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CONNECTEDNESS OF THE FINE TOPOLOGY

VARUN JINDAL AND ANUBHA JINDAL

Abstract. This paper studies the connectedness of the fine topology on $C(X,Y)$, the set of all continuous functions from a Tychonoff space X to a metric space (Y,d). Occasionally, we assume Y to be a normed linear space. We also determine the components of the space $H(\mathbb{R}^n)$, of all self homeomorphisms on the n-dimensional Euclidean space $\mathbb{R}^n$, where $H(\mathbb{R}^n)$ is considered as a subspace of the space $C(\mathbb{R}^n,\mathbb{R}^n)$ equipped with the fine topology.

1. Introduction

The set $C(X,Y)$ of all continuous functions from a topological space X into a metric space Y has a number of natural topologies, such as the point-open, compact-open, and uniform topologies (see, for example, [1], [2], [7], [17], [26] and [28]). When $Y=\mathbb{R}$, the space of real numbers, we write $C(X)$ instead of $C(X,\mathbb{R})$. Although the point-open and compact-open topologies on $C(X,Y)$ can be defined for any space Y, in order to define the uniform topology, it is necessary for Y to have some uniform structure. The *uniform topology on* $C(X,Y)$ *for a metric space* (Y,d) has basic open sets of the form

$$B_d(f,\delta)=\{h\in C(X,Y):\sup\{d(h(x),f(x)):x\in X\}<\delta\},$$

where $f\in C(X,Y)$ and δ is a positive constant. Let $C_d(X,Y)$ denote the space $C(X,Y)$ equipped with the uniform topology generated by the metric d on Y. However, sometimes none of the above topologies is strong enough to apply a function space to a given situation, in which case a finer topology may be needed so that we have a stronger convergence.

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One such topology is the fine topology.

The *fine topology on $C(X,Y)$ for a metric space* (Y,d) has basic open sets of the form

$$B_f(g,\varepsilon) = \{h \in C(X,Y) : d(h(x), g(x)) < \varepsilon(x) \text{ for all } x \in X\},$$

where $g \in C(X,Y)$ and $\varepsilon \in C_+(X) = \{f \in C(X) : f(x) > 0 \text{ for all } x \in X\}$. The space $C(X,Y)$ with the fine topology f is denoted by $C_f(X,Y)$. Various properties of the space $C_f(X,Y)$ have been studied particularly in [3], [4], [6], [9], [10], [11], [12], [13], [14], [16], [19], [21], [25] and [27].

The connectedness of the space $C_f(X,Y)$ is not well understood except in some particular cases. In order to understand connectedness of the space $C_f(X,Y)$ in a wider perspective, we first study the quasicomponents of the space $C_f(X,Y)$ for a metric space (Y,d). Then for a normed linear space $(Y, ||\cdot||)$, we characterize the connectedness of the space $C_f(X,Y)$. Our study shows that pseudocompactness and relative pseudocompactness play a vital role in the study of connectedness of the space $C_f(X,Y)$.

In Section 3, we concentrate on one important kind of function space, that is, the space $H(\mathbb{R}^n)$ of all self homeomorphisms on $\mathbb{R}^n$, the n-dimensional Euclidean space. Clearly, $H(\mathbb{R}^n)$ is a subset of $C(\mathbb{R}^n, \mathbb{R}^n)$. In this section, we characterize the components of the space $H_f(\mathbb{R}^n)$, where $H_f(\mathbb{R}^n)$ is the subspace of $C_f(\mathbb{R}^n, \mathbb{R}^n)$.

Throughout the rest of the paper, we use the following conventions. All spaces are completely regular and Hausdorff, that is, Tychonoff and are assumed to have more than one element (though we may specify that these spaces have additional properties). For a subset A of a topological space X the closure of A is denoted $\overline{A}$. For topological spaces, S and T that have the same underlying set, we use the notations $S = T$, $S \leq T$, and $S < T$ to mean that, respectively, the topology on S is equal to the topology on T, the topology on S is coarser than or equal to the topology on T, and the topology on S is strictly coarser than the topology on T. We denote the set of all natural numbers by $\mathbb{N}$.

2. Connectedness of $C_f(X,Y)$

In this section, we study the connectedness of the space $C(X,Y)$ equipped with the fine topology. In order to prove our first theorem, we need the following lemma.

Lemma 2.1. *If A is a C-embedded subset of X, then any continuous function $f : A \to (0, \infty)$ can be extended to a continuous function $F : X \to (0, \infty)$.*

Proof. Let $f : A \to (0, \infty)$ be a continuous function. Let $h : A \to \mathbb{R}$ be defined as $h(x) = \log(f(x))$ for all $x \in A$. Then h is continuous. Since A is C-embedded, there exists a continuous function $H : X \to \mathbb{R}$ such that $H(x) = h(x)$ for all $x \in A$. For $x \in X$, define $F : X \to (0, \infty)$ by $F(x) = \exp(H(x))$. Then F is the desired function. □

As mentioned in the introduction, connectedness of $C_f(X, Y)$ is very closely related to the pseudocompactness of X. So for $g \in C(X, Y)$, let

$$C(g) = \{h \in C(X, Y) : \overline{\{x \in X : d(g(x), h(x)) > 0\}} \text{ is pseudocompact}\}.$$

Note that for any $h, g \in C(X, Y)$, the set $\overline{\{x \in X : d(g(x), h(x)) > 0\}}$ is pseudocompact if and only if it is relatively pseudocompact (Theorem 2.1 in [20]). Recall that a subset A of a topological space X is called *relatively pseudocompact* if $f(A)$ is bounded in $\mathbb{R}$ for every $f \in C(X)$. Relatively pseudocompact sets are also sometimes called in literature bounded sets (see [18]). Also, recall that the largest connected subset of a space X containing $x \in X$ is called the *component* of $x \in X$ and the *quasicomponent* of X containing $x \in X$ is the intersection of all clopen (closed as well as open) subsets of X which contain x.

Theorem 2.2. *For any $g \in C(X, Y)$, the quasicomponent of $g \in C_f(X, Y)$ is contained in $C(g)$.*

Proof. Suppose $h \in C(X, Y) \setminus C(g)$. Therefore, $\overline{\{x \in X : d(g(x), h(x)) > 0\}}$ is not pseudocompact. Let $H : X \to \mathbb{R}$ be a continuous function such that H is unbounded on $\{x \in X : d(g(x), h(x)) > 0\}$. So, there exists a countable closed and discrete subset A of $\{x \in X : d(g(x), h(x)) > 0\}$ such that A is C-embedded in X (see 1.20 in [8]). Let $A = \{x_n : n \in \mathbb{N}\}$. By Lemma 2.1, we can find $\varepsilon \in C_+(X)$ such that $\varepsilon(x_n) = \frac{d(g(x_n), h(x_n))}{n}$ for all $n \in \mathbb{N}$. Define

$$\mathcal{G} = \{f \in C(X, Y) : \sup\{\frac{d(f(x), g(x))}{\varepsilon(x)} : x \in X\} \text{ is finite }\}.$$

Clearly, $h \notin \mathcal{G}$ and $g \in \mathcal{G}$. It is now enough to show that $\mathcal{G}$ is clopen in $C_f(X, Y)$.

We first show $\mathcal{G}$ is open in $C_f(X, Y)$. Let $p \in \mathcal{G}$. Find $M > 0$ such that $\frac{d(p(x), g(x))}{\varepsilon(x)} < M$ for all $x \in X$. Define $\delta \in C_+(X)$ such that $\delta(x) = M\varepsilon(x) - d(p(x), g(x))$ for all $x \in X$. Then for any $q \in B_f(p, \delta)$, we have

$$\begin{aligned} d(q(x), g(x)) &\leq d(q(x), p(x)) + d(p(x), g(x)) \\ &< M\varepsilon(x) \end{aligned}$$

for all $x \in X$. It follows that $q \in \mathcal{G}$. Hence, $\mathcal{G}$ is open.

If $p \in C(X,Y) \setminus \mathcal{G}$, then for any $\varepsilon \in C_+(X)$ consider the basic open set $B_f(p,\varepsilon)$. If $q \in B_f(p,\varepsilon)$, then q can not belong to $\mathcal{G}$. Because if there exists an $M > 0$ such that $\sup\{\frac{d(q(x),g(x))}{\varepsilon(x)} : x \in X\} \leq M$, then $\sup\{\frac{d(p(x),g(x))}{\varepsilon(x)} : x \in X\} \leq M+1$. But it contradicts that $p \notin \mathcal{G}$. Hence, $\mathcal{G}$ is closed. $\square$

Corollary 2.3. *For any $g \in C(X,Y)$, the component of $g \in C_f(X,Y)$ is contained in $C(g)$.*

Theorem 2.4. *Let X be pseudocompact, and let (Y,d) be a metric space. Then for any $g \in C(X,Y)$, we have $C(g) = C(X,Y)$. Moreover, for an uncountable metric space (Y,d), if $C(g) = C(X,Y)$ for each $g \in C(X,Y)$, then X is pseudocompact.*

Proof. Suppose X is pseudocompact. Therefore, for any $g \in C(X,Y)$, the regularly closed set $\overline{\{x \in X : d(g(x),h(x)) > 0\}}$ is pseudocompact for any $h \in C(X,Y)$. Hence, $C(g) = C(X,Y)$. Conversely, suppose that (Y,d) is uncountable and for any $g \in C(X,Y)$, we have $C(g) = C(X,Y)$. Choose a $g \in C(X,Y)$. If X is not pseudocompact, then we can find a countable set $A = \{x_n : n \in \mathbb{N}\}$ in X such that A is closed, discrete, and C-embedded in X. Choose $y_0 \in Y \setminus \{g(x_n) : n \in \mathbb{N}\}$. Define $h \in C(X,Y)$ such that $h(x) = y_0$ for all $x \in X$. Then $h(x_n) \neq g(x_n)$ for all $n \in \mathbb{N}$. So, $A \subseteq \overline{\{x \in X : d(g(x),h(x)) > 0\}}$. Consequently, $\overline{\{x \in X : d(g(x),h(x)) > 0\}}$ is not pseudocompact. Thus, we arrive at a contradiction that $h \in C(X,Y) \setminus C(g)$. $\square$

The following theorem says that connectedness of (Y,d) is a necessary condition for the connectedness of spaces $C_d(X,Y)$ and $C_f(X,Y)$.

Theorem 2.5. *For a space X and a metric space (Y,d), if $C_d(X,Y)$ is connected, then (Y,d) is also connected. In particular, Y is uncountable.*

Proof. For any $x \in X$, the map $E_x : C_d(X,Y) \to Y$ defined by $E_x(g) = g(x)$ is continuous and onto. Consequently, if $C_d(X,Y)$ is connected, then (Y,d) must be connected. Since every nontrivial connected metric space is uncountable, Y must be uncountable. $\square$

Since $C_d(X,Y) \leq C_f(X,Y)$, if $C_f(X,Y)$ is connected, then (Y,d) is connected and uncountable.

The following result now follows from Theorems 2.4 and 2.5.

Corollary 2.6. *For a space X and a metric space (Y,d), the following assertions are equivalent.*

(a) X is pseudocompact and $C(g)$ is the quasicomponent of $C_f(X,Y)$ containing $g \in C(X,Y)$.

(b) X is pseudocompact and $C(g)$ is the component of $C_f(X,Y)$ containing $g \in C(X,Y)$.

(c) $C_f(X,Y)$ is connected.

We now prove that the pseudocompactness of X is sufficient for the connectedness of $C_f(X,Y)$, whenever Y is a normed linear space. Let $(Y, ||\cdot||)$ be a normed linear space and d be the metric on Y induced by its norm, that is, $d(x,y) = ||x-y||$ for $x, y \in Y$.

Theorem 2.7. *For a normed linear space* $(Y, ||\cdot||)$ *and* $g \in C(X,Y)$*,* $C(g)$ *is the component (path component) of* $C_f(X,Y)$ *containing* g*.*

Proof. By Corollary 2.3, it suffices to show that $C(g)$ is connected. Let $h \in C(g)$ and $A = \{x \in X : d(g(x), h(x)) > 0\}$. So, $\overline{A}$ is pseudocompact. Define $\phi : [0,1] \to C(g)$ by $\phi(t) = tg + (1-t)h$. Since $d((tg + (1-t)h)(x), g(x)) = (1-t)d(g(x), h(x))$, the set

$$\overline{\{x \in X : d(g(x), (tg + (1-t)h)(x)) > 0\}},$$

is pseudocompact. Thus, ϕ is well-defined. Also $\phi(0) = h$ and $\phi(1) = g$. Let $B_f(\phi(t), \varepsilon)$ be any basic neighborhood of $\phi(t)$ in $C_f(X,Y)$. Find $M > 0$ such that $\frac{||g(x)-h(x)||}{\varepsilon(x)} \leq M$ for all $x \in A$. Choose $\delta > 0$ such that $\delta < \frac{1}{M}$. Then for any $x \in A$ and $t' \in [0,1] \cap (t-\delta, t+\delta)$, we have

$$\begin{aligned} d((tg + (1-t)h)(x), (t'g + (1-t')h)(x)) &= |t-t'| d(g(x), h(x)) \\ &< \delta ||g(x) - h(x)|| \leq \varepsilon(x); \end{aligned}$$

if $x \in X \setminus A$, then $d((tg + (1-t)h)(x), t'g + (1-t')h)(x)) = |t-t'|d(g(x), h(x)) = 0 < \varepsilon(x)$. So, $\phi(t') \in B_f(\phi(t), \varepsilon)$. Hence, ϕ is continuous. Thus, $C(g)$ is path connected. □

Corollary 2.8. *If* X *is pseudocompact and* $(Y, ||\cdot||)$ *is a normed linear space, then* $C_f(X,Y)$ *is connected.*

For a normed linear space $(Y, ||\cdot||)$, let $C^*(X,Y)$ be the subset of $C(X,Y)$ consisting of all those $f \in C(X,Y)$ such that the range of f is bounded in Y, that is, $\sup\{||f(x)|| : x \in X\}$ is finite. The set $C^*(X,Y)$ is a clopen subset of $C_d(X,Y)$ and when X is not pseudocompact, then $C^*(X,Y)$ is a proper subset of $C(X,Y)$ ([15]).

Our next theorem gives many equivalent characterizations of the connectedness of the space $C_f(X,Y)$. Let 0_X denote the constant zero function in $C(X,Y)$, that is, $0_X(x) = \mathbf{0}$ for all $x \in X$, where $\mathbf{0}$ denotes the zero element of the normed space Y.

Theorem 2.9. *For a space* X *and a normed linear space* $(Y, ||\cdot||)$*, the following assertions are equivalent.*

(a) X is pseudocompact.

(b) $C_d(X,Y)$ is a normed linear space.
(c) $C_d(X,Y) = C_f(X,Y)$.
(d) $C_f(X,Y)$ is a normed linear space.
(e) $C_f(X,Y)$ is path connected.
(f) $C_f(X,Y)$ is connected.
(g) $C_d(X,Y)$ is connected.
(h) $C_d(X,Y)$ is path connected.
(i) $C_f(X,Y)$ is a topological vector space.
(j) $C_f(X,Y)$ is locally pathwise connected.
(k) $C_f(X,Y)$ is locally connected.

Proof. $(a) \Rightarrow (b)$. If X is pseudocompact, then the function $||\cdot||_\infty$ defined by $||f||_\infty = \sup\{||f(x)|| : x \in X\}$ gives a norm on $C_d(X,Y)$.

$(b) \Rightarrow (a)$. If $C_d(X,Y)$ is a normed linear space, then it is connected. Therefore, X must be pseudocompact, otherwise $C^*(X,Y)$ is a proper clopen subset of $C(X,Y)$ which is both open and closed in $C_d(X,Y)$ ([15]).

$(a) \Leftrightarrow (c)$. It follows from Theorem 2.1 of [11].

$(c) \Rightarrow (d)$. If $C_d(X,Y) = C_f(X,Y)$, then X is pseudocompact. But then $C_d(X,Y)$ is a normed linear space.

$(d) \Rightarrow (j)$. Since every open ball in a normed linear space is convex and therefore path connected.

$(j) \Rightarrow (k)$, $(d) \Rightarrow (e)$ and $(e) \Rightarrow (f)$ are immediate.

$(k) \Rightarrow (a)$. Suppose $C_f(X,Y)$ is locally connected. If possible, suppose that X is not pseudocompact. Therefore, $C^*(X,Y)$ is a proper clopen subset of $C_f(X,Y)$. We show that every open set in $C_f(X,Y)$ containing 0_X is disconnected. Suppose that G is any open connected set containing 0_X. Since $C(0_X)$ is the component of $C_f(X,Y)$ containing 0_X, $G \subseteq C(0_X)$. Choose an $\varepsilon \in C_+(X)$ such that $B_f(0_X, \varepsilon) \subseteq G$ and a $g \in C(X,Y) \setminus C(0_X)$. If $h(x) = \frac{g(x)}{1+||g(x)||}$, then $\varepsilon h \in B_f(0_X, \varepsilon) \subseteq G$. Since $\{x \in X : g(x) \neq \mathbf{0}\} = \{x \in X : (\varepsilon h)(x) \neq \mathbf{0}\}$, $\varepsilon h \notin C(0_X)$. This contradicts that $G \subseteq C(0_X)$.

$(f) \Rightarrow (g)$. If $C_f(X,Y)$ is connected, then $C_d(X,Y)$ is connected, because $C_d(X,Y) \leq C_f(X,Y)$.

$(g) \Rightarrow (h)$. If $C_d(X,Y)$ is connected, then $C^*(X,Y) = C(X,Y)$ because $C^*(X,Y)$ is a clopen subset of $C_d(X,Y)$. Hence, X must be pseudocompact. But if X is pseudocompact, then $C_d(X,Y)$ is a normed linear space under the norm $||f||_\infty = \sup\{||f(x)|| : x \in X\}$. Consequently, $C_d(X,Y)$ is path connected.

$(h) \Rightarrow (a)$. Let $C_d(X,Y)$ be path connected. If X is not pseudocompact, then there exists a continuous function $f : X \to \mathbb{R}$ such that f is unbounded. Take any $y \in Y$ such that $y \neq 0$. Let $g : X \to Y$ be defined by $g(x) = f(x)y$. Then $g \in C(X,Y) \setminus C^*(X,Y)$. Hence, $C^*(X,Y)$ is a proper subspace of $C_d(X,Y)$. Since $C^*(X,Y)$ is both open as well as closed in $C_d(X,Y)$, we arrive at a contradiction.

$(a) \Rightarrow (i)$. If X is pseudocompact, then $C_d(X,Y)$ is a normed linear space under the norm $||f||_\infty = \sup\{||f(x)|| : x \in X\}$ and $C_d(X,Y) = C_f(X,Y)$.

$(i) \Rightarrow (a)$. Suppose X is not pseudocompact. So, there exists an $\varepsilon \in C_+(X)$ such that there is no $M > 0$ satisfying $\varepsilon(x) < M$ for all $x \in X$. In other words, if $\varepsilon' = \frac{1}{\varepsilon}$, then there is no $\delta > 0$ such that $\delta < \varepsilon'(x)$ for all $x \in X$. Let $y_0 \in Y$ be any non zero element and define $f_{y_0} : X \to Y$ by $f_{y_0}(x) = y_0$ for all $x \in X$. We show that the scaler multiplication is not continuous at $(0, f_{y_0}) \in \mathbb{R} \times C_f(X,Y)$. Consider the basic neighborhood $B_f(0_X, \varepsilon')$ of 0_X in $C_f(X,Y)$, where $0_X(x) = \mathbf{0}$ for all $x \in X$. Take a basic neighborhood $(-r,r) \times B_f(f_{y_0}, \varepsilon_1)$ of $(0, f_{y_0})$ in $\mathbb{R} \times C_f(X,Y)$, where $r > 0$ and $\varepsilon_1 \in C_+(X)$. Then for any non zero $\alpha \in (-r,r)$, αf_{y_0} does not belong to $B_f(0_X, \varepsilon')$. Because then $||\alpha f_{y_0}(x)|| = |\alpha|||y_0|| < \varepsilon'(x)$ for all $x \in X$. But this contradicts our choice of $\varepsilon' \in C_+(X)$. Consequently, if X is not pseudocompact, then $C_f(X,Y)$ is not a topological vector space. $\square$

We end this section with the following results on totally disconnectedness of $C_f(X,Y)$.

Theorem 2.10. *For a space X and a metric space (Y,d), if no open set in X is relatively pseudocompact, then $C_f(X,Y)$ is totally disconnected. Moreover, for a normed linear space $(Y, ||\cdot||)$, if $C_f(X,Y)$ is totally disconnected, then no open set in X is relatively pseudocompact.*

Proof. Suppose none of the open sets in X is relatively pseudocompact. Therefore, for any $f, g \in C(X,Y)$ the set $\{x \in X : d(f(x), g(x)) > 0\}$ is not relatively pseudocompact. It follows that the set

$$\overline{\{x \in X : d(f(x), g(x)) > 0\}},$$

cannot be pseudocompact (see Theorem 2.1 in [20]). Consequently, by Corollary 2.3, the component of $g \in C(X,Y)$ is $\{g\}$. Hence, $C_f(X,Y)$ is totally disconnected.

Now for a normed linear space $(Y, ||\cdot||)$, suppose that $C_f(X,Y)$ is totally disconnected and consider a nonempty open set G of X. Then G contains a nonempty cozero set S. Let $S = X \setminus Z(f)$, where $f : X \to [0,1]$ is a continuous function and $Z(f) = \{x \in X : f(x) = 0\}$ is the zero-set of f. Let $p : [0,1] \to Y$ be an arc. Define $g \in C(X,Y)$ as $g = p \circ f$ and $h \in C(X,Y)$ by $h(x) = p(0)$ for all $x \in X$. Then $S = \{x \in X : d(g(x), h(x)) > 0\}$.

Since $C(h) = \{h\}$ is the component of $C_f(X,Y)$ containing h, $\overline{S}$ is not pseudocompact. Therefore, S cannot be relatively pseudocompact. Thus, the open set G is not relatively pseudocompact. □

Theorem 2.11. *If (Y,d) is totally disconnected, then for any X the space $C_f(X,Y)$ is totally disconnected.*

Proof. Let G be the component of $C_f(X,Y)$ containing an element $g \in C(X,Y)$. Suppose $g_1 \in G \setminus \{g\}$. Choose $x_0 \in X$ such that $g_1(x_0) \neq g(x_0)$. Consider the continuous map $E_{x_0} : C_f(X,Y) \to (Y,d)$ defined as $E_{x_0}(h) = h(x_0)$. Since G is connected, the set $E_{x_0}(G) = \{h(x_0) : h \in G\}$ is connected in (Y,d), and therefore must be singleton. But that implies $g(x_0) = g_1(x_0)$, a contradiction. Hence $G = \{g\}$. □

3. Spaces of Homeomorphism on Euclidean Space $\mathbb{R}^n$

In this section, we study the components of the space $H(\mathbb{R}^n)$, of all self homeomorphisms on the n-dimensional Euclidean space $\mathbb{R}^n$, where $H(\mathbb{R}^n)$ is considered as a subspace of the space $C(\mathbb{R}^n,\mathbb{R}^n)$ equipped with the fine topology. The space $H(\mathbb{R}^n)$ with the subspace topology inherited from $C_f(\mathbb{R}^n,\mathbb{R}^n)$ is denoted by $H_f(\mathbb{R}^n)$ ([5], [24]). In order to characterize components and path components of the space $H_f(\mathbb{R}^n)$, we define an equivalence relation $\asymp$ on $H_f(\mathbb{R}^n)$. But in order to define this equivalence relation, we need to define the functions with compact support. A real-valued continuous function f defined on a locally compact space X is said to have a compact support on X if there exists a compact subset K of X such that $f(x) = 0$ for all $x \in X \setminus K$. In this case, K is said to be a support of f and the set $\overline{\{x \in X : f(x) \neq 0\}}$ is called the support of f.

Define an equivalence relation $\asymp$ on $H_f(\mathbb{R}^n)$ by $f \asymp g$ provided there exists a compact subset A of $\mathbb{R}^n$ such that $f(x) = g(x)$ for all $x \in \mathbb{R}^n \setminus A$, that is, if $f - g$ has compact support. Let $\mathcal{C}(f)$ denote the equivalence class of the equivalence relation $\asymp$ that contains the element $f \in H_f(\mathbb{R}^n)$. One can show that each such equivalence class is closed in $H_f(\mathbb{R}^n)$ that need not be open (see [22]).

The next result shows that the space $H_f(\mathbb{R}^n)$ is homogeneous.

Proposition 3.1. *The space $H_f(\mathbb{R}^n)$ is homogeneous.*

Proof. Let $g, h \in H(\mathbb{R}^n)$. Define a map $\eta : H_f(\mathbb{R}^n) \to H_f(\mathbb{R}^n)$ by $\eta(k) = k \circ g^{-1} \circ h$. So, $\eta(g) = h$. Also, note that the function $\varphi : H_f(\mathbb{R}^n) \to H_f(\mathbb{R}^n)$ defined by $\varphi(q) = q \circ h^{-1} \circ g$ is the inverse of η. We show that η is a homeomorphism. Consider any neighborhood $B_f(\eta(k),\varepsilon)$ of $\eta(k)$ in $H_f(\mathbb{R}^n)$, where $\varepsilon \in C_+(\mathbb{R}^n)$. Take the neighborhood $B_f(k,\delta)$ of k in $H_f(\mathbb{R}^n)$, where $\delta \in C_+(\mathbb{R}^n)$ is defined by $\delta(x) = (\varepsilon \circ h^{-1} \circ g)(x)$.

Then for any $t \in B_f(k, \delta)$ and for any $x \in \mathbb{R}^n$,

$$|\eta(t)(x) - \eta(k)(x)| = |t(g^{-1} \circ h(x)) - k(g^{-1} \circ h(x))| \\ < \delta(g^{-1} \circ h(x)) = \varepsilon(x).$$

So, $\eta(B_f(k, \delta)) \subseteq B_f(\eta(k), \varepsilon)$. Thus, η is continuous. A similar argument shows that the inverse of η, that is, φ is also continuous. Hence, η is a homeomorphism. □

Proposition 3.2. *For every $g, h \in H_f(\mathbb{R}^n)$, the equivalence classes $\mathcal{C}(g)$ and $\mathcal{C}(h)$ are homeomorphic as subspaces of $H_f(\mathbb{R}^n)$.*

Proof. By Proposition 3.1, the map $\eta : H_f(\mathbb{R}^n) \to H_f(\mathbb{R}^n)$ defined by $\eta(k) = k \circ g^{-1} \circ h$ is a homeomorphism on $H_f(\mathbb{R}^n)$ that takes g to h. To show that η maps $\mathcal{C}(g)$ into $\mathcal{C}(h)$, let $g_1 \in \mathcal{C}(g)$. So, there exists a compact subset K of $\mathbb{R}^n$ such that $g(x) = g_1(x)$ for all $x \in \mathbb{R}^n \setminus K$. Now $h^{-1}(g(K))$ is a compact subset of $\mathbb{R}^n$ and for each $x \in \mathbb{R}^n \setminus h^{-1}(g(K))$, we have $g^{-1}(h(x)) \notin K$. So, $\eta(g_1)(x) = g_1(g^{-1}(h(x))) = g(g^{-1}(h(x))) = h(x)$ for all $x \in \mathbb{R}^n \setminus h^{-1}(g(K))$. Hence, $\eta(g_1) \in \mathcal{C}(h)$ for each $g_1 \in \mathcal{C}(g)$. A similar argument shows that the inverse homeomorphism $\varphi : H_f(\mathbb{R}^n) \to H_f(\mathbb{R}^n)$ defined by $\varphi(q) = q \circ h^{-1} \circ g$ maps $\mathcal{C}(h)$ into $\mathcal{C}(g)$. It follows that $\mathcal{C}(g)$ and $\mathcal{C}(h)$ are homeomorphic to each other as subspaces of $H_f(\mathbb{R}^n)$. □

We conclude this paper by showing that the components and path components of the space $H_f(\mathbb{R}^n)$ are precisely the distinct equivalence classes $\mathcal{C}(f)$, where $f \in H_f(\mathbb{R}^n)$.

Theorem 3.3. *For any $m \in \mathbb{N}$, the components (and path components) of the space $H_f(\mathbb{R}^m)$ are precisely the members of the family $\{\mathcal{C}(f) : f \in H_f(\mathbb{R}^m)\}$.*

Proof. Because of Proposition 3.2, it suffices to prove that the component containing the identity homeomorphism e is $\mathcal{C}(e)$. Let $f \in H_f(\mathbb{R}^m) \setminus \mathcal{C}(e)$. Suppose, by way of contradiction, that f is in the component of $H_f(\mathbb{R}^m)$ containing e. Since $f \notin \mathcal{C}(e)$, for each $n \in \mathbb{N}$, there exists an $x_n \in \mathbb{R}^m \setminus \overline{B(0, n)}$ such that $f(x_n) \neq x_n$, where $B(0, n) = \{x \in \mathbb{R}^m : |x| < n\}$. For each n, let $\delta_n = \frac{|f(x_n) - x_n|}{n}$. It can be proved that the set $\{x_n : n \in \mathbb{N}\}$ does not have any accumulation point in $\mathbb{R}^m$. It follows that $\{x_n : n \in \mathbb{N}\}$ is closed and discrete in $\mathbb{R}^m$. Let $\varepsilon \in C_+(\mathbb{R}^m)$ such that $\varepsilon(x_n) = \delta_n$ for all $n \in \mathbb{N}$.

Consider the open cover $\{B_f(g, \varepsilon) : g \in H_f(\mathbb{R}^m)\}$ of $H_f(\mathbb{R}^m)$. Since f and e are in the same connected subset of $H_f(\mathbb{R}^m)$, $\{B_f(g, \varepsilon) : g \in H_f(\mathbb{R}^m)\}$ has a simple chain connecting e to f, say $B_f(g_1, \varepsilon), \ldots, B_f(g_k, \varepsilon)$, where $g_1 = e$ and $g_k = f$, and $B_f(g_i, \varepsilon) \cap B_f(g_j, \varepsilon) \neq \phi$ if and only if $|i - j| \leq 1$ (see Problem 6.3.1 in [7]). Let $n = 2k$, and for each

$i = 1, \dots, k-1$, let $y_i \in B_f(g_i(x_n), \varepsilon(x_n)) \cap B_f(g_{i+1}(x_n), \varepsilon(x_n)) = B_f(g_i(x_n), \delta_n) \cap B_f(g_{i+1}(x_n), \delta_n)$. Then we have

$$\begin{aligned} 2k\delta_n = |f(x_n) - x_n| &= |g_k(x_n) - g_1(x_n)| \\ &\leq |g_1(x_n) - y_1| + |y_1 - g_2(x_n)| + \cdots + |y_{k-1} - g_k(x_n)| \\ &< 2(k-1)\varepsilon(x_n) = 2(k-1)\delta_n. \end{aligned}$$

But this gives the contradiction $k < k-1$. Thus, the component of $H_f(\mathbb{R}^m)$ containing e is contained in $\mathcal{C}(e)$.

Now $\mathcal{C}(e)$ will be the component of $H_f(\mathbb{R}^m)$ containing e if it is connected. We show that $\mathcal{C}(e)$ is pathwise connected. Let $f \in \mathcal{C}(e)$. We show that f can be joined to e by a path in $\mathcal{C}(e)$. Since $f \in \mathcal{C}(e)$, there exists a compact subset A of $\mathbb{R}^m$ such that $f(x) = x$ for all $x \in \mathbb{R}^m \setminus A$. Since A is compact, there exists an $n \in \mathbb{N}$ such that $A \subseteq B(0, n)$. So, $f(x) = x$ for all $x \in \mathbb{R}^m$ for $|x| \geq n$. Define a map $p : [0,1] \to H_f(\mathbb{R}^m)$ by

$$p(t)(x) = \begin{cases} tf(\frac{x}{t}), & |x| < nt; \\ x, & |x| \geq nt. \end{cases}$$

Clearly, $p(t)(x) = x$ for $|x| \geq n$, implying $p(t) \in \mathcal{C}(e)$; so that p is a well-defined function from $[0,1]$ to $\mathcal{C}(e)$. It is easy to see that the map $\hat{p} : \mathbb{R}^m \times [0,1] \to \mathbb{R}^m$ defined by $\hat{p}(x,t) = p(t)(x)$ is continuous. Also, $p(0) = e$ and $p(1) = f$. Now we show that p is continuous when $\mathcal{C}(e)$ has the subspace topology inherited from the space $H_f(\mathbb{R}^m)$.

Let $B_f(p(t), \varepsilon)$ be a basic neighborhood of $p(t)$ in $H_f(\mathbb{R}^m)$. Also, let K be the closure of $B(0,n)$ in $\mathbb{R}^m$, so K is compact and $p(t)(x) = x$ for all $x \in \mathbb{R}^m \setminus K$. Now take $\delta = \inf\{\varepsilon(x) : x \in K\}$. Since K is compact, $\delta > 0$. Again, since $p(t)$ and $\hat{p}$ are continuous, for each $x \in K$ there exists a $\delta_x > 0$ such that

$$p(t)\big(B(x, \delta_x)\big) \subseteq B\big(p(t)(x), \frac{\delta}{2}\big)$$

and

$$\hat{p}\big(B(x, \delta_x) \times B(t, \delta_x)\big) \subseteq B\big(\hat{p}(x,t), \frac{\delta}{2}\big) = B\big(p(t)(x), \frac{\delta}{2}\big).$$

Since K is compact, there exist $x_1, \dots, x_n \in K$ such that $K \subseteq B(x_1, \delta_{x_1}) \cup \cdots \cup B(x_n, \delta_{x_n})$. Let $\eta = \min\{\delta_{x_1}, \dots, \delta_{x_n}\}$. We show that $p(B(t,\eta)) \subseteq B_f(p(t), \varepsilon)$. Let $s \in B(t, \eta)$. First of all, note that for all $x \in \mathbb{R}^m \setminus K$, $p(s)(x) = p(t)(x) = x$. If $x \in K$, then $x \in B(x_i, \delta_{x_i})$ for some $i = 1, \dots, n$. Now $s \in B(t,\eta) \subseteq B(t, \delta_{x_i})$, so

$$\begin{aligned} |p(s)(x) - p(t)(x)| &\leq |p(s)(x) - p(t)(x_i)| + |p(t)(x_i) - p(t)(x)| \\ &< \frac{\delta}{2} + \frac{\delta}{2} \leq \varepsilon(x). \end{aligned}$$

Therefore, $p(s) \in B_f(p(t), \varepsilon)$, and hence p is continuous. $\square$

Remark 3.4. In [23], the components (path components) of the space $H_f(\mathbb{R}^2)$ were characterized using a different equivalence relation on the space $H_f(\mathbb{R}^2)$.

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