

(3)

$$R^iP_\pi(T(x_{2n})) = \begin{cases} A(x_{2n}) & i = 0 \\ A(x_{2np}) & i = 1 \\ 0 & i > 1 \end{cases}$$

*Proof.* We first prove (2). Since $D$ is cofree as a coalgebra, it follows that $R^iP(D) = 0$ for $i > 0$ and the result follows. To prove (1) we observe that $C(x_n)^\pi$ is trivial. For (3) we construct a commutative diagram by piecing together injective extension sequences that appeared in the work [10]. The only non-trivial derived functor of the coalgebra $B(x_{2n})$ is the zero derived functor and it is the module of primitives of $B(x_{2n})$. This observation gives $R^0P(B(x_{2n})) = A(2n, 2np, \ldots)$. We observe that $T(x_{2n})^\pi \cong T(x_{2np})$, $B(y_{2n})^\pi = B(y_{2np})$ and $B(z_{2np})^\pi = B(z_{2np^2})$. There are injective extension sequences $T(x_{2n}) \to B(y_{2n}) \to B(z_{2np})$ and $T(x_{2np}) \to B(y_{2np}) \to B(z_{2np^2})$ and a commutative diagram:

$$\begin{array}{ccccc} A(2np^2, 2np^3, \ldots) & = & R^0P(B(z_{2np^2})) & \xrightarrow{\cong} & R^1P(T(x_{2np})) \\ \big\downarrow{\scriptstyle i_*} & & \big\downarrow{\scriptstyle i_*} & & \big\downarrow{\scriptstyle i_*} \\ A(2np, 2np^2, \ldots) & = & R^0P(B(z_{2np})) & \xrightarrow{\cong} & R^1P(T(x_{2n})) \end{array}$$

The horizontal isomorphisms come from the proof of Proposition 3.3 (iv) in §3 of [10]. The vertical map on the left is injective and the result follows by taking cokernels. □

**Definition 4.9.** $C$ is $\pi$-nice if $R^iP_\pi(C) = 0$ for $i > 1$.

It follows from part (1) of Lemma 4.4 that if $H$ is nice as a coalgebra then it is $\pi$-nice. We say that that the homology theory $E$ is $\pi$-nice if $E_*(\underline{E}_k)$ is $\pi$-nice for all $k$ where $\underline{E}_k$ are the spaces in the spectrum of $E$.

*Remark* 4.10. The construction of the functors $R^iP_\pi(-)$ first appeared in [25]. Although the constructions are different, the main results agree. A crucial idea is to eliminate the contribution of the decomposable primitives to the derived functors $R^iP(-)$.

For a fixed odd prime $p$ and a natural number $n$, there are the Morava $K$-theories, $K(n)$ [31]. The case of particular interest is $K(1)$. Recall, this is a multiplicative theory with unit and period $q = 2(p-1)$ that is related to $BP$ by a multiplicative spectrum map $BP \to K(1)$. This theory is a summand of complex $K$-theory mod $p$. The coefficient ring is

$\mathbb{Z}_p[v^{\pm}]$ where $|v| = q$. The Hopf Algebra structure of the spaces in the $\Omega$-spectrum of $K(1)$ were computed in [15]. There are elements:

(1) $e_1 \in K(1)_1(\underline{K(1)}_1)$
(2) $a_{(0)} \in K(1)_2(\underline{K(1)}_1)$
(3) $b_{(i)} \in K(1)_{2p^i}(\underline{K(1)}_2)$
(4) $[v^i] \in K(1)_0(\underline{K(1)}_{iq})$ where $i \in \mathbb{Z}$

The multiplicative structure induces a product:

$$\circ : K(1)_*(\underline{K(1)}_m) \otimes K(1)_*(\underline{K(1)}_n) \to K(1)_*(\underline{K(1)}_{m+n})$$

Let $J = (j_0, j_1, \ldots)$ with $j_i$ nonnegative and finite such that $b^J = b_{(0)}^{j_0}$. The Hopf ring structure is given by:

$$K(1)_*(\underline{K(1)}_n) \cong$$

$$\bigotimes_{j_0<p-1} \Lambda[a_{(0)} \circ b^J \circ e_1 \circ [v_1^k]] \bigotimes T_p[a_{(0)}^i \circ b^J \circ [v_1^k]] \bigotimes_{j_0<p-1} P[a_{(0)} \circ b^J \circ [v_1^k]]$$

where $\Lambda, T_p, P$ means an exterior, truncated (at $p$) and polynomial algebras respectively. Since circle multiplication by $[v^k]$ is an isomorphism between the Hopf algebras $K(1)_*(\underline{K(1)}_n)$ and $K(1)_*(\underline{K(1)}_{n+kq})$ it is omitted from now on.

Since $K(1)$ is a complex orientable theory and the generators of the $E$-face ring are even dimensional and come from products of $\mathbb{C}P^\infty$, a standard Atiyah-Hirzebruch Spectral Sequence argument coupled with the argument in the proof of Proposition 6.1 in [1] yields the following:

**Lemma 4.11.** $K(1)^*(B_T Z) \cong K(1)_*[[v_1, .., v_m]]/I$

We would like to apply the functors $R^i P_\pi(-)$ to $K(1)_*(B_T Z)$. But this does not possess a Hopf algebra structure. Let $K(1)_* \to K(1)_*(B_T Z)$ be the trivial coalgebra map and

$$R^i P_\pi(K(1)_*(B_T Z)) = R^i P_{K(1)_*}(K(1)_*(B_T Z))$$

For the next Lemma we recall that the ideal $I$ is generated by a single monomial of degree $2k$.

**Lemma 4.12.** *Given the Borel space $B_T Z$, the higher derived functors of $P_\pi$ are as follows:*

$$R^i P_\pi(B_T Z) = R^i P(K(1)_*(B_T Z)) = \begin{cases} K(1)_*(v_1, \ldots v_m) & i = 0, |v_j| = 2 \\ K(1)_*(r) & i = 1, |r| = 2k \\ 0 & \textit{otherwise} \end{cases}$$

*Proof.* Since $K(1)_*$ is the trivial coalgebra then $R^iP_\pi(K(1)_*) = 0$ for $i \geq 0$. This means that $R^iP_\pi(B_TZ) = R^iP(B_TZ)$. This observation coupled with the argument used in [6] gives the result. $\square$

## 5. Description of the $E_2$-Term of the Bousfield-Kan Spectral Sequence

For the convenience of the reader we give a brief description of the $E_2$-term of the of the Bousfield-Kan Spectral Sequence (BKSS), and then relate it to our $K(1)$-based version. The interested reader can find additional details in [9, 14, 18].

**Definition 5.1.** A cotriple $(F, \delta, \epsilon)$ over the category $\mathcal{C}$ is a functor $F : \mathcal{C} \to \mathcal{C}$ and natural transformations $\delta : F \to F^2$ and $\epsilon : F \to I$ such that the following diagrams commute:

$$\begin{tikzcd} F & F^2 \arrow[l, "F\epsilon"'] \arrow[r, "\epsilon_F"] & F \\ & F \arrow[ul, equal] \arrow[u, "\delta"] \arrow[ur, equal] & \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "F\delta"] \arrow[d, "\delta F"] & F^2 \arrow[d, "\delta"] \\ F^2 \arrow[r, "\delta"] & F^3 \end{tikzcd}$$

Given a cotriple $F$ and an element $C \in \mathcal{C}$, an $F$-coalgebra is a cotriple with a map $\psi : C \to G(C)$ such that the following diagrams commute

$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[dr, equal] & F(C) \arrow[d, "\epsilon"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} F(C) \arrow[r, "F\delta"] \arrow[d, "\delta F"'] & F^2(C) \arrow[d, "\delta"] \\ F^2(C) \arrow[r, "\delta"] & F^3(C) \end{tikzcd}$$

If we have a cotriple $F$ with a coalgebra structure for all $C \in \mathcal{C}$ then define the $F$-resolution of $C$

$$F(C) \underset{\overset{d^1}{\to}}{\overset{d^0}{\to}} \quad F^2(C) \begin{array}{c} \overset{d^0}{\to} \\ \overset{d^1}{\to} \\ \overset{d^2}{\to} \end{array} \cdots \tag{5.1}$$

where $d^i = F^{n-i}(\psi_{F^i}) : F^n(C) \to F^{n+1}(C)$ for $0 \leq i \leq n$.

Let $E = \{\underline{E}_k\}$ be a multiplicative spectrum with unit such that $E_*(\underline{E}_k)$ is a free $E_*$-module (this is satisfied by theories such as $E = K(1), BP$). For a space $X$ such that $E_*(X)$ is a free $E_*$-module, let $E(X) = \Omega^\infty(E \wedge \Sigma^\infty X)$ and $G(X) = G(E_*(X)) = E_*(E(X))$. This defines a cotriple over the category of free $E_*$-modules to itself. The diagram that follows is referred to as the $G$-resolution of $E_*(X)$.

$$G(X) \underset{d^1}{\overset{d^0}{\rightrightarrows}} \quad G^2(X) \overset{d^0}{\underset{d^2}{\overset{\xrightarrow{d^1}}{\rightrightarrows}}} \cdots \tag{5.2}$$

The following theorem was proven in [14]:

**Theorem 5.2.** *If $E_*(X)$ is a free $E_*$-module then*

$$E_2^{s,t} = Ext^s_{\mathcal{G}}(E_*(S^t), E_*(X))$$

*where $\mathcal{G}$ is the category of $G$-coalgebras.*

For a detailed description of Ext in the category of $G$-coalgebras, see §4 of [14].

**Notation 5.3.** From this point forward, the $G$ resolution of $X$ (5.2) will be denoted by $\mathbf{G}(X)$.

*Remark* 5.4. $G$ is a functor from the category of free $E_*$-modules to itself, and it first appeared in [9]. For a free $E_*$-module, Bendersky and his collaborators gave a non-functorial, explicit description of $G(M)$. For example, if $M \cong E_*\{x_1, ..., x_n\}$, then $G(M) = E_*(\prod \underline{E}_{|x_i|}) \cong \bigotimes E_*(\underline{E}_{|x_i|})$.

In general, the preferred method to compute the generalized higher derived functors would be to apply the $P_\pi$ functors to the $G$-resolution of $K(1)_*(X)$. However, $d^0$ may not commute with the inclusion map $i : G(X)^\pi \to G(X)$. Since every $K(1)_*$ module is free, for any $X$, $G(X) \cong \bigotimes_{n\in\Omega} K(1)_*(\underline{K(1)}_n)$ for some index set $\Omega$. As a result, it is sufficient to require:

**Hypothesis 5.5.** $d^0(G^n(X)^\pi) \in Im(i)$ where $d^0 : G^n(X) \to G^{n+1}(X)$ for all $n \geq 0$.

$K(1)$ satisfies hypothesis 5.5 for $n > 1$ [25]. In the case $n = 0$, let $X = B_TZ$, then Hypothesis 5.5 follows from the following trivial Lemma:

**Lemma 5.6.** *Let $i_1 : K(1)_* \to K(1)_*(B_TZ)$ be the inclusion into the ground ring of $K(1)_*(B_TZ)$, $i^\pi$ and $\eta$ the inclusion and unit map discussed in 4.6, then $d^0(i_1(x)) \in Im(i_2)$ and the following diagram commutes:*

$$\begin{tikzcd}
K(1)_* \arrow[r, "\eta"] \arrow[d, "i_1"] & G(B_TP)^\pi \arrow[d, "i^\pi"] \\
K(1)_*(B_TZ) \arrow[r, "d^0"] & G(B_TZ)
\end{tikzcd}$$

Recall that $G$ is a functor from the category of free $E_*$-modules to itself. When we want to stress a particular theory we will write $G_E$ instead of $G$. Let $\widetilde{\mathbf{G}}(X)$ be the resolution such that $\widetilde{\mathbf{G}}(X)_n = G^n_{BP}(X) \otimes_{BP_*} K(1)_*$.

**Lemma 5.7.** *If $BP_*(X)$ is a free $BP_*$-module then $\widetilde{\mathbf{G}}(X)$ is an acyclic resolution of the coalgebra $K(1)_*(X)$ by $K(1)_*$-coalgebras.*

*Proof.* Since $BP_*(X)$ is a free $BP_*$-module then $K(1)_*(X) \cong BP_*(X) \otimes K(1)_*$. By Remark 5.4, $G(X) \cong \bigotimes BP_*(\underline{BP}_k)$ and since $BP_*(\underline{BP}_k)$ is a free $BP_*$-module for all $k$ then $BP_*(\underline{BP}_k) \otimes_{BP_*} K(1)_* \cong K(1)_*(\underline{BP}_k)$ is a $K(1)_*$-coalgebra and $\widetilde{\mathbf{G}}(X) = \mathbf{G}(X) \otimes_{BP_*} K(1)_*$ is an acyclic resolution of $K(1)_*(X)$ by $K(1)_*$-coalgebras. □

**Definition 5.8.** Consider the $G$-resolution (5.2). We define $R^s_q P_\pi(X)$ to be the $s$ homology of the $G$-resolution after applying $R^q P_\pi$:

$$R^q P_\pi(G(X)) \xrightarrow{\partial^1} R^q P_\pi(G^2(X)) \xrightarrow{\partial^2} \cdots$$

here $\partial_n = \sum (-1)^i R^q P_\pi(d_i)$, then define $R^s_q \widetilde{P}_\pi(X)$ to be $s$ homology of the $\widetilde{G}$-resolution after applying $R^q P_\pi$:

$$R^q P_\pi(\widetilde{G}(X)) \xrightarrow{\partial^1} R^q P_\pi(\widetilde{G}^2(X)) \xrightarrow{\partial^2} \cdots$$

We have the following

**Theorem 5.9.** *There is a spectral sequence*

$$E^{i,j}_2 = R^j_i P_\pi(X) \Longrightarrow R^{i+j} P_\pi(K(1)_*(X))$$

*and if $BP_*(X)$ is a free $BP_*$-module then there is another spectral sequence*

$$\widetilde{E}^{i,j}_2 = R^j_i \widetilde{P}_\pi(X) \Longrightarrow R^{i+j} P_\pi(K(1)_*(X))$$

*Proof.* We construct spectral sequences using the double complexes

$$D^{i,j}_k = P_\pi S^{j+1} G^{i+1}_k(X)$$

where $G_1 = G_{K(1)}, G_2 = \widetilde{G}$ and $S$ is the cocommutative, coalgebra functor over $K(1)_*$. Fixing $i$ first, we obtain $D^{i,*}_k = P_\pi \mathbf{S}(G^{i+1}_k(X))$. Taking homology again we have $R^i_0 P_\pi(X)$ if $k = 1$ and $R^i_0 \widetilde{P}_\pi(X)$ if $k = 2$.

If we fix $j$, then we have $D^{*,j}_k = P_\pi S^{j+1}(\mathbf{G}_k(X))$. For $k = 1$, Lemma 5.7 we deduce that $E^{0,j}_1 = P_\pi S^{j+1}(K(1)_*(X))$ and is trivial for $i > 0$. Taking homology again gives $R^j P_\pi(K(1)_*(X))$. The same argument for $k = 2$ gives the result for the second spectral sequence. □

**Corollary 5.10.** *There is long exact sequence*
(5.3)
$$\cdots \to R_0^k P_\pi(X) \to R^k P_\pi(K(1)_*(X)) \to R_1^{k-1} P_\pi(X) \to R_0^{k+1} P_\pi(X) \to \cdots$$
*and if $BP_*(X)$ is a free $BP_*$-module, then*
(5.4)
$$\cdots \to R_0^k \widetilde{P}_\pi(X) \to R^k P_\pi(K(1)_*(X)) \to R_1^{k-1} \widetilde{P}_\pi(X) \to R_0^{k+1} \widetilde{P}_\pi(X) \to \cdots$$
*is a long exact sequence.*

*Proof.* From [30] and [25] we know that $BP_*(\underline{BP}_n)$ and $K(1)_*(\underline{K(1)}_n)$ are nice. It can be deduced from [16] that $R^i P_\pi(K(1)_*(\underline{BP}_k))$ and $R^i P_\pi(K(1)_*(\underline{K(1)}_k))$ are trivial for $i > 0$. Hence, by Remark 5.4, each spectral sequences from Theorem 5.9 has only two rows. The result follows. □

To compute $R_0^m P_\pi(X)$ the following is needed.

**Corollary 5.11.** *For all $X$ we have:*

(1) *The map $R_1^{k-1} P_\pi(X) \to R_0^{k+1} P_\pi(X)$ in the long exact sequence (5.3) is trivial.*
(2) *There is a short exact sequence*
$0 \to R_0^k P_\pi(X) \to R^k P_\pi(K(1)_*(X)) \to R_1^{k-1} P_\pi(X) \to 0.$
(3) $R_0^0 P_\pi(X) \cong P_\pi(K(1)_*(X))$.

*Proof.* To prove (1) it is sufficient to compare the degrees of the generators on both sides of the map. Let $x$ be a generator of $R_0^{k+1} P_\pi(X)$. New generators can be obtained from $x$ by multiplying by $v^k$ for $k \in \mathbb{Z}$. The spaces $K(1)_*(\underline{K(1)}_n)$ are a tensor product of polynomial, exterior and truncated polynomial algebras. This fact combined with the Universal coefficients theorem implies that the only portion of $G^{k+2}(X)$ contributing to $R^1 P_\pi(G^{k+2}(X)) \otimes \mathbb{Z}_p$ is the polynomial algebra and it is of the form $\otimes_{\alpha \in \Lambda} T(x_{2n_\alpha})$ for some index set $\Lambda$. The generators, $x_{2n_\alpha}$ can be written as the circle products, $a_{(0)} \circ b^{J_\alpha}$ with dimension $2(1 + \sum j_{i_\alpha} p^i)$ with $j_0 < p - 1$ and $n_\alpha = 1 + \sum j_i p^i$. Since $R^1 P(T(x_{2n_\alpha})) = \mathbb{Z}_p(2n_\alpha p)$, then there are generators in degree $2p(1 + \sum j_{i_\alpha} p^i)$. We inspect the degrees of the generators:
$$R^0 P_\pi(G^{i+1}(X)) \otimes \mathbb{Z}_p = P_\pi(G^{i+1}(X)) \otimes \mathbb{Z}_p.$$

One side has generators $a_{(0)} \circ b^K \circ e_1^\epsilon$ and the other has generators $b_{(0)} \circ b^K \circ e_1^\epsilon$ of degrees $2(1 + \sum_{k \geq 0} k_i p^i) + \epsilon$ with $k_0 < p - 1$ in both cases. Since the dimension of the generators have to agree we must have $\epsilon = 0$. Furthermore, the generators $a_{(0)} \circ b^K \circ e_1^\epsilon$ are divisible by $p$ while the generators $b_{(0)} \circ b^K \circ e_1^\epsilon$ are *not*. Parts (2) and (3) follow from part (1). □

**Lemma 5.12.** *If $BP_*(X)$ is a free $BP_*$-module that has only even dimensional generators, then*

$$R^s_q P_\pi(X) \cong R^s P_\pi(K(1)_*(X)).$$

*Proof.* The map of multiplicative spectra $BP \to K(1)$ induces a maps of resolutions:

(5.5)
$$\begin{tikzcd} & \widetilde{\mathbf{G}}(X) \arrow[dd] \\ \mathbf{G}_{BP}(X) \arrow[ur] \arrow[dr] & \\ & \mathbf{G}_{K(1)}(X) \end{tikzcd}$$

By [30] $H_*(\underline{BP}_{2n};\mathbb{Z}_{(p)})$ is a free $\mathbb{Z}_{(p)}$-module and this implies that $H^*(\underline{BP}_{2n};\mathbb{Z}_{(p)}) = Hom(H_*(\underline{BP}_{2n};\mathbb{Z}_{(p)}),\mathbb{Z}_{(p)})$. Since the generators are even dimensional, the Atiyah-Hirzebruch Spectral Sequence

$$H^*(\underline{BP}_{2n}; BP_*) \Longrightarrow BP^*(\underline{BP}_{2n})$$

collapses and we have $BP^*(\underline{BP}_{2n}) \cong H^*(\underline{BP}_{2n};\mathbb{Z}_{(p)}) \otimes BP_*$. By the same argument, $K(1)^*(\underline{BP}_{2n}) \cong H^*(\underline{BP}_{2n};\mathbb{Z}_{(p)}) \otimes K(1)_*$. Hence we obtain

$$K(1)^*(\underline{BP}_{2n}) \cong H^*(\underline{BP}_{2n},\mathbb{Z}_{(p)}) \otimes K(1)_* \cong$$
$$\cong H^*(\underline{BP}_{2n},\mathbb{Z}_{(p)}) \otimes BP_* \otimes_{BP_*} K(1)_* \cong BP_*(\underline{BP}_{2n}) \otimes_{BP_*} K(1)_*$$

Since $BP^*(\underline{BP}_{2n})$ is a free algebra we have that $K(1)^*(\underline{BP}_{2n})$ is a free algebra. From this we deduce that $K(1)_*(\underline{BP}_{2n})$ is a cofree coalgebra. Since $\widetilde{G}^k(X)$ is a tensor product of the cofree coalgebras $K(1)_*(\underline{BP}_{2n})$ we determine that $R^j_i \widetilde{P}_\pi(X)$ is trivial for $i > 0$ and $R^j_0 \widetilde{P}_\pi(X) = R^j P_\pi(K(1)_*(X))$. These facts give the following commutative diagram:

$$\begin{tikzcd} & R^1_0 \widetilde{P}_\pi(X) \arrow[r, "\cong"'] \arrow[d] & R^1 P_\pi(K(1)_*(X)) \arrow[d, equal] & & \\ 0 \arrow[r] & R^1_0 P_\pi(X) \arrow[r] & R^1 P_\pi(K(1)_*(X)) \arrow[r] & R^{k-1}_1 P_\pi(X) \arrow[r] & 0 \end{tikzcd}$$

The vertical maps are induced from diagram (5.5). It follows that the first horizontal map in the bottom is an isomorphism, giving the result. $\square$

**Proposition 5.13.** *For all $i$*

$$R^i_0 P_\pi(B_T Z) \cong R^i P_\pi(K(1)_*(B_T Z)).$$

*Proof.* $BP_*(B_TZ)$ is a free $BP_*$ module generated by even dimensional elements [1]. This result combined with Lemma 5.12 gives the result. □

## 6. Proof of the Main Result

In this section the determination of $Ext^{s,t}_{U_\pi}(R^n_0P_\pi(B_TZ))$ is made. Recall, the stabilization map $\sigma : K(1)_*(\underline{K(1)}_n) \to K(1)_*(K(1))$ is injective on the level of indecomposables (resp. dually, the primitives). Let

$$\sigma_\pi : P_\pi K(1)_*(\underline{K(1)}_n) \to K(1)_*(K(1))$$

be the composition of the inclusion of the $\pi$-primitives followed by the stabilization map $\sigma$. In [25] the injectivity of the map $\sigma_\pi$ was refereed to as Hypothesis 6.1 and it was established in Lemma 7.1 for $K(1)_*$ among other Morava $K$-Theories.

**Theorem 6.1.** *Let $U_\pi = P_\pi G$. Then $U_\pi$ is the functor of a cotriple on the category of $K(1)_*$-modules.*

The proof can be found in [25]. The structure of the Hopf Algebroid $\Gamma = K(1)_*(K(1))$ was computed in [33]:

$$\Lambda[\tau_0] \otimes K(1)_* \otimes_{BP_*} BP_*(BP)/(vt_i^{p^n} - v^{p^i}t_i)$$

where $|t_i| = 2(p^i - 1)$ and $v = v_1 \in BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$. It is possible to re-write the $h_i = c(t_i)$ using the canonical anti-isomorphism $c$. Since $h_i \equiv t_i$ mod decomposables we can change the $t_i$ for the $h_i$. Let $J = (j_1, j_2, \ldots)$ be a sequence of non-negative integers and $h^J = h_1^{j_1}h_2^{j_2}\cdots$. Finally, let $l(J) = \sum j_i$ and

$$<_i = \begin{cases} < & i = 0 \\ \leq & i = 1 \end{cases}$$

If $\mathcal{U}_\pi$ is the category of unstable $U_\pi$ comodules over $K(1)_*$ i.e., $K(1)_*$-modules $M$ with maps $M \to U_\pi$, then the construction of the unstable cobar complex in [14] can be mirrored to some degree.

**Lemma 6.2.** *Let $M$ be a $K(1)_*$-module. Then*

*i.* $U_\pi(M) = Span_{K(1)_*}\{\tau_0^i h^J \otimes m \in \Gamma \otimes M \mid i + 2l(J) <_{i_0} |m|\}$

*ii. If $C^{s,t}(M) = Hom_{\mathcal{U}_\pi}(K(1)_*, U_\pi^{s+1}(M)) \cong U_\pi^{s+1}(M)_t$ then the complex*

$$C^{0,t}(M) \to C^{1,t}(M) \to \cdots$$

*injects into the stable cobar complex.*

See §9 of [25] for the proof.

We now let $Ext^{s,t}_{\mathcal{U}_\pi}(M) = Ext^s_{\mathcal{U}_\pi}(K(1)_*(S^t), M)$.

**Corollary 6.3.** *Let $M$ be a $\Gamma$-comodule. Then*

$$Ext^{s,t}_{\mathcal{U}_\pi}(M) \cong Ext^{s,t}_{\Gamma}(M)$$

*where $Ext^{s,t}_{\Gamma}(M) = Ext^{s}_{\Gamma}(K(1)_*(S^t), M)$ are the derived functors of $\Gamma$-comodules.*

*Proof.* Recall, $Ext^{s,t}_{\Gamma}(M)$ can be calculated as the homology of the cobar complex (see [29]). Suppose $i = 0$, $m \in M$ and let $h^J$ be such that $2l(J) \geq |m|$. There is a $k \in \mathbb{N}$ such that $2l(J) < |m| + 2(p-1)k$. This implies $h^J \otimes v^k m \in U_\pi(M)$. Since the left and right action commute for $K(1)$, then $h^J \otimes v^k m = v^k h^J \otimes m$. A similar argument applies when $i = 1$. This proves that $U_\pi(M) = \Gamma \otimes M$ and the result follows by taking homology. □

Now we relate the $E_2$-term of the BKSS with the category of unstable $U_\pi$ comodules over $K(1)_*$ via the following:

**Theorem 6.4.** *There is a spectral sequence*

$$\overline{E}_2^{m,n,t} = Ext^{m,t}_{U_\pi}(R^n_0 P_\pi(X)) \Longrightarrow E_2^{n+m,t}(X)$$

*(converging to the $E_2$-term of the BKSS based on $K(1)$-theory).*

The proof can be found in [25]. We call this spectral sequence the **(Generalized) Composite Functor Spectral Sequence** (CFSS).

**Corollary 6.5.** *Suppose $P_\pi K(1)_*(X) = K(1)_*(X)$. Then the zero line of the composite functor for $K(1)$ injects into the stable Adams Spectral Sequence.*

*Proof.* The spectral sequence of Theorem 6.4 has an edge homomorphism

$$Ext^{m,t}_{U_\pi}(P_\pi K(1)_*(X)) = \bar{E}_2^{m,0,t} \longrightarrow E_2^{m,t}(X).$$

Since the construction commutes with stabilization, the following diagram commutes:

$$\begin{array}{ccccc}
Ext^{m,t}_{U_\pi}(P_\pi K(1)_*(X)) & \longrightarrow & Ext^{m,t}_{\mathcal{G}}(K(1)_*(X)) & = & E_2^{m,t}(X) \\
\cong\downarrow & & \sigma\downarrow & & \sigma\downarrow \\
Ext^{m,t'}_{\Gamma}(K(1)_*(X)) & \xrightarrow{\cong} & Ext^{m,t'}_{\Gamma}(K(1)_*(\Sigma^\infty X)) & = & E_2^{m,t'}(\Sigma^\infty X)
\end{array}$$

□

**Lemma 6.6.** *There is a long exact sequence*

$$\cdots \longrightarrow \overline{E}_2^{s,0,t} \longrightarrow E_2^{s,t}(B_T Z) \longrightarrow \overline{E}_2^{s-1,1,t} \xrightarrow{d_2} \overline{E}_2^{s+1,0,t} \longrightarrow \cdots$$

*where $\overline{E}_2^{s,m,t} = Ext^{s,t}_{U_\pi}(R^m P_\pi(B_T Z))$.*

*Proof.* Since we only have zero and first $P_\pi$ derived functors, the spectral sequence of Theorem 6.4 only has two rows. A standard argument gives the result. $\square$

In what follows, we use the notation of Lemma 6.2.

**Lemma 6.7.** *If $M$ is a $\Gamma$-comodule generated by $x_1, x_2 \dots, x_n$ with trivial coaction, then $Ext^{s,t}_{U_\pi}(M)$ is generated by the elements $v^{k_1}\tau_0 \otimes \cdots \otimes \tau_0 \otimes h_1^{k_2} \otimes x_{k_3}$ where $k_1 \in \mathbb{Z}$, $k_2 = 0, 1$ and $k_3 = 1, \dots, n$.*

*Proof.* Since the coaction is trivial, the calculation is the one made for the odd sphere [25]. $\square$

## Notational Conventions

From this point forward, the element $v^{k_1}\tau_0 \otimes \cdots \otimes \tau_0 \otimes h_1^{k_3} \otimes \iota$ where the $\tau_0$ appears $k_2$ times will be denoted by: $v^{k_1}\tau^{k_2}h_1^{k_3}\iota$. Note: $k_1 \in \mathbb{Z}$, $k_2 \in \mathbb{Z}^+ \cup \{0\}$ and $k_3 \in \{0, 1\}$. In the tables that follow, the indices may change to facilitate calculations.

For a fixed $m > 2$ and a fixed $k$ such that $1 < k < m$, there is a simplicial complex $\mathcal{L}$ that realizes $K(1)_*[[v_1, ...., v_m]]/\langle v_1 \cdots v_k\rangle$ (Lemma 3.16). Let $B_T Z = (\mathbb{C}P^\infty)^{\mathcal{L}}$ and define $l(B_T Z) = k$ to be the length of the square free monomial generator of the ideal $\langle v_1 \cdots v_k\rangle$. **When reference is made to $B_T Z$ it refers to this particular Borel space.**

The suspension of a space $X$ is defined as $X \times [0, 1]/\sim$ with $(x_1, 0) \sim (x_2, 0)$ and $(x_1, 1) \sim (x_2, 1)$ for all $x_1, x_2 \in X$. It will be denoted by $\Sigma$. The functor $X \to \Sigma(X)$ given by $x \mapsto (x, 0)$ induces a map of spectral sequences.

**Theorem 6.8.** *Let $p$ be an odd prime and $B_T Z = (\mathbb{C}P^\infty)^{\mathcal{L}}$ with $l(B_T Z) = k$. If $(p-1) \nmid (k-2)$ then the differential $d_2 : Ext^{s,t}_{U_\pi}(R^1 P_\pi(B_T Z)) \to Ext^{s+2,t}_{U_\pi}(R^0 P_\pi(B_T Z))$ in Lemma 6.6 is trivial.*

*Proof.* Since the $\Gamma$-coaction on $R^i P_\pi(B_T Z)$ is trivial, Lemma 6.7 gives the generators of the $Ext$ groups, and they are classified by the following tables where $s$ refers to the filtration and $t$ refers to the degree of the elements:

| | $Ext^{s,t}_{U_\pi}(R^0 P_\pi(B_T Z))$ | |
|---|---|---|
| element | $s$ | $t$ |
| $v^{k_1}\tau^{k_2}x_j$ | $k_2$ | $k_1(2p-2) + k_2 + 2$ |
| $v^{k_1}\tau^{k_2}h_1 x_j$ | $k_2 + 1$ | $(k_1+1)(2p-2) + k_2 + 2$ |

where $j < n+1$; and:

| | $Ext^{s,t}_{U_\pi}(R^1P_\pi(B_TZ))$ | |
|---|---|---|
| element | $s$ | $t$ |
| $v^{k_3}\tau^{k_4}x_{n+1}$ | $k_4$ | $k_3(2p-2)+k_4+2k$ |
| $v^{k_3}\tau^{k_4}h_1x_{n+1}$ | $k_4+1$ | $(k_3+1)(2p-2)+k_4+2k$ |

If $d_2$ were not trivial, then it maps elements from the second table to the first. Compare the degrees and the filtration of the generators in the tables. If the differential $d_2(v^{k_3}\tau^{k_4}h^{i_1}x_{n+1}) = rv^{k_1}\tau^{k_2}h^{i_2}x_j$ where $r \in \mathbb{Z}_p$ and $i_1, i_2 = 0, 1$, we obtain the relation

$$(k_3+i_1)(2p-2)+k_4+2k = (k_1+i_2)(2p-2)+k_2+2$$

since $d_2$ preserves the degree of the elements and raises filtration degree by 2 we have $k_2 = k_4+2$. Using these relations and simplifying we obtain $k-2 = (k_1+i_3-k_3-i_1)(p-1)$ or $(p-1)|(k-2)$. □

**Corollary 6.9.** *Let $p$ be an odd prime. If $(p-1)\nmid(k-2)$ then*

$$E_2^{s,t}(B_TZ) \cong Ext^{s,t}_{U_\pi}(R^0P_\pi(B_TZ)) \oplus Ext^{s-1,t}_{U_\pi}(R^1P_\pi(B_TZ)).$$

*Proof.* By Theorem 6.8 the long exact sequence of Lemma 6.6 breaks into a short exact sequence

$$0 \to Ext^{s,t}_{U_\pi}(R^0P_\pi(B_TZ)) \to E_2^{s,t}(B_TZ) \to Ext^{s-1,t}_{U_\pi}(R^1P_\pi(B_TZ)) \to 0.$$

Since the groups are free $\mathbb{Z}_p$-modules the splitting follows. □

Since $K(1)_*(S^2)$ has trivial coaction we can deduce explicitly, the generators of the group $Ext^{s,t}_{U_\pi}(P_\pi K(1)_*(S^2))$ by applying Lemma 6.7. In fact, the generators are: $v^{k_1}\tau^{k_2}h_1^{k_3}\iota_2$ where $\iota_2$ is the generator of the group $K(1)_*(S^2) = P_\pi K(1)_*(S^2)$. In what follows we recall the exponent for a fixed odd prime $p$. For a non-negative integer $k$, $\nu(k) = max\{j|\ p^j|k\}$. For example, if $p=3$ and $k=9$, then $\nu(9)=2$. Very often it is written as $k = ap^{\nu(k)}$ with $p \nmid a$ and $q = 2(p-1)$.

**Lemma 6.10.** *In $E_2^{s,t}(S^2)$, $v^{k_1}\iota_2$ supports a $d_{\nu(k_1)+2}$ differential.*

*Proof.* Since $P_\pi K(1)_*(S^2) = K(1)_*(S^2)$, by Theorem 6.5

$$Ext_{U_\pi}(R^0P_\pi(S^2)) = Ext_{U_\pi}(P_\pi K(1)_*(S^2))$$

injects into the stable Adams spectral sequence based on $K(1)$. Hence, the elements from $Ext_{U_\pi}(P_\pi K(1)_*(S^2))$ survive into $E_2^{s,t}(S^2)$ from the

Composite Functor Spectral Sequence. The suspension homomorphism $\Sigma : K(1)_*(S^2) \to K(1)_*(S^3)$ induces a homomorphism of Composite Functor Spectral Sequences $\Sigma_* : \overline{E}^{m,n,t}(S^2) \to \overline{E}^{m,n,t}(S^3)$ which is an isomorphism when $n = 0$. We observe that this isomorphism does not preserve degree; in fact, it shifts the degree by $+1$. A homomorphism of Bousfield-Kan spectral sequences is obtained by taking $v^{k_1}\tau^{k_2}h_1^{k_3}\iota_2$ to $v^{k_1}\tau^{k_2}h_1^{k_3}\iota_3$. From §11 of [25], it is known that $v^{k_1}\iota_3$ supports a $d_{\nu(k_1)+2}$ differential. The result follows by commutativity of $\Sigma_*$. $\square$

**Notation 6.11.** If $E$ is a homology theory, the $E$-completion of $X$ in the sense of [13] will be denoted by $X^\wedge_E$. When $E = K(1)$ we write $(X)^\wedge$ for $X^\wedge_E$.

**Theorem 6.12.** *Let $p$ be an odd prime.*

$$\pi_z((\mathbb{C}P^\infty)^\wedge) = \begin{cases} \mathbb{Z}_{p^{\nu(r)+1}} & z = 1 + rq, r \in \mathbb{Z} - \{0\} \\ \mathbb{Z}_p^\wedge & z = 1, 2 \\ 0 & \textit{otherwise} \end{cases}$$

*Proof.* Since $K(1)_*(\mathbb{C}P^\infty)$ is cofree, then by Lemma 4.8 we have

$$R^nP_\pi(K(1)_*(\mathbb{C}P^\infty)) = \begin{cases} K(1)_*(\iota_2^{\mathbb{C}P^\infty}) & n = 0, |\iota_2^{\mathbb{C}P^\infty}| = 2 \\ 0 & \text{otherwise} \end{cases}$$

It follows that the CFSS collapses and there is an isomorphism of the $E_2$-term of the BKSS $E_2^\bullet(\mathbb{C}P^\infty) = Ext_{U_\pi}(K(1)_*(\iota_2^{\mathbb{C}P^\infty}))$. The canonical map $i : S^2 \to \mathbb{C}P^\infty$ induces a map of Bousfield-Kan spectral sequences $\iota_* : E_2^\bullet(S^2) \to E_2^\bullet(\mathbb{C}P^\infty)$ that maps the class $v^{k_1}\tau_2^k h^i\iota_2$ to the class $v^{k_1}\tau_2^k\iota_2^{\mathbb{C}P^\infty}$. This map fits into a commutative diagram:

$$\begin{tikzcd} E_2^\bullet(S^2) \arrow[r] \arrow[d, Rightarrow] & E_2^\bullet(\mathbb{C}P^\infty) \arrow[d, Rightarrow] \\ \pi_*((S^2)^\wedge) \arrow[r] & \pi_*((\mathbb{C}P^\infty)^\wedge) \end{tikzcd}$$

Since the differentials commute with $i_*$, the result follows from Lemma 6.10. $\square$

*Remark* 6.13. Before proving the next statement regarding the homotopy groups of the unstable completion of $B_TZ$ we take note of the following.

- The spectral sequence can be applied to simply connected spaces so it is not the case that the result can be obtained by analyzing the spectral sequence for $S^1$.

- The spectral sequence for $\mathbb{C}P^\infty$ has towers that come from Hopf Algebroid elements in $\Gamma = K(1)_*(K(1))$ and there are differentials.

We can now describe the unstable $K(1)$-completion of a certain family of toric space. Recall, for a fixed $m > 2$ and a fixed $k$ such that $1 < k < m$, there is a simplicial complex $\mathcal{L}$ that realizes $K(1)_*[[v_1, ...., v_m]]/\langle v_1 \cdots v_k\rangle$ (Lemma 3.16). Let $B_TZ = (\mathbb{C}P^\infty)^{\mathcal{L}}$ and define $l(B_TZ) = k$ to be the length of the square free monomial generator of the ideal $\langle v_1 \cdots v_k\rangle$.

**Theorem 6.14.** *Let $p$ be an odd prime. For $B_TZ = (\mathbb{C}P^\infty)^{\mathcal{L}}$ with $l(B_TZ) = k$ and $(p-1) \nmid (k-2)$ the $K(1)$-Bousfield-Kan spectral sequence for $B_TZ$ converges completely in the sense of* [13] *and*

$$\pi_z((B_TZ)^\wedge) = \begin{cases} \mathbb{Z}^m_{p^{\nu(r)+1}} & z = 1 + rq, r \in \mathbb{Z} - \{0\} \\ \mathbb{Z}_{p^{\nu(r)+1}} & z = 2k - 1 + rq, r \in \mathbb{Z} - \{0\} \\ \mathbb{Z}^\wedge_p & z = 2k-2, 2k-1 \\ (\mathbb{Z}^\wedge_p)^m & z = 1, 2 \\ 0 & \textit{otherwise} \end{cases}$$

*Proof.* By §7 of [19], we have $(\prod_m \mathbb{C}P^\infty)^\wedge \cong \prod_m (\mathbb{C}P^\infty)^\wedge$. Since all of the calculations are $\mathbb{Z}_p$, the product of spaces gives a tensor product of coalgebras which in turn produces a direct sum when the derived functors are computed. Hence, we have $E_2(\prod_m S^2) \cong \bigoplus_m E_2(S^2)$ and $E_2(\prod_m \mathbb{C}P^\infty) \cong \bigoplus_m E_2(\mathbb{C}P^\infty)$. For $1 \le j \le m$, let $\iota_{(2,j)}$ be the generator of the $j^{th}$ $E_2$-term coming from the direct sum for the even spheres and similarly $\iota^{\mathbb{C}P^\infty}_{(2,j)}$ is the generator of the $j^{th}$ $E_2$-term coming from the direct sum for $\mathbb{C}P^\infty$. Since $\pi_2(B_TZ) \cong \mathbb{Z}^m$ [1, 20] there is a commutative diagram of $E_2$-terms

$$\begin{tikzcd} & E_2^{0,2}(\prod_m S^2) \arrow[dl] \arrow[d, "\prod_{1\le j\le m}\iota_{(2,j)}"] \\ E_2^{0,2}(B_TZ) \arrow[r, "g_*"'] & E_2^{0,2}(\prod_m \mathbb{C}P^\infty) \end{tikzcd}$$

The diagonal map comes from the generators of the second homotopy group of $B_TZ$ and $g_*$ is the induced map from the map $g : B_TZ \to (\mathbb{C}P^\infty)^m$. It follows that there exist generators $x_{(2,1)}, \ldots, x_{(2,m)}$ in $E^{0,2}(B_TZ)$ where the indexing set $(2, j)$ refers to the degree and the factor in the product since the generators of the cohomology of the Borel space come from those generators in the product of $\mathbb{C}P^\infty$. As noted above, there are generators $\iota^{\mathbb{C}P^\infty}_{(2,1)}, ..., \iota^{\mathbb{C}P^\infty}_{(2,m)}$ of $E_2^{0,2}(\prod_m \mathbb{C}P^\infty)$ such that $g_*(x_{(2,j)}) = \iota^{\mathbb{C}P^\infty}_{(2,j)}$. The result for stems $z = 1, 2, 1+rq, r \in \mathbb{Z}-\{0\}$ follows from Theorem 6.12 and the previous commutative diagram. Furthermore, in these stems there is an induced surjective map of completions:

$$(g_*)^\wedge : \pi_z((B_T Z)^\wedge) \to \pi_z(\prod_m (\mathbb{C}P^\infty)^\wedge).$$

We note that if there exists a non-trivial class in $\pi_z(\prod_m (\mathbb{C}P^\infty)^\wedge)$ that is not the image of a class in $\pi_z((B_T Z)^\wedge)$ then this would imply that in $E_2^{0,2}(\prod_m \mathbb{C}P^\infty)$ there is a generator that is not in the image of a generator in the term $E_2^{0,2}(B_T Z)$. These observations imply that the map $(g_*)^\wedge$ is surjective. Since $\mathbb{C}P^\infty$ and $B_T Z$ are simply connected, then by the Fibre Lemma (4.4) of §5 of [19], we have a fibration of completions coming from Theorem (3.18) and Remark (3.19)

$$(S^{2k-1})^\wedge \xrightarrow{(f)^\wedge} (B_T Z)^\wedge \xrightarrow{(g)^\wedge} \prod_m (\mathbb{C}P^\infty)^\wedge. \tag{6.1}$$

The following short exact sequence is obtained by applying $\pi_*(-)$ to (6.1) and noting that $(g)^\wedge$ is surjective.

$$\begin{aligned} 0 \longrightarrow \pi_z((S^{2k-1})^\wedge) \xrightarrow{(f_*)^\wedge} \pi_z(B_T Z)^\wedge) \xrightarrow{(g_*)^\wedge} \ldots \\ \ldots\ \pi_z(\prod_m (\mathbb{C}P^\infty)^\wedge) \longrightarrow 0. \end{aligned} \tag{6.2}$$

We recall the calculation of the homotopy groups of the unstable $K(1)$-completion of the odd sphere [25] with $n = k-1$. The smallest possible relation has degree four implying $k > 1$, so the result holds for $z = 2k-1$.

$$\pi_z((S^{2k-1})^\wedge) = \begin{cases} \mathbb{Z}_{p^{\nu(r)+1}} & z = 2k-2+rq, r \in \mathbb{Z} - \{0\} \\ \mathbb{Z}_p^\wedge & z = 2k-2, 2k-1 \\ 0 & \text{otherwise} \end{cases}$$

Since $\pi_z(\prod_m (\mathbb{C}P^\infty)^\wedge)$ has no homotopy in dimensions $z = 2k-1+rq$ (Theorem 6.12), then $(f_*)^\wedge$ in the short exact sequence (6.2) is an isomorphism in those dimensions. By [25] (pg. 697) there can not be a non-trivial differential coming out of the tower $\tau^{k_2} x_{2k}$ where multiplication by $\tau$ corresponds to multiplication by $p$ in homotopy. If we invert $v$, then there is a tower generated by the classes of the type $v^{-1}\tau^{k_2} h_1 x_{2k}$ of total degree $2k + k_2$ and filtration $k_2 + 2$ where one is added to the filtration because of the first derived functor. By the remark above, the classes in this tower are not in the image of a differential and the differentials out are zero too. This tower generates the group $\pi_{2k-2}((B_T Z)^\wedge) = \mathbb{Z}_p^\wedge$. □

The authors believe that the hypothesis of Theorem 6.14 is unnecessary but have not been able to prove it. Additionally, the authors do not know if the $K(1)$-completion agrees with the $K(1)$-localization of these spaces.

*Remark* 6.15. $K(1)$ is neither Landweber exact nor a connective theory so convergence is not guaranteed and the calculations of the $E_2$-term become much more involved. For this theory, the methods obtained in [13, 14] do not apply. Furthermore, Theorem 6.14 and Theorem 6.12 cannot be obtained using the results contained in those papers.

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