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TOPOLOGY PROCEEDINGS

Volume 55, 2020

Pages 295–313

http://topology.nipissingu.ca/tp/

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Electronically published on February 16, 2020

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 55 (2020)
Pages 295-313

E-Published on February 16, 2020

ON THE SET-THEORETIC STRENGTH OF A TOPOLOGICAL BANACH FIXED POINT THEOREM FOR CONTINUA

PAUL HOWARD AND ELEFTHERIOS TACHTSIS

Abstract. Juris Steprāns, Stephen Watson and Winfried Just [Canad. Bull. Math. **37** (4) (1994), 552–555] introduced and proved the following fixed point theorem for continua (i.e., for nonempty connected compact Hausdorff spaces): "*If T is a J-contraction of any continuum, then T has a unique fixed point*", using the full power of the Axiom of Choice (AC). We prove that the above result is deducible from each of the following weak choice principles: "Every continuum is weakly Loeb", the Boolean Prime Ideal Theorem, and the Principle of Dependent Multiple Choices, the latter being equivalent to "Every compact Hausdorff space is Baire" (and strictly weaker than the Axiom of Multiple Choice).

We also show that the above fixed point theorem is provable in ZF when restricted to well-ordered continua, and that the general version is deducible—in ZF—from "*If $T : X \to X$ is an onto J-contraction of a continuum X, then $|X| = 1$*".

Furthermore, we prove that "Every continuum is weakly Loeb" is not provable in ZF, and that it is strictly weaker than "Every continuum is Loeb" in ZFA; hence, it is strictly weaker than each of the Boolean Prime Ideal Theorem and the Axiom of Multiple Choice.

The question of whether or not the above fixed point theorem for continua is provable in ZF is still open.

2010 *Mathematics Subject Classification.* Primary 03E25, 03E35; Secondary 54D05, 54D30, 54H25.

Key words and phrases. Axiom of Choice, weak axioms of choice, Hausdorff space, connected space, compact space, continuum, Loeb space, weakly Loeb space, J-contractive open cover, J-contraction, fixed point, Fraenkel–Mostowski model of ZFA, Jech–Sochor First Embedding Theorem.

1. Introduction

It is a celebrated result of Stefan Banach from 1922 that any contraction of a complete metric space has a unique fixed point; see [1] or any standard textbook of general topology such as [6] and [21]. (Recall that a metric space is said to be complete if each of its Cauchy sequences converges in the space.) It is also well known that the proof of this beautiful and important topological result does *not* require any choice principle. In other words, Banach's contraction theorem is a theorem of ZF set theory.

In 1994, Steprāns, Watson and Just [17] proposed and proved an analogue of the Banach contraction principle for continua. In particular, they defined the J-contraction of a continuum (complete definitions will be given in Section 2) and showed the following remarkable result: "If X is a continuum and $T : X \to X$ is a J-contraction, then T has a unique fixed point". Furthermore, the authors in [17] justified their terminology "J-contraction" by showing that any contraction on a compact metric space is a J-contraction, and that any J-contraction of a compact metrizable space is a contraction for some admissible metric (see also the forthcoming Remark 2.9).

The proof of the main result of [17] (i.e., of the proposed fixed point theorem for continua) uses the full power of the Axiom of Choice, in striking contrast with Banach's relative result. It is thus an intriguing question whether one actually needs the full AC (or any choice principle at all) in order to establish the above result. To the best of our knowledge, there is no source of information (until now) with regard to the aforementioned question, which is a surprising fact.

It is exactly the goal of this paper to address this open question and to shed some light on it, as well as to provide strong motivation for further research on this deep topic. We will establish that the fixed point theorem for continua by Steprāns, Watson, and Just can be derived from each of "Every continuum is weakly Loeb", the Boolean Prime Ideal Theorem (BPI), and the Principle of Dependent Multiple Choices (DMC) which is (known to be) equivalent to "Every compact Hausdorff space is Baire" (see Theorem 3.6). Each of BPI and DMC is strictly weaker than AC in ZF set theory, and thus the above open question (of whether the full strength of AC is needed) receives a negative answer.

With regard to the deductive strength of "Every continuum is weakly Loeb" (where "weakly Loeb" means that the family of all nonempty closed subsets of the space admits a multiple choice function), we will observe that it is derivable from each of BPI and the Axiom of Multiple Choice (MC), and we will prove that it is strictly weaker than "Every continuum is Loeb" in ZFA (where "Loeb" means that the family of all nonempty closed subsets of the space admits a choice function)—see Theorem 3.8.

Furthermore, we will establish that "Every continuum is weakly Loeb" is not provable in ZF, which was unknown until now. To achieve our goal, we will first introduce a new permutation model $\mathcal{N}$ of ZFA and prove that, in $\mathcal{N}$, there exists a continuum which is not weakly Loeb (see Theorem 3.9); then the ZFA-independence result can be transferred to ZF using the Jech–Sochor First Embedding Theorem (see [11, Theorem 6.1]). We also note here that the above independence enhances the result of [13], [18] that the formally stronger statement "Every compact Hausdorff space is weakly Loeb" is not provable in ZF.

There remains the very interesting open question of whether or not the Steprāns–Watson–Just fixed point theorem for continua is provable in ZF. Besides this open problem, we will pose several other ones at the end of this paper which have naturally risen from the current research.

2. Notation, terminology, and known results

Below, we provide the terminology required for this paper.

Definition 2.1. ZF is Zermelo–Fraenkel set theory without AC.

ZFC is ZF + AC.

ZFA is ZF with the Axiom of Extensionality weakened in order to allow the existence of atoms.

ω denotes (as usual) the set of natural numbers.

If X is any set and $n \in \omega$, then $[X]^n$ denotes the set of n-element subsets of X. Furthermore, $[X]^{<\omega}$ denotes the set of finite subsets of X.

Definition 2.2. Let (X, d) be a metric space. A *contraction* is a continuous mapping $T : X \to X$ such that there is a non-negative real number $k < 1$ with the property that for all $x, y \in X$, $d(T(x), T(y)) \leq k \cdot d(x, y)$.

Let $(X, \mathcal{T})$ be a topological space and let $T : X \to X$ be a continuous mapping.

An open cover $\mathscr{U}$ of X is called *J-contractive* for T if for every $U \in \mathscr{U}$, there is $U' \in \mathscr{U}$ such that $T(\overline{U}) \subseteq U'$.

If $(X, \mathcal{T})$ is compact and Hausdorff, then T is called a *J-contraction* if any open cover $\mathscr{U}$ of X has a finite J-contractive open refinement $\mathscr{V}$ for T (i.e., $\mathscr{V}$ is an open cover of X (which is J-contractive for T) and for every $V \in \mathscr{V}$ there is $U \in \mathscr{U}$ such that $V \subseteq U$).

$(X, \mathcal{T})$ is called *Loeb* (*weakly Loeb*) if the family $\mathscr{C}_X$ of all nonempty closed subsets of X has a choice function (respectively, a multiple choice function, i.e., a function F on $\mathscr{C}_X$ such that for every $C \in \mathscr{C}_X$, $F(C)$ is a nonempty finite subset of C).

$(X, \mathcal{T})$ is called *Baire* if $\overline{\bigcap \mathscr{D}} = X$ for every nonempty countable family $\mathscr{D}$ of dense open subsets of X.

$(X, \mathcal{T})$ is called a *continuum* if it is a nonempty connected compact Hausdorff space.

The notions of "Loeb space" and "weakly Loeb space" were firstly introduced and studied (for the class of compact Hausdorff spaces in the basic Fraenkel model—Model $\mathcal{N}1$ in [9]) by Brunner [4].

Definition 2.3. Let X be a set.

X is called *Dedekind-finite* if there is no one-to-one mapping $f : \omega \to X$. If X is not Dedekind-finite, then X is called *Dedekind-infinite*.

If X is infinite (i.e., if there is no bijection $f : X \to n$ for any natural number n), then X is called *amorphous* if it cannot be partitioned into two infinite subsets. (In other words, an infinite set X is amorphous if the only subsets of X are the finite and the co-finite ones.)

Definition 2.4. (1) The *Axiom of Choice* (AC and Form 1 in [9]): Every family of nonempty sets has a choice function.

(2) The *Axiom of Multiple Choice* (MC and Form 67 in [9]): Every family of nonempty sets has a multiple choice function. (See Definition 2.2 for the meaning of the term 'multiple choice function'.)
(3) The *Boolean Prime Ideal Theorem* (BPI and Form 14 in [9]): Every Boolean algebra has a prime ideal.
(4) $\mathsf{AC}_{\mathsf{fin}}$ (Form 62 in [9]): Every family of nonempty finite sets has a choice function.
(5) The *Axiom of Countable Choice* $\mathsf{AC}^{\aleph_0}$ (Form 8 in [9]): Every countably infinite family of nonempty sets has a choice function.
(6) $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ (Form 10 in [9]): Every countably infinite family of nonempty finite sets has a choice function.
(7) The *Principle of Dependent Choices* (DC and Form 43 in [9]): If R is a binary relation on a nonempty set X such that for every $x \in X$ there is $y \in X$ with $x\ R\ y$, then there is a sequence $(x_n)_{n\in\omega}$ of elements of X such that for every $n \in \omega$, $x_n\ R\ x_{n+1}$.
(8) The *Principle of Dependent Multiple Choices* (DMC and Form [106 B] in [9]): If R is a binary relation on a nonempty set X such that for every $x \in X$ there is $y \in X$ with $x\ R\ y$, then there is a sequence $(F_n)_{n\in\omega}$ of nonempty finite subsets of X such that for every $n \in \omega$ and for every $x \in F_n$, there is $y \in F_{n+1}$ such that $x\ R\ y$.
(9) $\mathsf{DF} = \mathsf{F}$ (Form 9 in [9]): Every Dedekind-finite set is finite.
(10) Form 64 in [9]: There are no amorphous sets.

Next, we provide known results in this area for use in the forthcoming Section 3. We shall explicitly mention in the statements of these results whether their proofs have been conducted in ZF or in ZFC. Prior to this, let us recall some known facts on the relationships between some of the choice principles listed in Definition 2.4. We refer the reader to Howard and Rubin [9] for complete references in the literature, and to Jech's book [11] for the proofs of some of these results.

Theorem 2.5. *The following hold:*

(1) MC *is equivalent to* AC *in* ZF*, but not equivalent to* AC *in* ZFA*.*

(2) $\mathsf{AC} \to \mathsf{BPI} \to \mathsf{AC_{fin}}$*. None of these implications is reversible in* ZF*.*

(3) $\mathsf{DC} \to \mathsf{AC}^{\aleph_0} \to \mathsf{DF} = \mathsf{F} \to \mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$*. None of these implications is reversible in* ZF*. Furthermore, Form 64 (there are no amorphous sets) is strictly weaker than* $\mathsf{DF} = \mathsf{F}$*, and Form 64 and* $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ *are independent of each other in* ZF*.*

(4) DMC *is strictly weaker than* DC *in* ZFA*. Furthermore,* DMC *and* $\mathsf{AC}^{\aleph_0}$ *are independent of each other in* ZF*, and* DMC *is strictly weaker than* MC *in* ZFA*.*

(5) BPI *and P, where* $P \in \{\mathsf{AC}^{\aleph_0}, \mathsf{DC}, \mathsf{DMC}\}$*, are independent of each other in* ZF*.*

We note here that it is an open problem whether or not DMC implies DC in ZF.

Theorem 2.6. *The following hold:*

(1) ([14], [15], [16]*)* BPI *is equivalent to each of "Products of compact Hausdorff spaces are compact", "For every set I, $\{0,1\}^I$ is compact in the product topology", "For every set I, $[0,1]^I$ is compact in the product topology", and "Products of continua are continua".*

(2) ([13]*)* $\mathsf{BPI} \to$ *"Every compact Hausdorff space is Loeb"* $\to \mathsf{AC_{fin}}$*. The first implication is not reversible in* ZFA*.*

(3) ([13], [18]*)* $\mathsf{MC} \to$ *"Every compact Hausdorff space is weakly Loeb". Thus, "Every compact Hausdorff space is weakly Loeb" is strictly weaker than "Every compact Hausdorff space is Loeb" in* ZFA*. Furthermore, "Every compact Hausdorff space is weakly Loeb" is not provable in* ZF*.*

(4) ([7]*)* DMC *is equivalent to "Every compact Hausdorff space is Baire".*

For the proof of the equivalence of BPI with "Products of continua are continua", recall that any interval of the real line $\mathbb{R}$ is connected without using any form of choice, see for example [21].

With regard to Theorem 2.6(3), Keremedis and Tachtsis [13] constructed a Fraenkel–Mostowski ZFA-model in which there exists a compact metrizable space which is not weakly Loeb, and subsequently Tachtsis [18] constructed a symmetric ZF-model with the same property. Let us also note that the statements "There exists a compact Hausdorff weakly Loeb space which is not Loeb" and "There exists a compact Hausdorff space which is not weakly Loeb" are boundable (see [9, Note 103] for the definition of the term "boundable"), and since they both have ZFA-models (for the first existential statement, the second Fraenkel model (Model $\mathcal{N}2$ in [9]) may serve as a witness due to Theorem 2.6[(2),(3)]), it follows by the Jech–Sochor First Embedding Theorem (see [11, Theorem 6.1]) that they also have ZF-models.

Proposition 2.7. *(ZF) Let $(C_s)_{s\in S}$ be a family of subspaces of a topological space $(X,\mathcal{T})$ each of which is a continuum. If for any $s_1,s_2\in S$ there exists $s_3\in S$ such that $C_{s_3}\subseteq C_{s_1}\cap C_{s_2}$ (i.e., if $(C_s)_{s\in S}$ is directed by $\supseteq$—equivalently, if $(C_s)_{s\in S}$ is a filter base), then $\bigcap_{s\in S}C_s$ is a continuum.*

For a ZF-proof of Proposition 2.7, see Engelking [6, Theorem 6.1.18].

Proposition 2.8. *([17]) The following hold true in ZF:*

(1) If $T:X\to X$ is a J-contraction of a compact Hausdorff space and A is a closed subspace of X such that $T(A)\subseteq A$, then $T\restriction A$ is also a J-contraction.

(2) If $T:X\to X$ is a J-contraction of a compact Hausdorff space, then for any $n\in\omega\setminus\{0\}$, $T^n:X\to X$ is also a J-contraction.

(3) There is a J-contraction on any non-connected compact Hausdorff space X which has no fixed points.

(4) Any contraction on a compact metric space is a J-contraction.

(5) If $T:X\to X$ is an onto J-contraction and $\mathscr{U}$ is a finite J-contractive open cover of X, then there is a subcover $\mathscr{U}'$ of $\mathscr{U}$ such that $(\exists n\in\omega\setminus\{0\})(\forall U\in\mathscr{U}')T^n(\overline{U})\subseteq U$.

Remark 2.9.

(1) The proofs of the above results in [17] do not employ any form of choice as the interested reader may easily verify. There is only one point in the proof of (4) (which is Theorem 2 in [17]) that may give rise to the natural question of whether or not some choice was used in its proof. In particular, the authors prove their result using the renowned fact that every open cover of a compact metric space has a Lebesgue number (that is, a number $\epsilon>0$ such that any subset of the metric space having diameter less than ϵ is a subset of some element of the given open cover). In standard textbooks of topology such as Willard's [21] (see Theorem 22.5 therein), or Engelking's [6] (see Theorem 4.3.31 therein), the proof of the above fact is conducted in ZFC. However, minor adjustments to this proof reveal that no choice principle is necessary. Indeed, let $\mathscr{U}$ be an open cover of a compact metric space (X,d). By way of contradiction, we assume that for every $\epsilon>0$, the open cover $\mathscr{V}=\{B(x,\epsilon):x\in X\}$ of X (where $B(x,\epsilon)$ is the open ball of

radius ϵ centered at x) is not a refinement of $\mathscr{U}$. Thus for each $n \in \omega \setminus \{0\}$, the set $F_n = \{x \in X : B(x, \frac{1}{n}) \nsubseteq U \text{ for all } U \in \mathscr{U}\}$ is nonempty, and clearly the sequence $(F_n)_{n\in\omega\setminus\{0\}}$ is $\subseteq$-decreasing. By compactness of X, there exists $x \in \bigcap_{n\in\omega\setminus\{0\}} \overline{F_n}$. Similarly now to the proof given in [21], we may arrive at a contradiction. Indeed, we have $x \in U$ for some $U \in \mathscr{U}$ (since $\mathscr{U}$ is a cover of X), and thus for some $\delta > 0$, $B(x, \delta) \subseteq U$. Pick $n \in \omega \setminus \{0\}$ such that $\frac{1}{n} < \frac{\delta}{2}$, and let $m > n$ and $y \in F_m \cap B(x, \frac{\delta}{2})$ (this is possible since $x \in \overline{F_k}$ for all $k \in \omega \setminus \{0\}$). It is easy to see now that $B(y, \frac{1}{m}) \subseteq B(x, \delta) \subseteq U$, and hence $B(y, \frac{1}{m}) \subseteq U$, which is a contradiction (since $y \in F_m$). Thus, there exists $\epsilon > 0$ such that the collection $\mathscr{V} = \{B(x, \epsilon) : x \in X\}$ is an open refinement of $\mathscr{U}$, from which the result follows.

(2) The authors in [17] show that a converse of (4) of Proposition 2.8 also holds. In particular, in Theorem 3 of [17], it is shown that *if T is a J-contraction of a connected compact metrizable space X, then X admits a metric under which T is a contraction*. Their proof uses a result of Janos [10] ("Let X be a compact metrizable space and $f : X \to X$ a continuous one-to-one mapping with $\bigcap_{n\in\omega} f^n[X]$ a singleton. Given λ, $0 < \lambda < 1$, there exists a metric ρ on X such that the metric topology of (X, ρ) is identical with the original one and $\rho(f(x), f(y)) = \lambda\rho(x, y)$ for all $x, y \in X$"—see also Edelstein [5] for a formally stronger result), and Theorem 5 ("If T is a J-contraction of any connected compact Hausdorff space X, and $x \in X$, then $(T^n(x))_{n\in\omega}$ converges to the unique fixed point.") in their paper. We note that the proof of Theorem 5 requires no choice forms, whereas the proof of Janos' result (as given in [5]) uses the fact that compact metric spaces are second countable, which lies in strength between $\mathsf{AC}^{\aleph_0}$ and $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ (see Good and Tree [8], or Keremedis and Tachtsis [12]). Furthermore, Tietze's Extension Theorem and (its consequence) Urysohn's Lemma are used in the proof, but no choice is required for their applications due to the countable base for the given compact metrizable space (note that the (general) Urysohn Lemma, and hence the Tietze Extension Theorem, are not provable in ZF, see [9]—for a study on the set-theoretic strength of Urysohn and Tietze's results, see Brunner [2] and Tachtsis [19]). Now, in [12], it is shown that a compact metric space is second countable if and only if it is Loeb. Since BPI implies that every compact Hausdorff space is Loeb (Theorem 2.6(2)), we see that Theorem 3 in [17] as

well as Janos' result (and the formally stronger by Edelstein [5]) are deducible from each of $\mathsf{AC}^{\aleph_0}$ and BPI.

We do not know whether or not Theorem 3 of [17] or Janos' result are provable in ZF.

Next, we state the main results of Steprāns, Watson and Just (Proposition 4 and Theorem 4 in [17]), whose deductive strength shall be examined in the forthcoming Section 3.

Theorem 2.10. *([17]) (ZFC) The following hold:*

(1) If $T : X \to X$ is an onto J-contraction of a continuum X, then $|X| = 1$.

(2) If $T : X \to X$ is a J-contraction of a continuum X, then T has a unique fixed point.

A careful inspection of the proof of (1) (of Theorem 2.10) in [17], reveals that DC suffices for the proof. Thus the argument can be carried out in the axiomatic system $\mathsf{ZF}+\mathsf{DC}$, which is weaker than ZFC. On the other hand, the authors in [17] use AC for their proof of (2), since they apply Zorn's Lemma (a well-known equivalent of AC) in order to obtain a maximal filter base of subcontinua A of X with $T(A) \subseteq A$. We will show here that (1) suffices for the proof of (2), and hence (by the above observation) (2) can be proved in the weaker system $\mathsf{ZF} + \mathsf{DC}$ (see Theorem 3.4).

In the forthcoming Section 3, we will further enhance the above results by showing that their proofs can be also conducted in the weaker systems $\mathsf{ZF}+\mathsf{BPI}$ and $\mathsf{ZF}+\mathsf{DMC}$ (see Theorems 3.1 and 3.4). Furthermore, from our adjusted proofs of (1) and (2) of Theorem 2.10, we shall derive that—in ZF—if $(X, \mathcal{T})$ is a continuum with X a *well-ordered* set, and if $T : X \to X$ is a J-contraction of X, then T has a unique fixed point.

3. Main results

Theorem 3.1. *Each of "Every continuum is weakly Loeb" and* DMC *implies "If $T : X \to X$ is an onto J-contraction of a continuum X, then $|X| = 1$". Thus, each of "Every continuum is Loeb",* BPI*, "Every compact Hausdorff space is Baire",* MC*, and* DC *implies the above statement.*

Proof. Let $(X, \mathcal{T})$ be a continuum and let $T : X \to X$ be an onto J-contraction.

Assume that X is weakly Loeb. Let F be a multiple choice function for $\mathscr{C}_X$ (the family of all nonempty closed subsets of X). Towards a proof by contradiction, we assume that $|X| > 1$. Let $\mathscr{U}$ be a finite J-contractive open cover for T which does not contain X. (Since X is Hausdorff and has at least two elements, the collection $\mathscr{W} = \mathcal{T} \setminus \{\emptyset, X\}$ is an open cover of X (in fact, since X is connected and $|X| > 1$, it follows that X is dense-in-itself, i.e. it has no isolated points, and thus it is infinite).

Since T is a J-contraction, $\mathscr{W}$ has a finite J-contractive open refinement $\mathscr{U}$ for T.) By Proposition 2.8(5), there is a subcover $\mathscr{U}'$ of $\mathscr{U}$ and $n \in \omega \setminus \{0\}$ such that for all $U \in \mathscr{U}'$, $T^n(\overline{U}) \subseteq U$. By Proposition 2.8(2), T^n is also a (onto) J-contraction of X. For the sake of simplicity, and without loss of generality, we may assume that $n = 1$ and that $\mathscr{U}' = \mathscr{U}$.

Let $U \in \mathscr{U}$. Since X is connected, it is easy to see that there is $y_0 \in T(\overline{U} \setminus U)$ (if $\overline{U} \setminus U = \emptyset$, then $\{U, X \setminus U\}$ is a partition of X into two nonempty disjoint open sets—recall that $X \notin \mathscr{U}$—contradicting that X is connected). Furthermore, since T is continuous and X is Hausdorff (and thus singletons are closed), and $\overline{U} \setminus U$ is closed, it follows that $T^{-1}(\{y_0\}) \cap (\overline{U} \setminus U)$ is a nonempty, closed subset of X. Let

$$Y_1 = F(T^{-1}(\{y_0\}) \cap (\overline{U} \setminus U)).$$

Note that

$$(\forall y \in Y_1) T(y) = y_0.$$

Let

$$Y_2 = \bigcup \left\{ F(T^{-1}(\{y\})) : y \in Y_1 \right\}.$$

(For every $y \in Y_1$, $T^{-1}(\{y\}) \in \mathscr{C}_X$ since T is onto and continuous, and X is Hausdorff (and thus F can be applied)—Y_2 is closed, being a finite union of closed sets.) Note that

$$(\forall y \in Y_2) T(y) \in Y_1.$$

We continue in this fashion by induction (and using the multiple choice function F for $\mathscr{C}_X$), and obtain a sequence $(Y_n)_{n \in \omega}$ of nonempty finite subsets of X such that

$$Y_0 = \{y_0\},$$

and

$$(\forall n \in \omega)(\forall y \in Y_{n+1}) T(y) \in Y_n.$$

By the construction of the sequence $(Y_n)_{n \in \omega}$, we have $y_0 \in U$ and for every $n \in \omega \setminus \{0\}$, $Y_n \subseteq X \setminus U$. Indeed, $y_0 \in T(\overline{U} \setminus U) \subseteq T(\overline{U}) \subseteq U$. Now suppose, aiming at a contradiction, that for some $n \in \omega \setminus \{0\}$, there is $y \in Y_n \cap U$. Then for all $i \in \{1, 2, \ldots, n\}$, $T^i(y) \in U$ (since $T(\overline{U}) \subseteq U$), and thus $T^{n-1}(y) \in U$. However, $T^{n-1}(y) \in Y_1 \subseteq T^{-1}(\{y_0\}) \cap (\overline{U} \setminus U) \subseteq \overline{U} \setminus U$, and hence $T^{n-1}(y) \notin U$, a contradiction.

Now we argue that $(Y_n)_{n \in \omega}$ is disjoint. By way of contradiction, we assume that there exists $m < n$ such that $Y_m \cap Y_n \neq \emptyset$. Let $y \in Y_m \cap Y_n$. Now $y \in Y_m$ implies $T^m(y) = y_0$, and $y \in Y_n$ implies $T^m(y) \in Y_{n-m} \subseteq X \setminus U$, and thus $y_0 \in U \cap (X \setminus U) = \emptyset$, which is impossible.

Since $(Y_n)_{n\in\omega}$ is disjoint, it follows that the set $M = \bigcup\{Y_n : n \in \omega \setminus \{0\}\}$ is infinite, and thus by compactness of X, the closed set A of limit points of M is nonempty. Furthermore, since $y_0 \in U$ and $M \subseteq X \setminus U$, it follows that $y_0 \notin A$.

Similarly to the relative proof in [17], we have $T(A) \subseteq A$. (Let $x \in A$. If $T(x) \notin A$, then there exists an open neighborhood O of $T(x)$ such that $M \cap O$ is finite. Let $n_0 \in \omega \setminus \{0\}$ such that $Y_n \cap O = \emptyset$ for all integers $n \geq n_0$. Since T is continuous, $T^{-1}(O)$ is an open neighborhood of x, and thus meets M in an infinite set. Hence, there is $n \in \omega$ with $n > n_0$ and $Y_n \cap T^{-1}(O) \neq \emptyset$. Pick $y \in Y_n \cap T^{-1}(O)$. Then $T(y) \in Y_{n-1} \cap O$, a contradiction.)

Now let $\mathscr{V}$ be a J-contractive open cover for T such that no $V \in \mathscr{V}$ satisfies both $V \cap A \neq \emptyset$ and $y_0 \in V$. (We recall that compact Hausdorff spaces are normal without using any form of choice—the standard proof (see [21, Theorem 17.10]) goes through with minor changes. So there exist two disjoint open sets P and Q such that $A \subseteq P$ and $y_0 \in Q$. Then the open cover $\{P, Q, X \setminus (A \cup \{y_0\})\}$ has a J-contractive open refinement $\mathscr{V}$ for T, which clearly has the above property.) As in [17], we have $T(\mathrm{st}(A, \mathscr{V})) \subseteq \mathrm{st}(A, \mathscr{V})$, where $\mathrm{st}(A, \mathscr{V})$ is the star of A with respect to $\mathscr{V}$, i.e., $\mathrm{st}(A, \mathscr{V}) = \bigcup\{V \in \mathscr{V} : V \cap A \neq \emptyset\}$.

Since A is the (nonempty) set of limit points of M, and $\mathscr{V}$ is a cover of X, it follows that there exists $n \in \omega \setminus \{0\}$ and $y \in Y_n \cap \mathrm{st}(A, \mathscr{V})$. Then $y_0 = T^n(y) \in \mathrm{st}(A, \mathscr{V})$, which is a contradiction (recall that no $V \in \mathscr{V}$ satisfies both $V \cap A \neq \emptyset$ and $y_0 \in V$). This completes the proof.

Now we assume that DMC is true. As in the first part of the proof, we assume (towards a contradiction) that $|X| > 1$, and we let $\mathscr{U}$ be a finite J-contractive open cover for T which does not contain X and is such that for every $U \in \mathscr{U}$, $T(\overline{U}) \subseteq U$. Fix $U \in \mathscr{U}$ and $y_0 \in T(\overline{U} \setminus U)$. We let

$$W = \{(y_0, y_1, \ldots, y_n) : n \in \omega \setminus \{0\}, y_1 \in \overline{U} \setminus U, T(y_{i+1}) = y_i \text{ for all } i < n\}.$$

Then $W \neq \emptyset$ (see the first part of the proof). We define a binary relation R on W by stipulating for all $\mathbf{u}, \mathbf{w} \in W$,

$$\mathbf{u} \, R \, \mathbf{w} \iff \mathbf{u} = (y_0, y_1, \ldots, y_n) \text{ and } \mathbf{w} = (y_0, y_1, \ldots, y_n, y_{n+1}).$$

Since T is onto, we easily conclude that $\mathrm{dom}(R) = W$. Let $(F_n)_{n\in\omega}$ be a sequence of nonempty finite subsets of W, which satisfies the conclusion of DMC for (W, R). By induction, we construct a sequence $(G_n)_{n\in\omega}$ of nonempty finite subsets of X such that $G_0 = \{y_0\}$, $G_n \subseteq X \setminus U$ for all $n \in \omega \setminus \{0\}$, $G_n \cap G_m = \emptyset$ for all $n, m \in \omega$ with $n \neq m$, and for each $n \in \omega$ and $y \in G_{n+1}$, $T(y) \in G_n$.

Let $\mathbf{u} = (y_0, y_1, \ldots, y_n)$ (for some $n \in \omega \setminus \{0\}$) be any element of F_0. For $i < n+1$, let $G_i = \{y_i\}$. Then $y_1 \in \overline{U} \setminus U$ and $T(y_{i+1}) = y_i$ for all $i < n$. Let

$$U_0 = \{\mathbf{w} \in F_1 : \mathbf{u}\, R\, \mathbf{w}\}$$

(note that $U_0 \neq \emptyset$, and by definition of R, every element of U_0 is a $(n+1)$-tuple), and let

$$G_{n+1} = \pi_{n+1}[U_0]$$

(where π_{n+1} is the projection of the elements of U_0 onto their $(n+1)$-th coordinate). Clearly, for every $y \in G_{n+1}$, $T(y) = y_n \in G_n$. Now let $U_1 = \{\mathbf{y} \in F_2 : (\exists \mathbf{w} \in U_0)\mathbf{w}\, R\, \mathbf{y}\}$, and let

$$G_{n+2} = \pi_{n+2}[U_1].$$

We proceed in this fashion by induction, and thus obtaining the required sequence $(G_n)_{n\in\omega}$. We may follow now the proof of the first part in order to arrive at a contradiction.

The second assertion of the theorem follows from Theorems 2.5(4) and 2.6 ((2)–(4)). □

The first part of the proof of Theorem 3.1 easily yields the interesting ZF-result below.

Theorem 3.2. *(ZF) If $(X, \mathcal{T})$ is a continuum such that $\wp(X)$ (the power set of X) is Dedekind-finite, and if T is an onto J-contraction of X, then $|X| = 1$. (Equivalently, if $(X, \mathcal{T})$ is a continuum with $|X| > 1$ and T is an onto J-contraction of X, then $\wp(X)$ is Dedekind-infinite.)*

Proof. Let $(X, \mathcal{T})$ be a continuum with $|X| > 1$, and also let T, $\mathscr{U}$, U, and y_0 be as in the proof of Theorem 3.1. Since X is connected and Hausdorff, it is dense-in-itself, and thus is infinite. Now, the family $\mathcal{Y} = \{Y_n : n \in \omega\}$, where

- $Y_0 = \{y_0\}$,
- $Y_1 = T^{-1}(\{y_0\}) \cap (\overline{U} \setminus U)$, and
- $Y_n = T^{-1}(Y_{n-1})$ for all integers $n \geq 2$,

has the following properties:

(i) $Y_0 \subseteq U$;
(ii) $Y_n \subseteq X \setminus U$ for all $n \geq 1$; and
(iii) $\mathcal{Y}$ is a disjoint family of nonempty subsets of X

(see the argument for the relative sets Y_n $(n \in \omega)$ of the first part of the proof of Theorem 3.1). Thus, $\wp(X)$ is Dedekind-infinite, finishing the proof. □

Remark 3.3. We note that the existence of a continuum with a Dedekind-finite power set is relatively consistent with ZF. Indeed, it suffices to show that there is a permutation model in which such a continuum exists. Then using the Jech–Sochor First Embedding Theorem (see [11, Theorem 6.1]), one obtains a symmetric model of ZF which contains such a continuum. The required permutation model is a variation of the Mostowski linearly ordered model (Model $\mathcal{N}3$ in [9]). So we start with a model M of ZFA+AC with a continuum-sized set A of atoms using an ordering $\leq$ of A chosen so that $(A, \leq)$ is order-isomorphic to the set $\mathbb{R}$ of the real numbers with the usual ordering. Let G be the group of all order automorphisms of $(A, \leq)$. Let $\mathscr{F}$ be the filter of subgroups of G generated by the subgroups $\{\text{fix}_G(E) : E \in [A]^{<\omega}\}$, where $\text{fix}_G(E) = \{\phi \in G : (\forall e \in E)(\phi(e) = e)\}$. ($\mathscr{F}$ is a normal filter, see Jech [11, Section 4.2, p. 46] for the definition of the term 'normal filter'.) Let $\mathcal{N}$ be the permutation model determined by M, G, and $\mathscr{F}$.

It is clear that the linear order $\leq$ on A belongs to $\mathcal{N}$ (it has the empty set as a support, i.e., every permutation of A in G fixes $\leq$), and that A with the order topology is a connected Hausdorff space in $\mathcal{N}$. Similarly to the Mostowski linearly ordered model, one shows that $\wp(A)$, and hence A, is Dedekind-finite in $\mathcal{N}$ (see [9], [11]). Now letting $-\infty$ and ∞ be two distinct elements (not in A), and extending the ordering $\leq$ so that $-\infty < a < +\infty$ for all $a \in A$, we have that (in $\mathcal{N}$) $X = A \cup \{-\infty, \infty\}$ is a continuum with a Dedekind-finite power set. By Theorem 3.2, it follows that X does not admit an onto J-contraction in the model $\mathcal{N}$.

Recall that the Steprāns–Watson–Just fixed point theorem for continua (Theorem 2.10(2)) was proven in [17] using the full AC. We show next that this theorem is provable under the sole assumption of "If $T : X \to X$ is an onto J-contraction of a continuum X, then $|X| = 1$", which—by Theorem 3.1—is provable in set theories weaker than ZFC.

Theorem 3.4. *(ZF) Assume that the statement "If $T : X \to X$ is an onto J-contraction of a continuum X, then $|X| = 1$" is true. If T is a J-contraction of any continuum X, then T has a unique fixed point.*

Proof. Let $(X, \mathcal{T})$ be any continuum and $T : X \to X$ a J-contraction. By transfinite recursion we construct a $\subseteq$-decreasing family $\mathscr{B}$ of subcontinua of X such that for all $B \in \mathscr{B}$, $T(B) \subseteq B$.

Let $B_0 = X$. Assume that for some ordinal number α, we have constructed a $\subseteq$-decreasing sequence $(B_\nu)_{\nu<\alpha}$ of subcontinua of X (i.e., $\nu < \mu < \alpha \to B_\mu \subseteq B_\nu$) such that $T(B_\nu) \subseteq B_\nu$ for all $\nu < \alpha$.

If $a = \beta + 1$ is a successor ordinal number, then we let

$$B_\alpha = B_{\beta+1} = T(B_\beta).$$

Since T is continuous and B_β is a continuum, it follows that B_α is a continuum too. Furthermore, since $T(B_\beta) \subseteq B_\beta$, we have

$$T(B_\alpha) = T^2(B_\beta) \subseteq T(B_\beta) = B_\alpha,$$

and thus $T(B_\alpha) \subseteq B_\alpha$.

If α is a limit ordinal number, then we let

$$B_\alpha = \bigcap\{B_\nu : \nu < \alpha\}.$$

By Proposition 2.7, B_α is a subcontinuum of X. We assert that $T(B_\alpha) \subseteq B_\alpha$. Indeed, let $x \in B_\alpha$. Then $x \in B_\nu$ for all $\nu < \alpha$. Since for each $\nu < \alpha$, $T(B_\nu) \subseteq B_\nu$, we have $T(x) \in B_\nu$ for all $\nu < \alpha$, and thus $T(x) \in B_\alpha$.

Since the class of all ordinals is a proper one, there exists an ordinal number α such that $B_{\alpha+1} = B_\alpha$, i.e., $T(B_\alpha) = B_\alpha$. Then $T \restriction B_\alpha : B_\alpha \to B_\alpha$ is onto and, by Proposition 2.8(1), a J-contraction. By our hypothesis, there exists $x \in X$ such that $B_\alpha = \{x\}$, and hence x is a fixed point of T.

Now assume that there exists $y \in X$ such that $T(y) = y$. Since $y \in B_0$ and for all $\beta < \alpha + 1$, $T(B_\beta) \subseteq B_\beta$, it follows that $y \in B_\alpha = \{x\}$. Thus $y = x$. □

From the proof of Theorem 3.1, we obtain the following ZF-result.

Theorem 3.5. *(*ZF*) Let* $(X, \mathcal{T})$ *be a continuum with* X *well-ordered. If* T *is a* J*-contraction of* X*, then* T *has a unique fixed point.*

Furthermore, Theorems 3.1 and 3.4 yield the following result.

Theorem 3.6. *Each of "Every continuum is weakly Loeb" and* DMC *(and thus each of "Every continuum is Loeb",* BPI*, "Every compact Hausdorff space is Baire",* MC*, and* DC*) implies "If* T *is a* J*-contraction of any continuum* X*, then* T *has a unique fixed point". Thus, the latter statement implies neither* $\mathsf{AC}_{\mathsf{fin}}$ *(and hence neither* BPI*) nor* DMC *in* ZF*.*

Proof. In the basic Cohen model $\mathcal{M}1$ in [9], BPI is true, whereas DMC is false (see [9]). On the other hand, in Feferman's model $\mathcal{M}2$ in [9], DC is true, but $\mathsf{AC}_{\mathsf{fin}}$ (and hence BPI) is false (see [9]). The conclusion follows now from the first assertion of the theorem. □

Theorem 3.7. *The statement "If* T *is an onto* J*-contraction of a continuum* X*, then* $|X| = 1$*" (and hence, by Theorem 3.4, "If* T *is a* J*-contraction of any continuum* X*, then* T *has a unique fixed point") is true in the basic Fraenkel model, the second Fraenkel model, and the Mostowski linearly ordered model. Thus, it implies neither* $\mathsf{AC}_{\mathsf{fin}}^{\aleph_0}$ *nor* MC *in* ZFA*.*

Proof. The basic Fraenkel model, the second Fraenkel model, and the Mostowski linearly ordered model are labeled in [9] as $\mathcal{N}1$, $\mathcal{N}2$, and $\mathcal{N}3$, respectively. Now in $\mathcal{N}1$, "Compact Hausdorff spaces are weakly Loeb" is true, whereas DMC (and hence MC) is false.

On the other hand, MC is true in $\mathcal{N}2$, whereas $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ is false. Furthermore, BPI is true in $\mathcal{N}3$. (See [9] for complete references to the above facts.) The conclusion now follows from Theorem 3.6. □

Theorem 3.8. *The statement "Every continuum is weakly Loeb" does not imply "Every continuum is Loeb" in* ZFA. *In particular, in the basic Fraenkel model, every continuum is weakly Loeb, but there exists a continuum which is not Loeb. By Theorem 3.1, neither does "If T is an onto J-contraction of a continuum X, then $|X| = 1$" imply "Every continuum is Loeb" in* ZFA. *Hence, by Theorem 3.4, the same holds for "If T is a J-contraction of any continuum X, then T has a unique fixed point".*

Proof. For the reader's convenience, we first recall the description of the basic Fraenkel model, which is labeled as 'Model $\mathcal{N}1$' in [9]: We start with a model M of $\mathsf{ZFA} + \mathsf{AC}$ with a countably infinite set A of atoms. Let G be the group of all permutations of A. For any element x of M, $\mathrm{fix}_G(x)$ denotes the subgroup $\{\phi \in G : \forall y \in x(\phi(y) = y)\}$ of G and $\mathrm{Sym}_G(x)$ denotes the subgroup $\{\phi \in G : \phi(x) = x\}$ of G. Let $\mathscr{F}$ be the (normal) filter of subgroups of G generated by the filter base $\{\mathrm{fix}_G(E) : E \in [A]^{<\omega}\}$. An element x of M is called *symmetric* if $\mathrm{Sym}_G(x) \in \mathscr{F}$, and thus x is symmetric if there exists a finite subset $E \subset A$ such that $\mathrm{fix}_G(E) \subseteq \mathrm{Sym}_G(x)$. Under these circumstances, E is called a *support* of x. An element x of M is called *hereditarily symmetric* if x and all elements in the transitive closure of x are symmetric. $\mathcal{N}1$ is the permutation model determined by M, G, and $\mathscr{F}$, that is, $\mathcal{N}1$ consists exactly of the hereditarily symmetric elements of M.

It is well known that the set A of atoms is amorphous in $\mathcal{N}1$ (see [9], [11]). Furthermore, Brunner [4], [3] has respectively shown that, in $\mathcal{N}1$, the following are true:

(a) "Every compact Hausdorff space is weakly Loeb" (Form 116 in [9]), and

(b) "An amorphous power of a compact Hausdorff space, whose underlying set is well orderable, is compact in the product topology" (Form 127 in [9]).

Thus, by (a), "Every continuum is weakly Loeb" is true in $\mathcal{N}1$, and hence we only need to show that there exists a continuum in $\mathcal{N}1$ which is not Loeb. To this end, let

$$X = [0,1]^A$$

with the product topology (where $[0,1]$ has the topology inherited by the standard one on $\mathbb{R}$). Clearly $X \in \mathcal{N}1$, since it has empty support, i.e., any permutation of A fixes X. Since $[0,1]$ is connected Hausdorff and products of connected spaces are connected (without using any form of choice, see [21, Theorem 26.10]), it follows that X is connected Hausdorff.

Furthermore, since $[0,1]$ is well orderable in $\mathcal{N}1$ (i.e., $\mathrm{fix}_G([0,1]) \in \mathscr{F}$, see also [11, equation (4.2), p. 47]), it follows, by (b) above, that X is compact. Thus X is a continuum in $\mathcal{N}1$.

Now we show that $\mathscr{C}_X$ (the family of nonempty closed subsets of X) has no choice function in $\mathcal{N}1$. Assume the contrary and let g be a choice function for $\mathscr{C}_X$ in $\mathcal{N}1$. Let $E \subset A$ be a finite support of g. Let $F = \{a, b\} \in [A]^2$ such that $F \cap E = \emptyset$, and let

$$S_F = \{f \in X : f[F] \cap [0, 1/4] \neq \emptyset \text{ and } f[F] \cap [1/2, 1] \neq \emptyset\}.$$

Then $S_F \in \mathcal{N}1$ since F is a support of S_F. Furthermore, S_F is closed in X. Indeed, let $f \in X \setminus S_F$. Then $f[F] \cap [0, 1/4] = \emptyset$ or $f[F] \cap [1/2, 1] = \emptyset$. Assume that the first possibility occurs. Then $U = \pi_a^{-1}((1/4, 1]) \cap \pi_b^{-1}((1/4, 1])$ is an open neighborhood of f in $\mathcal{N}1$ which is disjoint from S_F. If the second possibility occurs, or if both occur, then similarly to the above argument, we may find again an open neighborhood of f in $\mathcal{N}1$ which is disjoint from S_F.

Now we consider the transposition $\pi = (a, b)$ (i.e., π interchanges a and b, and fixes all atoms $u \neq a, b$). Then $\pi \in \mathrm{fix}_G(E)$, and hence $\pi(g) = g$. Furthermore, $\pi(S_F) = S_F$ and we have the following:

$$(S_F, g(S_F)) \in g \Rightarrow \pi((S_F, g(S_F))) \in \pi(g) \Rightarrow (S_F, \pi(g(S_F))) \in g,$$

so $\pi(g(S_F)) = g(S_F)$. However, by the definition of π, it is clear that $\pi(g(S_F)) \neq g(S_F)$, and we have reached a contradiction. Hence, $\mathscr{C}_X$ has no choice function in $\mathcal{N}1$ as required.
(For the reader's further information, we note that an alternate argument can be given along the following lines: Consider $\mathcal{S} = \{S_F : F \in [A]^2\}$ where S_F is defined as above. Then $\mathcal{S} \in \mathcal{N}1$, since it has empty support, and $\mathcal{S} \subseteq \mathscr{C}_X$. If $\mathscr{C}_X$ has a choice function in $\mathcal{N}1$, then so does $\mathcal{S}$. Any choice function for $\mathcal{S}$ clearly yields a choice function for $[A]^2$ which, by Truss [20], implies that A is not amorphous, a contradiction.) ☐

Theorem 3.9. *The statement "Every continuum is weakly Loeb" is not provable in* ZF.

Proof. It suffices to prove that there exists a Fraenkel–Mostowski model of ZFA with a continuum which is not weakly Loeb. Then using the Jech–Sochor First Embedding Theorem (which provides embeddings of arbitrary long initial segments of ZFA-models into symmetric ZF-models, see [11, Theorem 6.1]), one obtains a symmetric model of ZF with the same property as the ZFA-model.

To this end, we start with a model M of $\mathsf{ZFA} + \mathsf{AC}$ with a set A of atoms indexed by spherical coordinates (ρ, θ, ϕ) of points on (the surface of) the unit sphere S^3 in $\mathbb{R}^3$. (Where ρ = distance from the origin $= \sqrt{x^2 + y^2 + z^2}$ (if the point has rectangular coordinates (x, y, z)),

θ is the angle from the positive x-axis to the line segment from $(0,0,0)$ to $(x,y,0)$ and ϕ is the angle from the positive z-axis to the line segment from $(0,0,0)$ to (x,y,z).) So the atoms are indexed by spherical coordinates $(1,\theta,\phi)$, where $0 \le \phi \le \pi$ and $0 \le \theta < 2\pi$. (But it will be useful to allow values of θ outside of this range. See the examples of permutations below.) In addition we will not usually mention the (constant) first coordinate. We will also identify atoms with their coordinates to make the exposition easier.

Assuming that $0 \le c \le \pi$ we will refer to the circle which is the intersection of the sphere and the cone $\phi = c$ as "the circle $\phi = c$".

Let G be the group of permutations ψ of A such that ψ is a continuous one-to-one function from A onto A (using the usual metric topology on S^3) and in addition, if $a = (1,\theta_1,\phi_1) = (\theta_1,\phi_1)$ is any point in A, then $\psi(a)$ has the same last coordinate as a. That is,

$$G = \{\psi : \psi \text{ is a continuous bijection on } A \text{ and there exists } f \text{ such that } \psi(\theta,\phi) = (f(\theta,\phi),\phi)\}$$

(So $\psi \in G$ fixes all of the circles $\phi = c$, $0 \le c \le \pi$.)

Note that for each $\psi \in G$, ψ^{-1} is also a one-to-one continuous function from A onto A since ψ is continuous, one-to-one and onto A and $A = S^3$ is compact. So ψ^{-1} is in G.

Here are some examples of elements of G. Only the third example will be used later in the proof.

$$\psi_0(\theta,\phi) = (\theta+\phi,\phi)$$

(For any fixed c between 0 and π, if $0 \le \theta \le 2\pi$ then $c \le \theta + c \le c + 2\pi$. So ψ restricted to the circle $\phi = c$ is a one-to-one function from the circle onto the circle even though it is not a function from the interval $[0,2\pi]$ to the interval $[0,2\pi]$.)

The second example is the following: for any c between 0 and π let

$$\psi_c(\theta,\phi) = \begin{cases} (\theta,\phi) & \text{if } 0 \le \phi \le c \\ (\theta + (\phi - c),\phi) & \text{if } c < \phi \le \pi \end{cases}.$$

(So ψ_c fixes the circles $\phi = d$ pointwise for $0 \le d \le c$ and moves every point on the circles $\phi = d$ if $c < d < \pi$.)

For the third example, let $a = (\theta_1,\phi_0)$ and $b = (\theta_2,\phi_0)$ be two atoms with the same ϕ coordinate and let c be in the interval $(0,\phi_0)$. Then $\psi_{(a,b,c)}$ defined below will be an element of G which fixes the circles $\phi = d$ pointwise for $0 \le d \le c$ and takes a to b.

$$\psi_{(a,b,c)}(\theta,\phi) = \begin{cases} (\theta,\phi) & \text{if } 0 \le \phi \le c \\ (\theta + (\theta_2 - \theta_1)\dfrac{\phi - c}{\phi_0 - c},\phi) & \text{if } c < \phi \le \pi \end{cases}.$$

The (normal) ideal $\mathcal{I}$ of supports is the ideal of subsets of A generated by sets of the form $U_c = \{(\theta, \phi) : \phi < c\}$ for $0 < c < \pi$. (U for "upper". U_c is the part of the sphere above the circle $\phi = c$.) Let $\mathcal{N}$ be the permutation model determined by M, G, and $\mathcal{I}$.[1]

Define a topology $\mathscr{T}$ on A as follows: $\mathscr{T}$ consists of all sets V which are open in the usual topology on $A = S^3$ and which are in $\mathcal{N}$. It is reasonably clear that $\mathscr{T}$ is a topology in $\mathcal{N}$ and it is not hard to show that under this topology A is connected (since S^3 with the usual topology is connected) and compact (since S^3 is compact).

It is also true that A with the topology $\mathscr{T}$ is Hausdorff: If $a_1 = (\theta_1, \phi_1)$ and $a_2 = (\theta_2, \phi_2)$ are two points in A different from $(0, \pi)$, then there is a $\phi_0 < \pi$ such that $\phi_1 < \phi_0$ and $\phi_2 < \phi_0$. Then we can find disjoint open sets V_1 and V_2 in the usual topology on S^3 such that $a_1 \in V_1$ and $a_2 \in V_2$ and V_1 and V_2 are both contained in U_{ϕ_0}. The sets V_1 and V_2 will then have support U_{ϕ_0} and are therefore in the model. On the other hand, if $a_1 = (0, \pi)$ and $a_2 = (\theta_2, \phi_2)$ with $0 < \phi_2 < \pi$, then we choose c so that $\phi_2 < c < \pi$. Then $\{(\theta, \phi) : c < \phi \leq \pi\}$ and U_c will be disjoint open sets in $\mathscr{T}$ which separate a_1 and a_2, respectively. (Note that U_c is a support of $\{(\theta, \phi) : c < \phi \leq \pi\}$.)

We now show by contradiction that the space $(A, \mathscr{T})$ is not weakly Loeb in $\mathcal{N}$.

Assume that f is a multiple choice function on the non-empty closed subsets of A which is in $\mathcal{N}$ and has support U_c where $0 < c < \pi$. Choose ϕ_0 so that $c < \phi_0 < \pi$ and let x be the circle $\phi = \phi_0$. Let a and b be two points on x chosen so that $a \in f(x)$ and $b \notin f(x)$. Then $a = (\theta_1, \phi_0)$ and $b = (\theta_2, \phi_0)$ for some θ_1 and θ_2 in the interval $[0, 2\pi)$. Define $\psi_{(a,b,c)}$ as in our third example. Then

(1) $\psi_{(a,b,c)}$ fixes x. (Every element of G fixes x.)
(2) $\psi_{(a,b,c)}$ fixes f. (Since $\psi_{(a,b,c)}$ fixes U_c, the support of f.)
(3) $\psi_{(a,b,c)}(f(x)) \neq f(x)$. (Since $\psi_{(a,b,c)}(a) \notin f(x)$ while $a \in f(x)$.)

It follows from items (1) and (2) that $f(x) = x$, but this contradicts (3). □

4. Open questions

Below, we list some open questions on the set-theoretic of the Steprāns–Watson–Just fixed point theorem for continua, as well as of relative statements studied in this paper, which suggest directions for further research on this intriguing topic. Firstly, let us provide some notation for the following two results of [17].

[1] The model $\mathcal{N}$ is introduced for the first time in this paper.

(SWJ-1): If T is an onto J-contraction of a continuum X, then $|X| = 1$.

(SWJ-2): If T is a J-contraction of a continuum X, then T has a unique fixed point.

By Theorem 3.4, we have (SWJ-1) $\rightarrow$ (SWJ-2).

(1) Does (SWJ-2) imply (SWJ-1)?
(2) Is there a model of ZF in which (SWJ-1), or (SWJ-2), is false? Are these statements provable in ZF?
(3) Does $\mathsf{AC}^{\aleph_0}$ imply either of (SWJ-1) and (SWJ-2)?
(4) Does DMC imply "Every continuum is weakly Loeb"? (Recall that "Every continuum is weakly Loeb" does not imply DMC (see the proof of Theorem 3.7), and that DMC is equivalent to "Every compact Hausdorff space is Baire" (see Theorem 2.6(4)).)
(5) What weak choice principles are implied by "Every continuum is Loeb"? Does the latter principle imply $\mathsf{AC}_{\mathsf{fin}}$? (Recall that "Every compact Hausdorff space is Loeb" does imply $\mathsf{AC}_{\mathsf{fin}}$ (Theorem 2.6(2)).)
(6) Is there a model of ZFA, or of ZF, in which every continuum is (weakly) Loeb, but there exists a compact Hausdorff space which is not (weakly) Loeb?

Acknowledgement. We are very thankful to the anonymous referee for several useful comments and suggestions which improved the quality and the exposition of the paper.

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