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ON THE SET-THEORETIC STRENGTH OF A TOPOLOGICAL BANACH FIXED POINT THEOREM FOR CONTINUA

PAUL HOWARD AND ELEFTHERIOS TACHTSIS

Abstract. Juris Steprāns, Stephen Watson and Winfried Just [Canad. Bull. Math. **37** (4) (1994), 552–555] introduced and proved the following fixed point theorem for continua (i.e., for nonempty connected compact Hausdorff spaces): "*If T is a J-contraction of any continuum, then T has a unique fixed point*", using the full power of the Axiom of Choice (AC). We prove that the above result is deducible from each of the following weak choice principles: "Every continuum is weakly Loeb", the Boolean Prime Ideal Theorem, and the Principle of Dependent Multiple Choices, the latter being equivalent to "Every compact Hausdorff space is Baire" (and strictly weaker than the Axiom of Multiple Choice).

We also show that the above fixed point theorem is provable in ZF when restricted to well-ordered continua, and that the general version is deducible—in ZF—from "*If $T : X \to X$ is an onto J-contraction of a continuum X, then $|X| = 1$*".

Furthermore, we prove that "Every continuum is weakly Loeb" is not provable in ZF, and that it is strictly weaker than "Every continuum is Loeb" in ZFA; hence, it is strictly weaker than each of the Boolean Prime Ideal Theorem and the Axiom of Multiple Choice.

The question of whether or not the above fixed point theorem for continua is provable in ZF is still open.

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