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TOPOLOGICAL CONJUGACY FOR THE MORSE MINIMAL SYSTEM: AN EXAMPLE

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Abstract. We give an example of a subshift that is topologically conjugate to the Morse minimal system but is not generated by a constant-length substitution. This shows that the property of being generated by a constant-length substitution is not a topological conjugacy invariant. Our proof illustrates the usefulness of techniques developed by Coven, Dekking, and Keane [3].

1. Introduction and Preliminaries

Given a finite alphabet $\mathcal{A}$, let $\mathcal{A}^{\mathbb{Z}}$ denote the set of doubly-infinite sequences $x = (x_i)_{i\in\mathbb{Z}}$ on $\mathcal{A}$, and let $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be the left shift: $\sigma(x)_i = x_{i+1}$. If a subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is σ-invariant and closed in the product topology, then the pair (X, σ) is a *subshift.* The subshift is *minimal* if $X = \overline{\{\sigma^n(x) : n \in \mathbb{Z}\}}$ for every $x \in X$. Given another subshift (Y, σ), a continuous function $\psi : X \to Y$ such that $\psi \circ \sigma = \sigma \circ \psi$ is called a *factor map* if it is surjective, and a *topological conjugacy* if it is a homeomorphism. For more background on subshifts, see [10].

Given an integer $L \geq 2$, a *substitution* of constant length L is a mapping $\theta : \mathcal{A} \to \mathcal{A}^L$. Higher powers $\theta^k : \mathcal{A} \to \mathcal{A}^{L^k}$ are defined recursively: if $\theta(a) = a_1 \cdots a_\ell$, then $\theta^2(a) := \theta(a_1) \cdots \theta(a_\ell)$; $\theta^3(a) := \theta^2(a_1) \cdots \theta^2(a_\ell)$; and so on. The substitution is *primitive* if, for all $a, b \in \mathcal{A}$, there exists $k \in \mathbb{N}$ such that $\theta^k(a)$ contains b. It is well-known that a primitive

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substitution generates a unique minimal subshift (X_θ, σ). A sequence $x \in \mathcal{A}^{\mathbb{Z}}$ is an element of X_θ if and only if, for every finite-length word w that appears in x, there exist $a \in \mathcal{A}$ and $k \in \mathbb{N}$ such that w appears in $\theta^k(a)$.

In this paper we focus on the *Morse substitution* given by $\theta(0) = 01$ and $\theta(1) = 10$. For this example we ask: if $\psi : X_\theta \to Y$ is a topological conjugacy, must Y be generated by a constant-length substitution? We show that the answer is 'no' by proving the following.

Theorem 1.1. *Let θ be the Morse substitution, and let Y be the image under the factor map $\psi : X_\theta \to Y$ defined by*

$$\psi(x)_i = x_i + x_{i+1}x_{i+2} \mod 2.$$

Then ψ is a topological conjugacy, and Y is not generated by a constant-length substitution.

One reason this may be somewhat surprising is that topological conjugacy is a very strong equivalence relation; indeed, to say that two subshifts are topologically conjugate is often interpreted to mean that they are dynamically identical. One might expect, then, that the property of being generated by a substitution would be preserved under topological conjugacy.

Moreover it is known that, under a topological conjugacy $\psi : X_\theta \to Y$, points in Y must inherit much of the combinatorial structure of points in X_θ. For example, in [5], Coven, Keane, and LeMasurier provide a precise sense in which patterns of symbols (and substitution blocks) occurring in a given point $x \in X_\theta$ must be "mirrored" by the point $\psi(x) \in Y$. In particular, the well-known "no BBb" property, which characterizes points in X_θ, must (in a certain sense) carry over to points in Y. More general "structure preservation" theorems for arbitrary primitive, constant-length substitutions appear in [4].

Our motivation for working through the example in Theorem 1.1 stems from work of Coven, Dekking, and Keane [3] on the topological conjugacy class of a given substitution minimal system. A related question is whether the topological conjugacy class of the Morse minimal system is "typical," or whether there are substitution subshifts whose topological conjugacy classes consist entirely of subshifts that can themselves be generated by constant-length substitutions.

Other authors have studied similar questions. In [1], Berstel gives a different example of a subshift that is topologically conjugate to a constant-length substitution subshift but not generated by a constant-length substitution. His example is generated by a nonconstant-length substitution.

In [9], Durand proved that any subshift factor of constant-length substitution subshift, while not necessarily generated by a substitution itself, must be topologically conjugate to a substitution subshift. But he left open the question of whether the factor must be conjugate to a *constant-length* substitution subshift. In [13], Müllner and Yassawi answered this question by proving that any factor of a constant-length substitution subshift must be conjugate to a constant-length substitution subshift.

2. Proof of Theorem 1.1

First we verify that the factor map $\psi : X_\theta \to Y$ from Theorem 1.1 is one-to-one and hence a topological conjugacy.

Lemma 2.1. *The factor map $\psi : X_\theta \to Y$ is one-to-one.*

Proof. Because the Morse substitution θ is primitive and aperiodic, it is *recognizable*, as defined by Mossé. (See [11], [12], and also [14, Theorem 5.8].) It follows from recognizability that, for each $k \in \mathbb{N}$, each $x \in X_\theta$ can be written uniquely as a concatenation of blocks of the form $\theta^k(0)$ and $\theta^k(1)$. This is true in particular for $k = 2$, and, if $X_0 \subset X_\theta$ represents the set of $x \in X_\theta$ such that $x_0x_1x_2x_3$ is one of the substitution blocks $\theta^2(0)$ or $\theta^2(1)$ in this unique concatenation structure, then

$$X_\theta = X_0 \cup \sigma(X_0) \cup \sigma^2(X_0) \cup \sigma^3(X_0)$$

is a partition of X_θ, and, for $0 \le i \le 3$, the induced system $(\sigma^i(X_0), \sigma^4)$ is minimal. (See [7] for background on this partition structure.) For $0 \le i \le 3$, define $W_i = \{x_0x_1x_2x_3 \mid x \in \sigma^i(X_0)\}$. Then

- $W_0 = \{0110, 1001\} \longrightarrow \psi(W_0) = \{11, 10\}$;
- $W_1 = \{1101, 0011, 0010, 1100\} \longrightarrow \psi(W_1) = \{11, 01, 00\}$;
- $W_2 = \{1010, 0110, 1001, 0101\} \longrightarrow \psi(W_2) = \{10, 11, 01\}$; and
- $W_3 = \{0100, 1100, 0011, 1011\} \longrightarrow \psi(W_3) = \{01, 11\}$.

Suppose $x, y \in X_\theta$ with $x \ne y$. We may assume that at least one of x or y is in X_0. (If not, pick n so that at least one of $\sigma^n(x)$ or $\sigma^n(y)$ is in X_0. Then apply the argument below to $\sigma^n(x)$ and $\sigma^n(y)$ to obtain $\psi(\sigma^n(x)) \ne \psi(\sigma^n(y))$, which implies $\sigma^n(\psi(x)) \ne \sigma^n(\psi(y))$, so $\psi(x) \ne \psi(y)$.)

Without loss of generality, assume $x \in X_0$. Note that ψ is one-to-one on W_0, so, if $y \in X_0$ as well, then (since $x \ne y$) there exists $n \in \mathbb{Z}$ such that $x_{[4n,4n+3]}$ and $y_{[4n,4n+3]}$ are distinct words in W_0, which implies $\psi(x) \ne \psi(y)$. Otherwise, $x \in X_0$ and $y \in \sigma^i(X_0)$ for some $i \in \{1, 2, 3\}$. In this case, we can find $z_0z_1z_2z_3 \in W_i$ such that $\psi(z_0z_1z_2z_3) \notin \psi(W_0)$; indeed, for $i \in \{1, 2, 3\}$, $\psi(W_i)$ contains at least one word not in $\psi(W_0)$. But since $(\sigma^i(X_0), \sigma^4)$ is minimal, there exists $n \in \mathbb{Z}$ such that $y_{[4n,4n+3]} = z_0z_1z_2z_3$. Then $\psi(y)_{[4n,4n+1]} \notin \psi(W_0)$, whereas $\psi(x)_{[4n,4n+1]} \in \psi(W_0)$. Therefore $\psi(x) \ne \psi(y)$. $\square$

We now show that there cannot exist a constant-length substitution ζ that generates Y. By [14, Proposition 5.5], we may assume that ζ is primitive.

Lemma 2.2. *If Y is generated by a primitive substitution ζ of length L, then for any $y \in Y$, there exists $M \in \{0, \dots, L-1\}$ such that there are at most two words of the form*

$$y_p y_{p+1} \cdots y_{p+L-1}$$

where $p \equiv M \mod L$.

Proof. This follows from [3, Lemma 7.1], but for completeness we sketch the argument here. First, define $\mathcal{G}_1^y$ to be the directed graph whose vertex set is $V_1 = \{0, 1\}$, and whose edge set is

$$E_1 = \{(a, b) \ : \ ab \text{ is a word of length 2 in } y\}.$$

Next, for each $M \in \{0, \dots, L-1\}$ define $\mathcal{G}_{L,M}^y$ to be the directed graph whose vertex set is

$$V_{L,M} = \{y_p \cdots y_{p+L-1} \ : \ p \equiv M \mod L\}$$

and whose edge set is

$$E_{L,M} = \{(u, v) \ : \ uv \in W_{2L,M}\},$$

where $W_{2L,M} := \{y_p \cdots y_{p+2L-1} \ : \ p \equiv M \mod L\}$. By [3, Lemma 7.1], there must exist $M \in \{0, \dots, L-1\}$ such that ζ gives a graph epimorphism $\zeta : \mathcal{G}_1^y \to \mathcal{G}_{L,M}^y$. Since $\mathcal{G}_1^y$ has just two vertices, $\mathcal{G}_{L,M}^y$ can have at most two vertices. This completes the proof. $\square$

Lemma 2.3. *If Y is generated by a primitive substitution ζ of length L, then $L = 2^k$ for some $k \in \mathbb{N}$.*

Proof. This follows from Cobham's Theorem, which states that the lengths of any two primitive, constant-length substitutions that generate topologically conjugate systems must be powers of the same integer. (See, for example, [2], [6], and [8].) $\square$

To complete the proof of Theorem 1.1, we assume that the length of ζ is a power of 2 and exhibit a point $y \in Y$ for which the conclusion of Lemma 2.2 fails to hold. For this, first observe that, for each n, $\theta^{2n}(0)$ begins and ends with the symbol 0. It follows that the doubly-infinite sequence $x = (x_i)_{i \in \mathbb{Z}}$ given by

$$x = \lim_{n \to \infty} \theta^{2n}(0.0) = \cdots 0110\,1001\,1001\,0110\,.\,0110\,1001\,1001\,0110 \cdots$$

is a well-defined element of X_θ. (The symbol x_0 is the first 0 to the right of the decimal point.) We may then define

$$y = \psi(x) = \cdots 1110\,1011\,1001\,1110\,.\,1110\,1011\,1001\,1110 \cdots.$$

By definition of ψ, each symbol y_s in y is determined by the word $x[s, s+2] := x_s x_{s+1} x_{s+2}$ of length 3 in x. This is what it means to say that ψ is a 3-*block code*. By adding extraneous symbols, we can also think of ψ as an r-block code for any $r \geq 3$; indeed, a word $x[s, s+r-1]$ of length $r \geq 3$ in x can be truncated to its first three symbols, $x[s, s+2]$, which in turn determine the symbol y_s.

Therefore, given any $k \in \mathbb{N}$, we can think of ψ as a (2^k+1)-block code. Then ψ maps (2^k+1)-blocks to 1-blocks, and (2^{k+1})-blocks to (2^k)-blocks. This enables us to define, for $i, j \in \{0, 1\}$,

$$A_i(k) = \psi(\theta^{k+1}(i)) \quad \text{and} \quad B_{ij}(k) = \psi(\theta^k(C_{ij})),$$

where C_{ij} is the 2-block in the middle of $\theta(ij)$. Then $A_i(k)$ and $B_{ij}(k)$ each have length 2^k.

Remark 2.4. It follows from the "Morse Dynamical Characterization Theorem" from [5] that $y = \psi(x)$ can be written uniquely as a concatenation of alternating A- and B-blocks, where the sequence of subscripts on the A-blocks is the sequence x.

Example 2.5. For $k = 1$, $A_0 = 11$, $A_1 = 10$, $B_{01} = 10$, $B_{10} = 01$, $B_{00} = 10$, and $B_{11} = 11$. Therefore

$$\begin{aligned} x &= \cdots 0110\,.\,0110\ 1001\ 1001\ 0110 \cdots \\ y = \psi(x) &= \cdots 1110\,.\,1110\ 1011\ 1001\ 1110 \cdots \\ &= \cdots A_0 B_{00} . A_0 B_{01}\ A_1 B_{11}\ A_1 B_{10}\ A_0 B_{01} \cdots . \end{aligned}$$

Lemma 2.6. *For all* $k \in \mathbb{N}$,
$A_0(k+1) = A_0(k) B_{01}(k)$ *and* $A_1(k+1) = A_1(k) B_{10}(k)$,
$B_{01}(k+1) = A_1(k) A_0(k)$ *and* $B_{10}(k+1) = A_0(k) A_1(k)$.

Proof. By the definition of the A- and B-blocks,

$$\psi(\theta^k(0110)) = \psi(\theta^{k+1}(0)\theta^{k+1}(1)) = A_0(k) B_{01}(k) A_1(k).$$

And since $x[0, 2^{k+2}-1] = \theta^k(0110)$, $y[0, 2^{k+1}-1] = A_0(k) B_{01}(k)$. On the other hand, $x[0, 2^{k+2}-1] = \theta^k(0110) = \theta^{k+2}(0)$, and $A_0(k+1) = \psi(\theta^{k+2}(0))$, so $y[0, 2^{k+1}-1] = A_0(k+1)$. So $A_0(k+1) = A_0(k) B_{01}(k)$.

Similar arguments prove the other three equalities. $\square$

Lemma 2.7. *The blocks $A_i(k)$ and $B_{ij}(k)$ satisfy the following.*

(1) *For all $k \in \mathbb{N}$, and for $0 \leq M \leq 2^k - 1$,*
 a. $A_0(k)[M, 2^k - 1] \neq A_1(k)[M, 2^k - 1]$

b. $A_0(k)[M, 2^k - 1] \neq B_{01}(k)[M, 2^k - 1]$

c. $A_1(k)[M, 2^k - 1] \neq B_{10}(k)[M, 2^k - 1]$

(2) *For k even: $B_{10}(k) = A_0(k)$, and $B_{01}(k)$ and $A_1(k)$ differ only in their last 2-block.*
(3) *For k odd: $B_{01}(k) = A_1(k)$, and $B_{10}(k)$ and $A_0(k)$ differ only in their last 2-block.*

Proof. We can expand the A- and B-blocks by repeatedly applying Lemma 2.6:

$$\begin{array}{rcll} A_0(k) & = & A_0(k-1)B_{01}(k-1) & k \geq 2 \\ & = & A_0(k-1)A_1(k-2)A_0(k-2) & k \geq 3 \\ & = & A_0(k-1)A_1(k-2)A_0(k-3)B_{01}(k-3) & k \geq 4 \\ & = & \vdots & \end{array}$$

$$\begin{array}{rcll} A_1(k) & = & A_1(k-1)B_{10}(k-1) & k \geq 2 \\ & = & A_1(k-1)A_0(k-2)A_1(k-2) & k \geq 3 \\ & = & A_1(k-1)A_0(k-2)A_1(k-3)B_{10}(k-3) & k \geq 4 \\ & = & \vdots & \end{array}$$

If k is even, $A_0(k)$ ends in $B_{01}(1) = 10$ and $A_1(k)$ ends in $B_{10}(1) = 01$, and if k is odd, $A_0(k)$ ends in $A_0(1) = 11$ and $A_1(k)$ ends in $B_{10}(1) = 01$. This proves (1)a.

Comparing the expansions of $B_{01}(k)$ and $B_{10}(k)$

$$\begin{array}{rcll} B_{01}(k) & = & A_1(k-1)A_0(k-1) & k \geq 2 \\ & = & A_1(k-1)A_0(k-2)B_{01}(k-2) & k \geq 3 \\ & = & A_1(k-1)A_0(k-2)A_1(k-3)A_0(k-3) & k \geq 4 \\ & = & \vdots & \end{array}$$

$$\begin{array}{rcll} B_{10}(k) & = & A_0(k-1)A_1(k-1) & k \geq 2 \\ & = & A_0(k-1)A_1(k-2)B_{10}(k-2) & k \geq 3 \\ & = & A_0(k-1)A_1(k-2)A_0(k-3)A_1(k-3) & k \geq 4 \\ & = & \vdots & \end{array}$$

to the expansions for the A-blocks gives (1)b and c.

If k is even, $A_0(k)$ and $B_{10}(k)$ agree in their initial $(2^k - 2)$-blocks, with $A_0(k)$ ending in $B_{01}(1) = 10$ and $B_{10}(k)$ ending in $A_1(1) = 10$, so $A_0(k) = B_{10}(k)$. A similar observation proves the second part of (2), and (3) is proved in the same way. $\square$

We now complete the proof of Theorem 1.1. By Lemma 2.3, Y can only be generated by a primitive substitution of length $L = 2^k$ for some k. But by Remark 2.4, y has alternating $A_i(k)$ and $B_{ij}(k)$-blocks at positions congruent to $0 \mod 2^k$, and using Lemma 2.7 it is straightforward to check that, for each $M \in \{0, \ldots, 2^k - 1\}$, there are at least three words of the form $y_p \cdots y_{p+2^k-1}$ where $p \equiv M \mod 2^k$. This contradicts the conclusion of Lemma 2.2.

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