http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 55, 2020

Pages 243–264

http://topology.nipissingu.ca/tp/

The Topological Group of Autohomeomorphisms of Homogeneous Functionally Alexandroff Spaces

by

Sami Lazaar and Houssem Sabri

Electronically published on December 6, 2019

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 55 (2020)
Pages 243-264

E-Published on December 6, 2019

THE TOPOLOGICAL GROUP OF AUTOHOMEOMORPHISMS OF HOMOGENEOUS FUNCTIONALLY ALEXANDROFF SPACES

SAMI LAZAAR AND HOUSSEM SABRI

Abstract. In this paper, we study topological properties of the group of autohomeomorphisms of homogeneous functionally Alexandroff spaces.

In particular, when the functionally Alexandroff space $(X, \mathcal{P}(f))$ is a connected homogeneous space where the map f is κ-to-one for some finite cardinal number $\kappa > 1$, we prove that its autohomeomorphism group $\mathcal{H}(X)$ equipped with the permutation topology is a second countable, totally disconnected, locally compact (s.c.t.d.l.c.), non discrete Hausdorff group. Furthermore, in this case, or even if κ is an infinite cardinal number, we prove that the topology of pointwise convergence, the compact-open topology and the permutation topology which equip the topological group $\mathcal{H}(X)$ are equivalents.

Finally, one of the main results of this paper is to prove that the group $\mathcal{H}(X)$ is not locally compact when κ is an infinite cardinal number. Consequently, we deduce that the group of all isometries of a regular tree is locally compact if and only if the tree is locally finite.

1. Introduction

A topological space X is said to be Alexandroff space if any arbitrary intersection of open sets is open. They have been introduced in 1937 by P. S. Alexandrov in [3] under the name Diskrete Räume "discrete space".

2010 *Mathematics Subject Classification.* 54C99, 54A05, 54H11, 22F30.

Key words and phrases. Functionally Alexandroff space, Homogeneous spaces, Homeomorphism group, Topological groups.

Thus, a topological space is Alexandroff if and only if every point has a least neighborhood. For instance, any finite space is an Alexandroff space. Accordingly, such topologies are often applied in many areas such as computer sciences, social and natural sciences, digital topology, etc. (see, for instance, [16], [14], [15]). For some elementary properties of Alexandroff spaces, we refer to [3, 2, 24].

It is well known that there is a categorical isomorphism between the category of Alexandroff spaces and the category of quasi-ordered sets. Taking a quasi-order $\leq$ on a set X, one can define an Alexandroff topology $\tau_{\leq}$ defined by choosing the open subsets equal to upper subsets of $(X, \leq)$. Hence, the corresponding closed subsets are the lower subsets.

In this paper, we deal with a class of Alexandroff spaces that were recently introduced and are called *functionally Alexandroff spaces*. This construction was independently introduced by Shirazi and Golestani [25] in 2011 under the name *functional Alexandroff spaces* and by Echi [9] in 2012 under the name *primal spaces*. Let X be a set and $f : X \to X$ a function. We can define an Alexandroff topology denoted by $\mathcal{P}(f)$ on X by taking the family $\{A \subset X : f(A) \subset A\}$ (i.e. the f-invariant subsets of X) as the family of closed sets for this topology. Many results and properties have been obtained and studied in such a space (see, for example, [18, 8, 20]). In this paper, the topological space $(X, \mathcal{P}(f))$ will be called a *functionally Alexandroff space*.

By the Alexandroff specialization theorem which characterizes an Alexandroff topology in terms of a quasi-order, the closure of $x \in (X, \mathcal{P}(f))$ is the lower set $\downarrow \{x\} = \{y \in X : y \leq x\}$. The smallest open neighborhood of x, denoted by $\mathcal{V}_f(x)$ is given by the corresponding upper set. We summarize some elementary properties in the following proposition. For a proof, see [25]. Throughout the paper, $\mathbb{N}$ denote the set of naturals with zero.

Proposition 1.1 (Elementary properties)**.** *Let $(X, \mathcal{P}(f))$ be a functionally Alexandroff space and let $(X, \leq_f)$ denote the corresponding specialization qoset.*

(1) *The space $(X, \mathcal{P}(f))$ is an Alexandroff space.*
(2) *For any $x \in X$, the closure of $\{x\}$ is the orbit $\downarrow x = \{f^n(x) : n \in \mathbb{N}\}$. That is,*

$$y \leq_f x \iff \exists n \in \mathbb{N} : y = f^n(x).$$

(3) *The Kuratowski closure operator is given by*

$$\mu_f(A) = \bigcup_{n \in \mathbb{N}} f^n(A)$$

for each subset $A \subseteq X$.

(4) *For any $x \in X$, the smallest open neighbourhood $\mathcal{V}_f(x)$ of x is*

$$\mathcal{V}_f(x) = \uparrow x = \{y \in X : \exists n \in \mathbb{N},\ f^n(y) = x\}.$$

(5) *For any $x, y \in X$ we have either $\mathcal{V}_f(x) \cap \mathcal{V}_f(y) = \emptyset$ or $\mathcal{V}_f(x)$ and $\mathcal{V}_f(y)$ are comparable, i.e. $x \leq_f y$ or $y \leq_f x$.*

(6) *The map $f : (X, \mathcal{P}(f)) \to (X, \mathcal{P}(f))$ is a closed continuous map.*

(7) *The space $(X, \mathcal{P}(f))$ is compact if and only if there exist $x_1, x_2, \ldots,$ $x_n \in X$ such that $X = \bigcup_{i=1}^n \mathcal{V}_f(x_i)$.*

(8) *For each $n \in \mathbb{N}$ we have $\mathcal{P}(f) \subseteq \mathcal{P}(f^n)$.*

(9) *Any connected component of X is open (therefore clopen).*

(10) *The binary relation defined on X by*

$$x \sim y \iff \exists p, q \in \mathbb{N},\ f^p(x) = f^q(y)$$

is an equivalence relation on X and the equivalence class of a point x is the connected component of X containing x.

A point $x \in (X, P(f))$ is called periodic of period $p \in \mathbb{N} - \{0\}$ if the points x, $f(x)$, $f^2(x)$, ..., $f^{p-1}(x)$ are distinct and $f^p(x) = x$ (in such case, we say that f has a p-cycle). A point with period 1 is called a fixed point of f, and hence is a minimal point in the corresponding quasi-ordered set.

A topological space Y is called homogeneous if for any points $a, b \in Y$ there exists a homeomorphism $\varphi : Y \to Y$ such that $\varphi(a) = b$. That is, homogeneous spaces are those spaces Y on which $Aut(Y)$ acts transitively.

Let κ denote a positive cardinal number. The following theorem from [18] gives us a complete characterization of homogeneity in functionally Alexandroff spaces.

Theorem 1.2. *Let $(X, P(f))$ be a functionally Alexandroff space.*

(1) *If f has a p-cycle, $(X, \mathcal{P}(f))$ is homogeneous if and only if all elements are periodic with same period p. The topology $\mathcal{P}(f)$ is therefore a partition topology in which every equivalence class has p elements. Here the equivalence relation $\sim$ is defined by*

$$x \sim y \iff x \in cl\{y\} \text{ and } y \in cl\{x\}.$$

(2) *If f has no cycle, $(X, \mathcal{P}(f))$ is homogeneous if and only if f is a κ-to-one function. Moreover, the group of autohomeomorphisms of $(X, \mathcal{P}(f))$ denoted by $\mathcal{H}(X)$ is:*

$$\mathcal{H}(X) = \{q \in \mathcal{S}_X : q \circ f = f \circ q\}$$

where $\mathcal{S}_X$ denotes the symmetric group of the set X.

We know by [18] that functionally Alexandroff spaces $(X, \mathcal{P}(f))$ are locally connected and the component of any $a \in X$ is given by $C_a = \bigcup_{n \geq 0} \mathcal{V}_f(f^n(a))$.

Let I be an indexing set of the family $\{C_i : i \in I\}$ of components of X and let $f_i = f_{|C_i} : C_i \to C_i$ be the well defined restriction of f to C_i. Then, $X = \coprod_{i\in I} C_i$ and the topology $\mathcal{P}(f)$ is the disjoint union topology, i.e. the space $(X, \mathcal{P}(f))$ is homeomorphic to the disjoint union of its components $(C_i, P(f_i))$. Thus, the space $(X, \mathcal{P}(f))$ is homogeneous if and only if all its components $\{C_i : i \in I\}$ are homeomorphic and homogeneous (see [18]).

In this paper, we study the topological properties of the autohomeomorphism group $\mathcal{H}(X)$ of homogeneous functionally Alexandroff space $(X, \mathcal{P}(f))$ as a topological group. We should mention that the class of spaces under consideration provide enough structure that very nice algebraic results are possible. For more details, the study of the algebraic structure was developed in the paper [19].

2. Definitions and preliminary results

In this section, we begin by summarizing some basic definitions, results and examples about topological groups, the permutation topology and totally disconnected, locally compact groups (t.d.l.c. groups).

The first references to the *permutation topology* go back to the papers [21] and [17]. This topology gives the possibility to apply ideas and results from topological groups theory in the theory of permutation groups.

Definition 2.1. Let G be a group acting on a set Ω. The *permutation topology* is the topology defined by the subbase $\mathcal{B} = \{\mathcal{S}(a,b) : a, b \in \Omega\}$ where

$$\mathcal{S}(a,b) = \{g \in G : g.a = b\}.$$

Definition 2.2. [22] Let Y be a topological space and F be a family of functions from X to Y. Let Y^X denote the usual product space (i.e. the Tychonoff topology). The *pointwise convergence* topology on F is the relative topology on F as a subspace of Y^X. The subbase elements of the pointwise convergence topology are:

$$p_x^{-1}(U) = \{f : f \in F \text{ and } f(x) \in U\}$$

for each $x \in X$ and U an open subset of Y.

The compact-open topology is also defined in very similar way.

Definition 2.3. [13] Let X and Y be two topological spaces and F be a family of functions from X to Y. For each compact subset K of X and an open subset U of Y, define

$$V(K,U) = \{f : f \in F \text{ and } f(K) \subseteq U\}.$$

Then, the family of all such subsets $V(K,U)$ is a subbase for the *compact-open topology* on F.

We also mention the following theorem.

Theorem 2.4 ([13]). *The compact-open topology is finer than that of pointwise convergence. The space $F \subseteq Y^X$ endowed with the compact-open topology is a Hausdorff space if the space Y is Hausdorff and is regular when Y is regular and the members of F are continuous.*

The following remark establishes an equivalence between the previous three topologies.

Remark 2.5. Let G be a group acting on a set Ω ($G \curvearrowright \Omega$).

(1) The group G equipped with the permutation topology is a topological group.
(2) Suppose Ω is endowed with the discrete topology. Each element $g \in G$ can be considered as bijection $\sigma_g : \Omega \to \Omega$. Then, the permutation topology is the same as the topology of pointwise convergence, and it is also equal to the compact-open topology.

We recall that a topological space is called *totally disconnected* if its connected components are the one-point sets.

By the definition of the permutation topology and using basic properties of topological groups (see [23], [11] and [5]), we have the following facts.

Proposition 2.6. *Let $G \curvearrowright \Omega$ be equipped with the permutation topology.*

(1) *The stabilizer of a point is a clopen subgroup of G.*
(2) *A subgroup H of G is open if and only if H contains the pointwise stabilizer of some finite set of points.*
(3) *The permutation topology on G is Hausdorff (or equivalently T_0) if and only if the action of G is faithful.*
(4) *The permutation topology on G is totally disconnected if and only if the action of G is faithful.*
(5) *If the set Ω is countable, then G is a second countable topological group.*

2.1. **Graphical notation.** The graphical representation of the space $(X, \mathcal{P}(f))$ will consist of the set X as the set of vertices and

$$EX = \{(x, f(x)) : x \in X \text{ and } f(x) \neq x\}$$

as the set of edges. The vertex $f(x)$ is represented below the vertex x. If $(X, \mathcal{P}(f))$ is connected and the map f has no cycles, the graph representation of $(X, \mathcal{P}(f))$ is in fact an infinite tree and if in addition f is κ-to-one, the tree is a $(\kappa+1)$-*regular tree* where each vertex v has κ incoming and 1 outgoing edges.

So, let us set the following notation.

Notation 2.7. We denote by $\mathcal{T}_f$ the described tree of a connected homogenous space $(X, \mathcal{P}(f))$ where f is a cycle-free, κ-to-one function. We note that if κ is countable, then the set of vertices $V\mathcal{T}_f$ is countably infinite. See Figure 1 as an illustration when $\kappa = 2$.

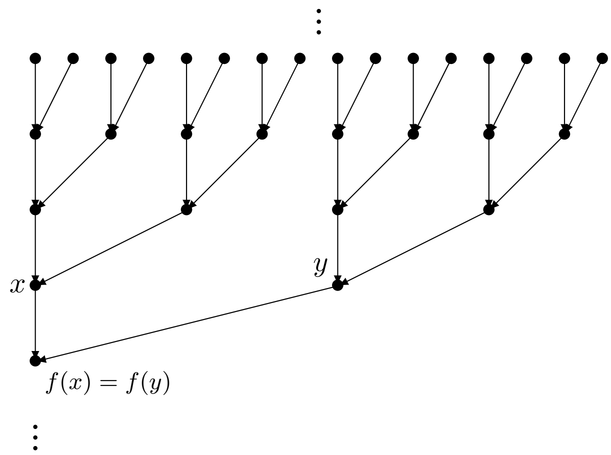

FIGURE 1. The tree $\mathcal{T}_f$ of a connected homogeneous functionally Alexandroff space where f is 2-to-one

Recall that for any tree $\mathcal{T}$, $Aut(\mathcal{T})$ denotes the group of all isometries of $\mathcal{T}$ considered as a metric space (where inversion of vertices is allowed). We denote by $\mathcal{T}^n$ the n-regular tree, also called homogeneous tree (i.e. each vertex has valency n). For more details on locally finite homogeneous trees and on the group $Aut\,(\mathcal{T}^n)$ and its topology see the first chapter of [1]. If f is κ-to-one, it is worth noting that the group $\mathcal{H}(X)$ is a proper subgroup of $Aut(\mathcal{T}^{\kappa+1})$.

The automorphism groups of locally finite connected graphs (especially trees) appear naturally and gives a large and significant class of examples of second countable t.d.l.c. non-discrete topological groups (see [4] for a rich illustration). For more information about t.d.l.c groups we refer to [28] and the proceedings [7].

Example 2.8. Let Γ be a graph. The graph is called *locally finite* if each vertex has a finite number of neighboring vertices (i.e. each vertex degree is finite). The graph is called *connected* if there is a path between any two vertices. A connected graph Γ can be equipped with the distance metric. The distance between two vertices on Γ is the minimum length over all lengths of paths connecting them. An automorphism of a graph is a permutation of the set of vertices which preserve the edge–vertex connectivity. We denote by $Aut(\Gamma)$ the group of automorphisms of Γ. It is clear that if Γ is connected then the automorphism group is the same as the isometry group of Γ viewed as a metric space.

The group $Aut(\Gamma)$ induces a faithful action on Γ (more precisely on the set of vertices of Γ). Hence, $Aut(\Gamma)$ is a topological group under the permutation topology. Since the metric topology on Γ is discrete, then by Remark 2.5 the permutation, pointwise convergence and compact-open topologies are equivalent. Also, they are equal to the compact convergence topology, because it is known that when (Y, d) is a metric space, then the compact-open topology and the compact convergence topology coincide on $\mathcal{C}(X, Y)$ [22]. Also, it is not difficult to see that if the graph Γ is connected and locally finite, then the set of vertices is at most countable. Hence, in this case, the topological space $Aut(\Gamma)$ is finite or second countable.

To summarize, if Γ is a connected locally finite graph, then $Aut(\Gamma)$ is a totally disconnected, locally compact Polish group. The key idea of the proof is that the stabilizer of any vertex is a profinite group, hence compact. For more details and proofs, we refer to [29] and [26].

3. Topologies on The Group $\mathcal{H}(X)$

In this section, we prove our first results about the topological group $\mathcal{H}(X)$ where $(X, \mathcal{P}(f))$ is a homogeneous space. Many cases are to discuss according to the nature of the map f.

Let $(X, \mathcal{P}(f))$ be a homogeneous functionally Alexandroff space. Since autohomeomorphisms of $(X, \mathcal{P}(f))$ are permutations of X, then by Proposition 2.6, $\mathcal{H}(X)$ endowed with the permutation topology is a totally disconnected Hausdorff topological group.

The first natural question one might ask is when the autohomeomorphism group $\mathcal{H}(X)$ is the trivial discrete topological group. Before giving the answer, we need the following result which can be deduced from [18, Theorem 2.7]

Proposition 3.1. *Let $(X, \mathcal{P}(f))$ be a homogeneous space where f is a κ-to-one map and $\kappa > 1$. Let $Z \in X$ and a, b be two distinct points of $f^{-1}[Z]$ (i.e. $f(a) = f(b) = Z$). Then, there exists a Z-branch transposition $q_{a,b} \in \mathcal{H}(X)$ (not unique) verifying the following conditions:*

$$\begin{cases} q_{a,b}(a) = b \\ q_{a,b}(b) = a \\ q_{a,b}(x) = x : x \in X - (\mathcal{V}_f(a) \cup \mathcal{V}_f(b)) \end{cases}$$

See Figure 2.

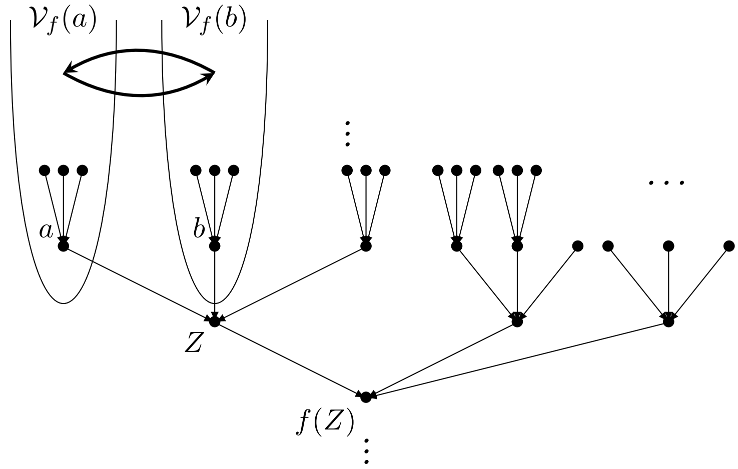

FIGURE 2. Z-branch transposition

Proof. Without loss of generality, we suppose that $(X, \mathcal{P}(f))$ is connected. Let $(Y_{\Bbbk}, f_{\Bbbk})$ be the connected homogeneous functionally Alexandroff space defined in [18, Example 2.6] where every point is covered by κ others. Let $Z \in X$ and let $(\overline{\eta}, 0) \in Y_{\Bbbk}$. Then, by [18, Theorem 2.7], there exists an order-isomorphism map

$$h : \mathcal{V}_f(Z) \mapsto \mathcal{V}_{f_{\Bbbk}}((\overline{\eta}, 0)).$$

For each $x \in f^{-1}[Z]$, we denote by $x^* = h(x)$.

Similarly, there exists an order-isomorphism map $\mathcal{V}_{f_{\Bbbk}}(a^*) \mapsto \mathcal{V}_f(b)$ and an order-isomorphism map $\mathcal{V}_{f_{\Bbbk}}(b^*) \mapsto \mathcal{V}_f(a)$ and for each $x^* \in f_{\Bbbk}^{-1}[(\overline{\eta}, 0)] - \{a^*, b^*\}$, $h^{-1}_{|\mathcal{V}_{f_{\Bbbk}}(h(x))}$ is an order-isomorphism map from $\mathcal{V}_{f_{\Bbbk}}(x^*)$ to $\mathcal{V}_f(x)$. Since distinct points in $f_{\Bbbk}^{-1}[(\overline{\eta}, 0)]$ have disjoint minimal neighborhoods (by Proposition 1.1), then their union gives a well defined order-isomorphism map

$$s : \mathcal{V}_{f_{\Bbbk}}((\overline{\eta}, 0)) \mapsto \mathcal{V}_f(Z)$$

such that:

$$\begin{cases} s(a^*) = b \\ s(b^*) = a \\ s(h(y)) = y \ ; \ \forall y \in \mathcal{V}_f(Z) - (\mathcal{V}_f(a) \cup \mathcal{V}_f(b)) \\ s((\overline{\eta}, 0)) = Z \end{cases}$$

The composition of the two order-isomorphisms $s \circ h$ is an order-isomorphism map $\mathcal{V}_f(Z) \mapsto \mathcal{V}_f(Z)$ such that: $s \circ h(a) = b$, $s \circ h(b) = a$ and fixes all points in $\mathcal{V}_f(Z) - (\mathcal{V}_f(a) \cup \mathcal{V}_f(b))$. That is, $s \circ h$ transposes the a and b branches and fixes all other branches. Finally, the order-isomorphism $s \circ h$ can be extended to an autohomeomorphism $q_{a,b} \in \mathcal{H}(X)$ by:

$$\begin{cases} q_{a,b_{|\mathcal{V}_f(Z)}} = s \circ h \\ q_{a,b}(x) = x : x \in X - V_f(Z) \end{cases}$$

which completes the proof. □

Hence, we get the following proposition.

Proposition 3.2. *Let $(X, \mathcal{P}(f))$ be a homogeneous space. The following properties hold.*

(1) *If f has a periodic point or f is a cycle-free 1-to-one map, then: $\mathcal{H}(X)$ is discrete $\iff$ $(X, \mathcal{P}(f))$ has finitely many connected components.*

(2) *If f is a κ-to-one map where $\kappa \geq 2$, then $\mathcal{H}(X)$ is not discrete.*

Proof. Let $(X, \mathcal{P}(f))$ be a homogeneous space.

(1) In the first case, by Theorem 1.2, all points are periodic with same period p. Let $\{C(x_i) : i \in I\}$ denote the family of components of X. Since $C(x_i) = \{x_i, f(x_i), ..., f^{p-1}(x_i)\}$, if I is finite we have:

$$\bigcap_{i \in I} \bigcap_{k=0}^{p-1} \mathcal{S}(f^k(x_i), f^k(x_i)) = \{id\}$$

and consequently, $\{id\}$ is open. By homogeneity of topological groups, all singletons are open and therefore $\mathcal{H}(X)$ is a finite discrete topological group.

If I is infinite and $\{id\}$ is an open subset, then there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of X such that $\bigcap_{j=1}^{n} \mathcal{S}(x_j, x_j) = \{id\}$. Now, take two distinct points y and z in $X - \bigcup_{i=1}^{n} C(x_i)$. The autohomeomorphism g which maps y to z and fixes all points x_i is clearly not the identity and $g \in \bigcap_{j=1}^{n} \mathcal{S}(x_j, x_j)$, which completes the first case.

For a connected homogeneous $(X, \mathcal{P}(f))$ where f is a cycle-free 1-to-one map, $\mathcal{H}(X)$ is uniquely homogeneous and an autohomeomorphism which fixes some point is the identity (see [18] for a proof). Hence, the result follows.

(2) Let $\kappa \geq 2$ and suppose that $\mathcal{H}(X)$ is discrete. Then $\{id\}$ is open and therefore there exist $x_1, ..., x_n \in X$ such that $\bigcap_{i=1}^{n} \mathcal{S}(x_i, x_i) = \{id\}$. Since the map f has no cycle, then the quasi-order on $(X, \mathcal{P}(f))$ is a partial order. In fact, if $x \leq y$ and $y \leq x$, then $x = f^n(y)$ and $y = f^m(x)$ for some $n, m \in \mathbb{N}$ and consequently $x = f^{n+m}(x)$. If $n + m > 0$, then x is a periodic point which

is absurd. Thus, $n = m = 0$ and $x = y$ which shows that the quasi-order is a partial order. Now, since n is finite, applying Proposition 1.1, there exists an $x \in \{x_i : i = 1...n\}$ such that $\mathcal{V}_f(x) \cap \{x_i : i = 1...n\} = \{x\}$. Since $\kappa \geq 2$, then there exist two distinct points $a, b \in X$ such that $f(a) = f(b) = x$. By Proposition 3.1, there exists an order isomorphism $q \in \mathcal{H}(X)$ which interchanges $\mathcal{V}_f(a)$ and $\mathcal{V}_f(b)$ and the restriction of q to $X - (\mathcal{V}_f(a) \cup \mathcal{V}_f(b))$ is the identity map. Thus $q \in \bigcap_{i=1}^{n} \mathcal{S}(x_i, x_i)$ and $q \neq id$ (because $q(a) = b$).

□

Thus, we can see that the most interesting case to study is the case where the map f is a κ-to-one where $\kappa \geq 2$. In this case, by Proposition 2.6, if X is countable, then the topological group $\mathcal{H}(X)$ is second countable. The following lemma shows that X is countable if and only if κ is countable and $(X, \mathcal{P}(f))$ has at most countably infinitely many connected components.

Lemma 3.3. *Let f be a κ-to-one map and C a connected component of the homogeneous $(X, \mathcal{P}(f))$. Then the set C is countable if and only if κ is countable.*

Proof. The statement is trivial when $\kappa > \aleph_0$.

Let $\kappa \leq \aleph_0$ and $x \in C$. We have $\mathcal{V}_f(x) = \bigcup_{n \geq 0} f^{-n}[x]$. Since

$$\left|f^{-1}[x]\right| = \kappa \text{ and } f^{-n}[x] = \bigcup_{y \in f^{-1}[x]} f^{-n+1}[y]$$

we deduce that $\mathcal{V}_f(x)$ is countable.

Finally, since C is connected, then $C = \bigcup_{n \geq 0} \mathcal{V}_f(f^n(x))$ is countable. □

Hence, we deduce that if κ is countable and the homogeneous space $(X, \mathcal{P}(f))$ has at most countably infinitely many connected components, then the topological group $\mathcal{H}(X)$ is second countable. The converse is true. First, it is not difficult to see that if $\mathcal{H}(X)$ is second countable, then the homogeneous space $(X, \mathcal{P}(f))$ has at most countably infinitely many connected components. By a simple cardinality argument, we show that κ must be countable.

Proposition 3.4. *If κ is an uncountable cardinal number, then $\mathcal{H}(X)$ is not second countable.*

Proof. Let β be a countable base for $\mathcal{H}(X)$. Given $x \in X$, we show that there exists $y \in X$ such that for any $B \in \beta$; $B \nsubseteq S(x, y)$.

Since the family β is countable, then it can be denoted by:

$$\beta = \{B_0, B_1, \cdots, B_n, \cdots\}$$

For each $i \in \mathbb{N}$, let $g_i \in B_i$ and we choose the set $A_i = X - \{g_i(x)\}$. Hence, $\bigcap_{i\in\mathbb{N}} A_i = X - \bigcup_{i\in\mathbb{N}} \{g_i(x)\}$ is non empty. Thus, for a $y \in \bigcap_{i\in\mathbb{N}} A_i$ we have:

$$\forall B \in \beta : B \nsubseteq S(x, y).$$

□

3.1. **Comparison with the other topologies.** Despite the similarities with the topological group of isometries of a connected graph, the main big difference with $\mathcal{H}(X)$ is that the topology $\mathcal{P}(f)$ which equips the set X is not discrete like the metric topology which equips a connected graph. Hence, we can not use the Remark 2.5 to easily deduce that the permutation topology defined on $\mathcal{H}(X)$ is equivalent to the pointwise convergence and the compact-open topologies. However, we have the following results.

Denote by $\mathcal{J}$ the permutation topology, $\mathcal{J}_p$ the pointwise convergence topology and $\mathcal{J}_c$ the compact-open topology on $\mathcal{H}(X)$.

Proposition 3.5. *Let $(X, \mathcal{P}(f))$ be a homogeneous space where the map f has a cycle of period p. Then:*

(1) *If all points are fixed (i.e. $p = 1$), then $\mathcal{J} = \mathcal{J}_p = \mathcal{J}_c$.*
(2) *If $p \geq 2$, then $\mathcal{J}_p = \mathcal{J}_c \subsetneq \mathcal{J}$.*

Proof.

(1) If $p = 1$, then $\mathcal{P}(f)$ is the discrete topology on X. By Remark 2.5, we get $\mathcal{J} = \mathcal{J}_p = \mathcal{J}_c$.
(2) Let $p \geq 2$. First, we have $\mathcal{J}_p \subseteq \mathcal{J}$. In fact, we only need to study subbasis elements in $\mathcal{J}_p$ of the form $p_x^{-1}(\mathcal{V}_f(a))$. Let $a \in X$, so:

$$\forall x \in X, p_x^{-1}(\mathcal{V}_f(a)) = \bigcup_{y \in \mathcal{V}_f(a)} \mathcal{S}(x, y).$$

Thus, $\mathcal{J}_p \subseteq \mathcal{J}$.

The smallest neighborhood of each point is finite of cardinal p. Thus, if K is compact in $(X, \mathcal{P}(f))$, then K must be finite. Hence for any open subset U we have $V(K, U) = \bigcap_{x\in K} p_x^{-1}(U)$. Thus $\mathcal{J}_p = \mathcal{J}_c$.

Now, we prove that $\mathcal{J} \neq \mathcal{J}_p$. Suppose that $\mathcal{J} \subseteq \mathcal{J}_p$. Let a, b be two distinct points in the same connected component. Then, for a $g \in \mathcal{S}(a, b)$ there exists a family $\{x_i, y_i \in X : i = 1...n\}$ such that

$$g \in \bigcap_{i=1}^{n} p_{x_i}^{-1}(\mathcal{V}_f(y_i)) \subseteq \mathcal{S}(a,b).$$

If x_i is in the same component of a ($x_i \sim a$), then we have $g(x_i) \in \mathcal{V}_f(g(a)) = \mathcal{V}_f(b) = \mathcal{V}_f(a)$ and on the other hand we have $g(x_i) \in \mathcal{V}_f(y_i)$. So, $\mathcal{V}_f(a) \cap \mathcal{V}_f(y_i) \neq \emptyset$ and then $\mathcal{V}_f(a) = \mathcal{V}_f(y_i) = \mathcal{V}_f(x_i)$. In the same way, if x_i is not the connected component of a, then $\mathcal{V}_f(a) \cap \mathcal{V}_f(y_i) = \emptyset$. So, the map h defined by:

$$h(x) = \begin{cases} x \text{ if } x \in \mathcal{V}_f(a) \\ g(x) \text{ if } x \notin \mathcal{V}_f(a) \end{cases}$$

is a well defined autohomeomorphism of X such that if $x_i \sim a$, then $h(x_i) = x_i \in \mathcal{V}_f(a) = \mathcal{V}_f(y_i)$. If $x_i \nsim a$, then $h(x_i) = g(x_i) \in \mathcal{V}_f(y_i)$. Thus, $h \in \bigcap_{i=1}^{n} p_{x_i}^{-1}(\mathcal{V}_f(y_i))$ but $h(a) = a \neq b$ so $h \notin \mathcal{S}(a,b)$ which completes the proof.

□

Finally, in the remaining case where the homogeneous space $(X, \mathcal{P}(f))$ has no periodic point, we prove the following result.

Theorem 3.6. *Let $(X, \mathcal{P}(f))$ be homogeneous where the map f is a cycle-free κ-to-one function for some positive cardinal number $\kappa \geq 1$. Then:*

(1) *If $\kappa = 1$, then $\mathcal{J}_p = \mathcal{J}_c \subsetneq \mathcal{J}$.*
(2) *If $\kappa > 1$, then $\mathcal{J}_p = \mathcal{J}_c = \mathcal{J}$.*

Proof. Let us begin by showing that $\mathcal{J}_p \subseteq \mathcal{J}_c \subseteq \mathcal{J}$. Since the compact-open topology is always finer than the pointwise convergence topology, it just remains to show that $\mathcal{J}_c \subseteq \mathcal{J}$. Let $V(K,U)$ be a neighborhood of g where K is a compact set in X and U is an open set in X. Since K is compact, we may establish a finite cover of the form:

$$K \subseteq \bigcup_{x \in K} \mathcal{V}_f(x) \subseteq \bigcup_{i=1}^{n} \mathcal{V}_f(x_i).$$

Let $H = \bigcap_{i=1}^{n} \mathcal{S}(x_i, g(x_i))$ be an open neighborhood of g in $\mathcal{J}$. Let h be an autohomeomorphism in H and $y \in K$. Then, for some $i \in \{1, ..., n\}$ we have:

$$h(y) \in h(\mathcal{V}_f(x_i)) = \mathcal{V}_f(h(x_i)) = \mathcal{V}_f(g(x_i)).$$

But, we have $g(x_i) \in U$ and $\mathcal{V}_f(g(x_i))$ is the smallest neighborhood of $g(x_i)$, so $\mathcal{V}_f(g(x_i)) \subseteq U$. Thus, $h(y) \in U$ (for any $y \in K$) and then, $h(K) \subseteq U$. Hence, $V(K,U)$ is a neighborhood of all its points relative to $\mathcal{J}$. Thus, $\mathcal{J}_c \subseteq \mathcal{J}$ which completes the proof of the statement.

(1) Let $\kappa = 1$.

In this case the specialization order restricted to each component of $(X, \mathcal{P}(f))$ is a countably infinite chain and is order isomorphic to $\mathbb{Z}$. So, the homogeneous space $(X, \mathcal{P}(f))$ can be illustrated by Figure 3.

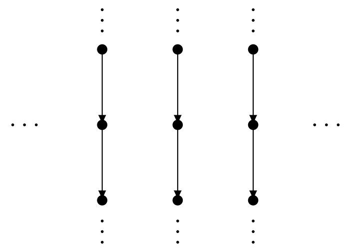

FIGURE 3. The homogeneous space $(X, \mathcal{P}(f))$ where f is 1-to-one.

We first show that $\mathcal{J}_p = \mathcal{J}_c$. Let $V(K, U)$ be a neighborhood of g where K is a compact set in X and U is an open set in X. Because each $\mathcal{V}_f(a)$ is a chain order isomorphic to $\mathbb{N}$, then we may establish a finite cover of the form:

$$K \subseteq \bigcup_{i=1}^{n} \mathcal{V}_f(x_i),$$

where each $x_i \in K$.

Let $H = \bigcap_{i=1}^{n} p_{x_i}^{-1} \left(\mathcal{V}_f \left(g(x_i)\right)\right)$ be an open neighborhood of g in $\mathcal{J}_p$. Let h be an autohomeomorphism in H and $y \in K$. Then, for some $i_0 \in \{1, ..., n\}$ we have: $y \in \mathcal{V}_f(x_{i_0}) \iff y \geq x_{i_0}$. But $\forall i$, $h(x_i) \in \mathcal{V}_f \left(g(x_i)\right) \iff h(x_i) \geq g(x_i)$, then $h(y) \geq g(x_{i_0}) \iff h(y) \in \mathcal{V}_f \left(g(x_{i_0})\right)$.

Since $g(x_{i_0}) \in U$ and $\mathcal{V}_f \left(g(x_{i_0})\right)$ is the smallest neighborhood of $g(x_{i_0})$, then $h(y) \in U$ (for any $y \in K$). Thus, $V(K, U)$ is a neighborhood of all its points relative to $\mathcal{J}_p$ and we conclude that $\mathcal{J}_p = \mathcal{J}_c$.

Now, it just remains to show that $\mathcal{J}_p \neq \mathcal{J}$. Suppose that $\mathcal{J} \subseteq \mathcal{J}_p$ and take $a \in X$. Then there exist $x_1, x_2, \ldots, x_n \in X$ and open subsets $U_1, U_2, \ldots, U_n$ of X such that:

$$\mathcal{S}(a, a) \supseteq \cap \bigcap_{i=1}^{n} p_{x_i}^{-1} \left(U_i\right).$$

Fix $h \in \bigcap_{i=1}^{n} p_{x_i}^{-1}(U_i)$. Since $h(a) = a$, then h fixes all elements in the component of x, that is $h_{|C(a)} = id$.

- Suppose that for each $1 \leq i \leq n$, $x_i \notin C(a)$. We define $g \in \mathcal{H}(X)$ a translation of the component $C(a)$ by:

$$\begin{cases} g(x) = f(x) \ \forall x \in C(a), \\ g(x) = x \ \forall x \notin C(a). \end{cases}$$

  Then, $g \circ h(x_i) = h(x_i)$ so the autohomeomorphism $g \circ h \in \bigcap_{i=1}^{n} p_{x_i}^{-1}(U_i)$ but $g \circ h(a) = f(a) \neq a$ which contradicts the fact that $\bigcap_{i=1}^{n} p_{x_i}^{-1}(U_i) \subseteq \mathcal{S}(a, a)$.

- If for some $1 \leq i \leq n$, $x_i \in C(a)$. So, $\forall i = 1 \ldots n$, if $x_i \in C(a)$, then $U_i \supseteq \mathcal{V}_f(x_i)$. Thus:

$$\mathcal{S}(a,a) \supseteq \bigcap_{i=1}^{n} \left\{p_{x_i}^{-1}(U_i) : x_i \notin C(a)\right\} \bigcap_{i=1}^{n} \left\{p_{x_i}^{-1}(\mathcal{V}_f(x_i)) : x_i \in C(a)\right\}.$$

  Now, for each $x_i \in C(a)$, there exists $q_i \in \mathbb{Z}$ such that $x_i = f^{q_i}(a)$. Let $q = \min\{q_i \ ; \ i\}$ and let $b = f^q(a)$. Then, $\forall i \in \{1..n : x_i \in C(a)\}$

$$p_b^{-1}(\mathcal{V}_f(b)) \subseteq p_{x_i}^{-1}(\mathcal{V}_f(x_i)),$$

$$\text{then} \quad p_b^{-1}(\mathcal{V}_f(b)) \subseteq \bigcap_{i=1}^{n} \left\{p_{x_i}^{-1}(\mathcal{V}_f(x_i)) : x_i \in C(a)\right\}.$$

  Hence, as in the previous case, we can define $g \in \mathcal{H}(X)$ a translation of $C(a)$ by:

$$\begin{cases} g(a) = f^{-1}(a), \\ g(z) = z \ \forall z \notin C(a). \end{cases}$$

  Then, the autohomeomorphism

$$g \circ h \in \bigcap_{i=1}^{n} \left\{p_{x_i}^{-1}(U_i) : x_i \notin C(a)\right\} \cap p_b^{-1}(\mathcal{V}_f(b)) \subseteq \mathcal{S}(a,a),$$

  because $g \circ h(b) = g(b) = g(f^q(a)) = f^q(g(a)) = f^{q-1}(a) \in \mathcal{V}_f(b)$ and $g \circ h(x_i) = h(x_i)$. But, $g \circ h \notin \mathcal{S}(a, a)$ which is a contradiction.

(2) Let $\kappa \geq 2$.

We show that $\mathcal{J} \subseteq \mathcal{J}_p$. Let $g \in \mathcal{S}(x, y)$. Since $\kappa > 1$, there exists some $z \in f^{-1}[f(x)] - \{x\}$. Hence:

$$g \in p_x^{-1}(\mathcal{V}_f(y)) \cap p_z^{-1}(\mathcal{V}_f(g(z))) \subseteq \mathcal{S}(x, y).$$

In fact, let $h \in p_x^{-1}(\mathcal{V}_f(y)) \cap p_z^{-1}(\mathcal{V}_f(g(z)))$, that is:

$$\begin{cases} h(x) \in \mathcal{V}_f(y) \\ h(z) \in \mathcal{V}_f(g(z)) \end{cases} . \tag{3.1}$$

Moreover, by Theorem 1.2 we have:

$$f(x) = f(z) \implies f \circ h(x) = f \circ h(z). \tag{3.2}$$

Since, $\mathcal{V}_f(y) \cap \mathcal{V}_f(g(z)) = \emptyset$ and by (3.1) and (3.2), we deduce that the autohomeomorphism h must map x to y, that is $h \in \mathcal{S}(x, y)$. Or, $p_x^{-1}(\mathcal{V}_f(y)) \cap p_z^{-1}(\mathcal{V}_f(g(z)))$ is open in $\mathcal{J}_p$, so $\mathcal{S}(x, y)$ is a neighborhood of all its points relative to $\mathcal{J}_p$. Thus, $\mathcal{J} \subseteq \mathcal{J}_p$ and finally $\mathcal{J} \subseteq \mathcal{J}_p \subseteq \mathcal{J}_c \subseteq \mathcal{J}$. So, for $\kappa \geq 2$ the topology $\mathcal{J}$ placed on $\mathcal{H}(X)$ is exactly the topology of pointwise convergence and the compact-open topology.

□

Remark 3.7. The topological group $\mathcal{H}(X)$ is an example which shows that the opposite direction of the Theorem 2.4 is not true. In fact, let $(X, \mathcal{P}(f))$ be homogeneous where f is a cycle-free 2-to-one function. We take the space F of Theorem 2.4 to be $F = \mathcal{H}(X) \subseteq X^X$ (hence the maps of F are by default continuous) endowed with the compact-open topology which is equal to the permutation topology and is a Tychonoff space, but X (equipped with the $\mathcal{P}(f)$ topology) is neither Hausdorff nor regular.

4. Local Compactness

The local compactness is a very important concept to study in topological groups. One of the major results [10] is the existence of a Haar measure on any locally compact Hausdorff topological group which allows defining an integral for functions defined on such groups. Moreover, what is more interesting here is the fact that our topological groups are totally disconnected. In fact, for connected locally compact groups many general and deep results have been discovered. For instance, the connected case is completely described as the inverse limit of Lie groups, thanks to the solution to Hilbert's fifth problem. On the other hand, the totally disconnected case resists to a general theory. As mentioned in [28]:

> Most recent theory of t.d.l.c. groups views the t.d.l.c. groups as simultaneously geometric groups and topological groups and investigates the connection between the geometric structure and topological structure.

For more information we also refer to [6] and [27].

As a first observation, a discrete space is locally compact, and thus we deduce the following result.

Proposition 4.1. *Let $(X, \mathcal{P}(f))$ be homogeneous with finitely many connected components. If f has a cycle of period p or f is a cycle-free, 1-to-one map, then $\mathcal{H}(X)$ is locally compact.*

Proof. By Proposition 3.2, $\mathcal{H}(X)$ is discrete and the result follows. $\square$

Proposition 4.2. *Let $(X, \mathcal{P}(f))$ be homogeneous with infinitely many connected components. Then $\mathcal{H}(X)$ is not locally compact.*

Proof. If $\mathcal{H}(X)$ is locally compact, then there exists an open set $U = \bigcap_{i=1}^{n} \mathcal{S}(x_i, x_i)$ (where $x_i \in X$) and a compact K such that $id \in U \subseteq K$.

Let $Y = X - \bigcup_{i=1}^{n} C(x_i)$ where $C(x_i)$ is the component which contains x_i. For each $g \in K$, $\exists y_g \in Y$ such that $g(y_g) \in Y$. Since K is compact, we may realise a finite cover of the form:

$$K \subseteq \bigcup_{j=1}^{p} \mathcal{S}(y_j, g(y_j)) \subseteq \bigcup_{g \in K} \mathcal{S}(y_g, g(y_g)).$$

Hence, $\bigcap_{i=1}^{n} \mathcal{S}(x_i, x_i) \subseteq \bigcup_{j=1}^{p} \mathcal{S}(y_j, g(y_j))$ which is absurd. Indeed, because of the homogeneity of $(X, \mathcal{P}(f))$ and since Y is infinite, $\exists h \in \mathcal{H}(X)$ such that:

$$\begin{cases} h_{|C(x_i)} = id_{|C(x_i)}, & \forall i = 1 \dots n \\ h(y_j) \neq g(y_j), & \forall j = 1 \dots p \end{cases}$$

which completes the proof. $\square$

Now, we conclude the topological study with the important case where $(X, \mathcal{P}(f))$ is a connected homogeneous functionally Alexandroff space where the map f is a κ-to-one function and $\kappa \geq 2$.

As said in Section 2.1, we can think of a connected homogeneous $(X, \mathcal{P}(f))$, where f is a κ-to-one, cycle-free function as a $(\kappa+1)$-regular tree $\mathcal{T}_f$. The image of an edge $(x, f(x))$ of $\mathcal{T}_f$ by an autohomeomorphism $g \in \mathcal{H}(X)$ is the edge $(g(x), f(g(x)))$. Hence, the permutation topology defined on $\mathcal{H}(X)$ is indeed the relative topology as a subgroup of $(Aut(\mathcal{T}_f), \mathcal{J})$, the automorphisms of the homogeneous tree $\mathcal{T}_f$, endowed with the permutation topology $\mathcal{J}$.

By Example 2.8, if κ is finite, then the topological group $Aut(\mathcal{T}_f)$ is locally compact since $\mathcal{T}_f$ is a connected, locally finite tree. Local compactness is *closed hereditary*, i.e. any closed subspace of a locally compact space is locally compact. We show that $\mathcal{H}(X)$ is a closed subspace of $Aut(\mathcal{T}_f)$ and so we get the following:

Proposition 4.3. *Let f be a cycle-free, κ-to-one function and $(X, \mathcal{P}(f))$ be connected. If κ is finite, then the topological group $\mathcal{H}(X)$ is locally compact.*

Proof. Let $\{g_n; n \in D, \geq\}$ be a net of elements of $\mathcal{H}(X)$ which converges to g. If $\mathcal{S}(x, g(x))$ is a neighborhood of g, then $\exists m \in D$ such that:

$$\text{if } n \in D, n \geq m \Rightarrow g_n \in \mathcal{S}(x, g(x)).$$

Similarly, $\mathcal{S}(f(x), g(f(x)))$ is a neighborhood of g and $\exists p \in D$ such that:

$$\text{if } n \in D, n \geq p \Rightarrow g_n \in \mathcal{S}(f(x), g(f(x))).$$

But, there exists $q \in D$ such that $q \geq p$ and $q \geq m$, so $g_q \in \mathcal{H}(X)$ and $g_q \in \mathcal{S}(x, g(x)) \bigcap \mathcal{S}(f(x), g(f(x)))$. Since $g_q \circ f(x) = f \circ g_q(x)$, we get $g \circ f(x) = f \circ g(x)$. This is true for any $x \in X$ so g commutes with f, which is equivalent to saying that $g \in \mathcal{H}(X)$. □

Now, the straightforward question is what happens if the tree is not locally finite. There is no direct answer; it depends on the structure of the tree (or the graph). For instance, let $\mathcal{T}$ be a rooted tree with a root vertex o adjacent to vertices x_n for each $n \geq 1$. Suppose that each vertex x_n has degree n and suppose that for each rooted subtree with root x_n distinct vertices within each level have a distinct degree, see Figure 4. The tree $\mathcal{T}$ is not locally finite, however $Aut(\mathcal{T}) = \{id\}$ is compact.

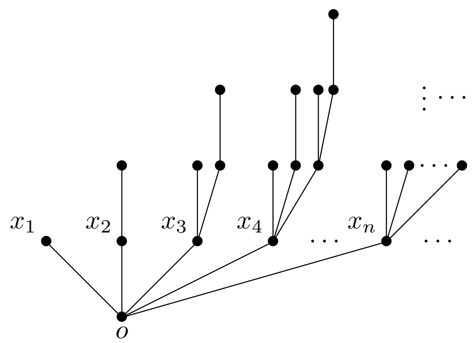

FIGURE 4. A non locally finite tree with trivial automorphism group.

We prove that if κ is an infinite cardinal number, then $\mathcal{H}(X)$ is not locally compact and consequently the automorphism group of a non locally finite regular tree is not locally compact.

First, we begin by defining the level partition of the tree $\mathcal{T}_f$.

Definition 4.4. Let $(X, \mathcal{P}(f))$ be a connected homogeneous functionally Alexandroff space, where f is a cycle-free, κ-to-one function. Let y and z be two points in X. We say that y and z are in the same level in the tree $\mathcal{T}_f$ if $\exists n \in \mathbb{N}$ such that $f^n(y) = f^n(z)$.

Let us define a basic bi-infinite line:

$$\mathcal{L} = \{..., v_{-2}, v_{-1}, v_0, v_1, v_2, ...\}$$

such that $v_{-n} = f^n(v_0)$ and $v_n \in f^{-1}(\{v_{n-1}\})$ $(\forall n \geq 1)$ where v_0 is an arbitrary point of X. Since $(X, \mathcal{P}(f))$ is connected, then $X = \coprod_{i\in\mathbb{Z}} L_i$ where

$$L_i = \{y \in X : y \text{ and } v_i \text{ are in the same level}\}.$$

Then, the family $\{L_i : i \in \mathbb{Z}\}$ is called a *level partition* of the tree $\mathcal{T}_f$ according to the bi-infinite line $\mathcal{L}$. Figure 5 gives a detailed illustration of the level partition.

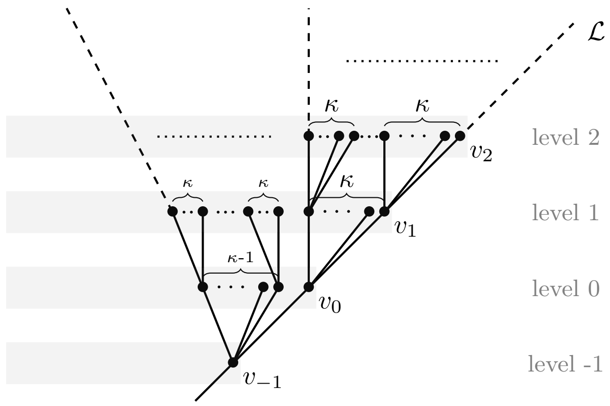

FIGURE 5. The level partition of the tree $\mathcal{T}_f$.

We denote by $L_{\mathfrak{w}(x)}$ the level containing x. More precisely, $\forall x \in X$, $\exists (p,q) \in \mathbb{N}^2$ such that $f^p(v_0) = f^q(x)$ then, the level of x is $\mathfrak{w}(x) = q - p$.

Notation 4.5. We fix the following notations:

- For $x \in (X, \mathcal{P}(f))$, we denote by $\mathbb{S}t(x)$ the stabilizer of the point x in $\mathcal{H}(X)$. Hence, $\mathbb{S}t(x) = \mathcal{S}(x,x)$ is a clopen subgroup of $\mathcal{H}(X)$.
- The metric distance defined on the tree $\mathcal{T}_f$ will be denoted by:

$$d_{\mathcal{T}_f}(u,v)$$

  for any two vertices u and v. Recall that, $d_{\mathcal{T}_f}(u,v)$ is equal to the length of the unique path connecting u to v.

Finally, we prove that the topology $\mathcal{J}$ is not locally compact when κ is infinite.

Theorem 4.6. *Let $(X, \mathcal{P}(f))$ be a connected homogeneous functionally Alexandroff space, where f is a cycle-free, κ-to-one function. If κ is infinite, then $\mathcal{H}(X)$ is not locally compact.*

Proof. We show that identity has no compact neighborhood. Since the topology $\mathfrak{J}$ on $\mathcal{H}(X)$ is described in terms of a sub-basis, one could use the Alexander subbase theorem [12] which states that a topological space is compact if and only if every sub-basic open cover has a finite subcover. Let A be a compact neighborhood of the identity and let $U = \bigcap_{i=1}^{n} \mathbb{S}t(u_i)$ be an open neighborhood of the identity such that $U \subseteq A$. Let $\{\mathcal{S}(x_i, y_i) : i \in I\}$ be a subfamily of the subbase covering A. Since $id \in A$ then, there must exist some $i \in I$ such that $x_i = y_i$. Consequently, we denote by the following a finite subcover of A

$$U \subseteq A \subseteq \bigcup_{j \in J_1} \{\mathcal{S}(x_j, y_j)\} \cup \bigcup_{i \in I_1} \{\mathbb{S}t(z_{1,i})\},$$

where J_1 and I_1 are finite subset of I such that $x_j \neq y_j$ $(\forall j \in J_1)$. We can also suppose that for each distinct $a, b \in I_1$, $\mathcal{V}_f(z_{1,a}) \cap \mathcal{V}_f(z_{1,b}) = \emptyset$ or otherwise we obtain $\mathbb{S}t(z_{1,a}) \subset \mathbb{S}t(z_{1,b})$ (or conversely).

Each autohomeomorphism $g \in \mathbb{S}t(z_{1,i})$ is an order-isomorphism. Fix the point $z_{1,i}$. Then:

$$x \in f^{-1}[\{z_{1,i}\}] \Rightarrow g(x) \in f^{-1}[\{z_{1,i}\}].$$

Conversely, for any point v, if for some $x \in f^{-1}[\{v\}]$, $g(x) \in f^{-1}[\{v\}]$, then $g \in \mathbb{S}t(v)$ because $g(v) = g(f(x)) = f(g(x)) = v$.

Hence, we conclude that:

$$\mathbb{S}t(z_{1,i}) = \bigcup_{x \in f^{-1}[\{z_{1,i}\}]} \left\{\mathcal{S}(x, z_{1,i}^{-})\right\},$$

for some $z_{1,i}^{-} \in f^{-1}[\{z_{1,i}\}]$. Thus, we get:

$$A \subseteq \bigcup_{j \in J_1} \{\mathcal{S}(x_j, y_j)\} \cup \bigcup_{i \in I_1} \left(\bigcup_{x \in f^{-1}[\{z_{1,i}\}]} \left\{\mathcal{S}(x, z_{1,i}^{-})\right\} \right).$$

Again, by compactness of A, for each $i \in I_1$ there exists a finite subset $K_i \subset f^{-1}[\{z_{1,i}\}]$ such that we obtain a new finite subcover of A

$$A \subseteq \bigcup_{j \in J_1} \{\mathcal{S}(x_j, y_j)\} \cup \bigcup_{i \in I_1} \left(\bigcup_{x \in K_i} \left\{\mathcal{S}(x, z_{1,i}^{-})\right\} \right).$$

Again, since $id \in A$ then, there must exist some $x \in K_i$ (for some $i \in I_1$) such that $x = z_{1,i}^{-}$. Hence after arrangement of terms the finite subcover of A may be written

$$A \subseteq \bigcup_{j \in J_2} \{\mathcal{S}(x_j, y_j)\} \cup \bigcup_{i \in I_2} \{\mathbb{S}t(z_{2,i})\},$$

where J_2 and I_2 are finite subsets such that $x_j \neq y_j$ $(\forall j \in J_2)$ and $z_{2,i} = z_{1,i}^{-}$ $(\forall i \in I_2 \subseteq I_1)$. We note that, $\mathfrak{w}(z_{2,i}) = \mathfrak{w}(z_{1,i}) + 1$. Hence:

$$\min \{\mathfrak{w}(z_{2,i}) : i \in I_2\} > \min \{\mathfrak{w}(z_{1,i}) : i \in I_1\}.$$

Recall that the $\mathfrak{w}(\bullet)$ is the level of the point $\bullet$ after fixing from the beginning some vertex v_0 in the tree $\mathcal{T}_f$, see Definition 4.4.

This process can be indefinitely repeated and therefore we get a strictly increasing sequence $\{\min \{\mathfrak{w}(z_{p,i}) : i \in I_p\} : p = 1, 2, ...\}$. So, there exists some integer p such that $\min \{\mathfrak{w}(z_{p,i}) : i \in I_p\} > \max \{\mathfrak{w}(u_i) : i = 1..n\}$. Hence, we have

$$\bigcap_{i=1}^{n} \mathbb{S}t(u_i) \subseteq A \subseteq \bigcup_{j \in J_p} \{\mathcal{S}(x_j, y_j)\} \cup \bigcup_{i \in I_p} \{\mathbb{S}t(z_{p,i})\},$$

where $x_j \neq y_j$ $(\forall j \in J_p)$ and as said before for all distinct points $a, b \in I_p$, $\mathcal{V}_f(z_{p,a}) \cap \mathcal{V}_f(z_{p,b}) = \emptyset$.

Now, since J_p, I_p are finite and κ is infinite then:

$$\forall i \in I_p,\ \exists z_{p,i}^{*} \in f^{-1}\left[\{f(z_{p,i})\}\right] - \{y_j : j \in J_p\} \cup \{z_{p,i}\}.$$

Also, if for some $j \in J_p$, we have $x_j \in \mathcal{V}_f(t_{j,i})$ where $t_{j,i} \in f^{-1}[\{f(z_{p,i})\}]$ then, there exists $t_{j,i}^{*} \in f^{-1}[\{f(z_{p,i})\}]$ such that $y_j \notin \mathcal{V}_f(t_{j,i}^{*})$. By the homogeneity of $(X, \mathcal{P}(f))$, using the similar construction outlined in Proposition 3.1 then, for each $i \in I_p$ there exists $g_i \in \mathcal{H}(X)$ which verifies the following conditions:

$$\begin{cases} g_{i|X - \mathcal{V}_f(f(z_{p,i}))} = id_{|X - \mathcal{V}_f(f(z_{p,i}))} \\ g_i(z_{p,i}) = z_{p,i}^{*} \\ g_i(t_{j,i}) = t_{j,i}^{*} \text{ (if there is some } x_j \in \mathcal{V}_f(f(z_{p,i})) - \{f(z_{p,i})\}) \end{cases}.$$

One may check that, for each $i \in I_p$ we have:

- $g_i \in \bigcap_{i=1}^{n} \mathbb{S}t(u_i)$, because $\mathfrak{w}(u_j) < \min \{\mathfrak{w}(z_{p,i}) : i \in I_p\})$.
- $g_i \notin \mathbb{S}t(z_{p,i})$.
- $g_i(x_j) \neq y_j$ $(\forall j \in J_p)$.

Since for every distinct $a, b \in I_p$, $\mathcal{V}_f(z_{p,a}) \cap \mathcal{V}_f(z_{p,b}) = \emptyset$, then the autohomeomorphism

$$g = \circ\{g_i : i \in I_p\} \in \mathcal{H}(X),$$

is well defined and verifies

$$\begin{cases} g \in \bigcap_{i=1}^{n} \mathbb{S}t(u_i) \\ g \notin \bigcup_{j \in J_p} \{\mathcal{S}(x_j, y_j)\} \cup \bigcup_{i \in I_p} \{\mathbb{S}t(z_{p,i})\} \end{cases}$$

which completes the proof. □

It turns out that the permutation topology $\mathfrak{J}$ of the group of isometries of a n-regular tree is locally compact if and only if n is finite.

Acknowledgement

The authors thank the referee for the excellent report improving both the presentation and the mathematical content of this paper.

References

[1] Figá-Talamanca Alessandro and Nebbia Claudio, *Harmonic Analysis and Representation Theory for Groups Acting on Homogenous Trees.* London Mathematical Society Lecture Note Series. Cambridge University Press, 1991.

[2] Paul Alexandroff and Heinz Hopf, *Topologie.* Die Grundlehren der mathematischen Wissenschaften 45. Springer-Verlag Berlin Heidelberg, reprint der erstausgabe berlin 1935 edition, 1974.

[3] Pavel S. Alexandroff, *Diskrete räume*, Mat. Sb. (N.S.), **44**(2) (1937), 501–519.

[4] Marc Burger and Shahar Mozes, *Groups acting on trees: from local to global structure*, Publications Mathématiques de l'IHÉS, **92** (2000), 113–150.

[5] Nicolas Bourbaki, *Topologie générale. Chapitres 1 à 4.* Springer-Verlag Berlin Heidelberg, 2007.

[6] Pierre-Emmanuel Caprace and Tom De Medts, *Simple locally compact groups acting on trees and their germs of automorphisms*, Transformation Groups, **16** (2) (2011), 375–411.

[7] Pierre-Emmanuel Caprace and Nicolas Monod, editors. *New Directions in Locally Compact Groups.* Cambridge University Press, 2018.

[8] Intissar Dahane, Sami Lazaar, Tom Richmond, and Tarek Turki, *On resolvable primal spaces*, Quaestiones Mathematicae, **0**(0) (2018).

[9] Othman Echi, *The category of flows of set and top*, Topology and its Applications, **159**(9) (2012), 2357–2366.

[10] Alfred Haar, *Der massbegriff in der theorie der kontinuierlichen gruppen*, Annals of Mathematics, **34**(2) (1933), 147–169.

[11] Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis. Volume I, Structure of Topological Groups Integration theory Group Representations.* Springer-Verlag Berlin Heidelberg, 1963.

[12] K.D. Joshi, *Introduction to General Topology.* John Wiley and Sons (Asia) Pte Ltd, 1983.

[13] John L. Kelley, *General Topology.* Graduate Texts in Mathematics. Springer-Verlag New York Inc., 1st ed. 2nd printing edition, 1975.

[14] Efim Khalimsky, Ralph Kopperman, and Paul R. Meyer, *Computer graphics and connected topologies on finite ordered sets*, Topology and its Applications, **36**(1) (1990), 1–17.

[15] T. Y. Kong, R. Koperman, and P. R. Meyer, *A topological approach to digital topology*, The American Mathematical Monthly, **98** (1991), 901–917.

[16] V. Krishnamurthy, *On the number of topologies on a finite set*, The American Mathematical Monthly, **73**(2) (1966), 154–157.

[17] Abraham Karrass and Donald Solitar, *Some remarks on the infinite symmetric groups*, Mathematische Zeitschrift, **66**(1) (1956), 64–69.

[18] Sami Lazzar, Tom Richmond, and Houssem Sabri, *Homogeneous functionally alexandroff spaces*, Bulletin of the Australian Mathematical Society, **97**(2) (2018), 331–339.

[19] ______, *The autohomeomorphism group of connected homogeneous functionally Alexandroff spaces*, Communications in Algebra, **47**(9) (2019), 3818-3829.

[20] Sami Lazaar, Tom Richmond, and Tarek Turki, *Maps generating the same primal space*, Quaestiones Mathematicae, **40**(1) (2017), 17–28.

[21] I. Maurer, *Les groupes de permutations infinies*, Gazeta Matematica şi Fizica. Seria A, **7** (1955), 400–408.

[22] James R. Munkres, *Topology*. Prentice Hall, 2 edition, 2000.

[23] Lev S. Pontryagin, *Topological Groups*. Princeton University Press, 1946. Translated from the Russian By Emma Lehmer.

[24] Tom Richmond, *Quasiorders, principal topologies, and partially ordered partitions*, International Journal of Mathematics and Mathematical Sciences, **22**(2) (1998), 221–234.

[25] Fatemah A. Z. Shirazi and Nasser Golestani, *Functional Alexandroff spaces*, Hacettepe Journal of Mathematics and Statistics, **40**(4) (2011), 515–522.

[26] Vladimir I. Trofimov, *Automorphism groups of graphs as topological groups*, Mathematical notes of the Academy of Sciences of the USSR, **38**(3) (1985), 717–720.

[27] Baumgartner Udo, *Totally disconnected, locally compact groups as geometric objects*, In Goulnara N. Arzhantseva, José Burillo, Laurent Bartholdi, and Enric Ventura, editors, Geometric Group Theory, pages 1–20. Birkhäuser Basel, 2007.

[28] Phillip Wesolek, *Elementary totally disconnected locally compact groups*, Proceedings of the London Mathematical Society, **110**(6) (2015), 1387–1434.

[29] Wolfgang Woess, *Topological groups and infinite graphs*, Discrete Mathematics, **95**(1) (1991), 373–384.

(Lazaar) Taibah University, College of Science, Department of Mathematics, Tayba, Medina 42353, Saudi Arabia
Email address: salazaar72@yahoo.fr

(Sabri) Faculty of Science of Tunis, Department of Mathematics, 2092 Campus Universitaire El Manar, Tunisia
Email address: sabrihoussem@gmail.com