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PRESERVATION OF A NEIGHBORHOOD BASE OF A DISCRETE SET BY FORCING

AKIRA IWASA

Abstract. Let $\langle X, \tau \rangle$ be a topological space and let A be a discrete subset of X. We study under what circumstances a neighborhood base of A remains a neighborhood base of A in forcing extensions.

1. Introduction

In [3], we studied the preservation of a neighborhood base of a set by ccc forcings. This note is a sequel to [3] and this time we consider the case where sets are *discrete*; that is, we study under what circumstances a neighborhood base of a discrete set remains a neighborhood base of the set in forcing extensions. Let us repeat some definitions from [3].

Definition 1.1. For a ground model $\mathbf{V}$ and a forcing $\mathbb{P}$, $\mathbf{V}^{\mathbb{P}}$ is the forcing extension of $\mathbf{V}$ by the forcing $\mathbb{P}$. For a topological space $\langle X, \tau \rangle$ in $\mathbf{V}$, $\langle X, \tau^{\mathbb{P}} \rangle$ is the topological space in $\mathbf{V}^{\mathbb{P}}$ such that $\tau^{\mathbb{P}}$ is the topology on X generated by τ in $\mathbf{V}^{\mathbb{P}}$.

Definition 1.2. (1) Let $\langle X, \tau \rangle$ be a topological space and let $A \subseteq X$. We denote the set of all neighborhoods of A by

$$\mathcal{N}_\tau(A) := \{H \subseteq X : (\exists U \in \tau)(A \subseteq U \subseteq H)\}.$$

If $A = \{x\}$, then we write $\mathcal{N}_\tau(x)$ for $\mathcal{N}_\tau(\{x\})$.

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(2) Let $\langle X, \tau \rangle$ be a topological space and let $A \subseteq X$. We say that a family $\mathcal{B}$ of subsets of X is a neighborhood base of A if for each $B \in \mathcal{B}$, the interior of B contains A, and for every open set U containing A, there is $B \in \mathcal{B}$ such that $B \subseteq U$.

(3) Let $\langle X, \tau \rangle$ be a topological space and let $A \subseteq X$. We say that a forcing $\mathbb{P}$ preserves a neighborhood base of A if for some neighborhood base $\mathcal{B}$ of A, $\mathcal{B}$ remains a neighborhood base of A in the space $\langle X, \tau^{\mathbb{P}} \rangle$. Or equivalently, we say that a forcing $\mathbb{P}$ preserves a neighborhood base of A if $\mathcal{N}_\tau(A)$ remains a neighborhood base of A in the space $\langle X, \tau^{\mathbb{P}} \rangle$; that is, a forcing $\mathbb{P}$ preserves a neighborhood base of A if

$$(\forall W \in \tau^{\mathbb{P}} \text{ with } A \subseteq W)(\exists H \in \mathcal{N}_\tau(A))(H \subseteq W).$$

If a forcing $\mathbb{P}$ does not preserve a neighborhood base of A, then we say that $\mathbb{P}$ destroys a neighborhood base of A.

The proposition below follows easily.

Proposition 1.3. *Let $\langle X, \tau \rangle$ be a topological space and let $A \subseteq X$. Let $\mathbb{P}$ be a forcing. The following are equivalent:*

(1) *$\mathbb{P}$ preserves a neighborhood base of A.*

(2) *Every neighborhood base of A remains a neighborhood base of A in the space $\langle X, \tau^{\mathbb{P}} \rangle$.*

A subset A of a space X is called *discrete* if for each $a \in A$, there is an open subset U of X such that and $U \cap A = \{a\}$. The purpose of this note is to study under what circumstances a neighborhood base of a discrete set remains a neighborhood base of the set in forcing extensions. The main result is Corollary 4.16, where we characterize a discrete set whose neighborhood base is preserved by forcings which have the cardinal covering property (Definition 2.1), assuming that the space is hereditarily normal and the Generalized Continuum Hypothesis holds.

2. Cardinal covering property

We work on forcings which satisfy the cardinal covering property (defined below). In what follows, $|A|^{\mathbf{V}}$ is the cardinality of A taken in $\mathbf{V}$, and $|A|^{\mathbf{V}^{\mathbb{P}}}$ is the cardinality of A taken in $\mathbf{V}^{\mathbb{P}}$. (See [5, VII Definition 6.4(1)] for the definition of preserving cardinals.)

Definition 2.1. For a cardinal κ, we say that a forcing $\mathbb{P}$ has the κ-cardinal covering property if

(1) $\mathbb{P}$ preserves cardinals $\leq \kappa$ and

(2) for every $A \in \mathbf{V}^{\mathbb{P}}$ such that $A \subseteq \mathbf{V}$ and $|A|^{\mathbf{V}^{\mathbb{P}}} < \kappa$, there is a set $B \in \mathbf{V}$ such that $A \subseteq B$ and $|A|^{\mathbf{V}^{\mathbb{P}}} = |B|^{\mathbf{V}^{\mathbb{P}}}$.

If $\mathbb{P}$ has the κ-cardinal covering property for every cardinal κ, then we say that $\mathbb{P}$ has the cardinal covering property.

Remark 2.2. Regarding $|B|^{\mathbf{V}^{\mathbb{P}}}$ and $|B|^{\mathbf{V}}$ as ordinals in $\mathbf{V}^{\mathbb{P}}$, we have that $|B|^{\mathbf{V}^{\mathbb{P}}} = |B|^{\mathbf{V}}$ in Definition 2.1(2). This is because if $|B|^{\mathbf{V}^{\mathbb{P}}} < |B|^{\mathbf{V}}$, then $|B|^{\mathbf{V}}$ would not be a cardinal in $\mathbf{V}^{\mathbb{P}}$; since $\mathbb{P}$ preserves cardinals $\leq \kappa$, we have $|B|^{\mathbf{V}} > \kappa$, but, since $|B|^{\mathbf{V}^{\mathbb{P}}} < \kappa$, this would imply that $\mathbb{P}$ does not preserve the cardinal κ.

We prove a lemma below; for the definitions for κ-closedness and κ^+-chain condition, see [5, VII Definition 6.12 and Definition 6.7].

Lemma 2.3. *Let κ be a cardinal and let $\mathbb{P}$ be a forcing.*

(1) *If $\mathbb{P}$ is κ-closed, then $\mathbb{P}$ has the κ-cardinal covering property.*
(2) *If $\mathbb{P}$ is κ-closed and has the κ^+-chain condition, then $\mathbb{P}$ has the cardinal covering property.*

Proof. (1) Suppose that $\mathbb{P}$ is κ-closed. By [5, VII Corollary 6.15], $\mathbb{P}$ preserves cardinals $\leq \kappa$. If $A \in \mathbf{V}^{\mathbb{P}}$, $A \subseteq \mathbf{V}$ and $|A|^{\mathbf{V}^{\mathbb{P}}} < \kappa$, then $A \in \mathbf{V}$ by [5, VII Lemma 6.14]. Therefore, $\mathbb{P}$ has the κ-cardinal covering property.

(2) Suppose that $\mathbb{P}$ is κ-closed and has the κ^+-chain condition. By (1), $\mathbb{P}$ has the κ-cardinal covering property. In particular, $\mathbb{P}$ preserves cardinals $\leq \kappa$. Forcings which have the κ^+-chain condition preserve cardinals $\geq \kappa^+$ ([5, VII Lemma 6.9]). Consequently, $\mathbb{P}$ preserves cardinals. Let $A \in \mathbf{V}^{\mathbb{P}}$ and $A \subseteq \mathbf{V}$. We will find a set B as in Definition 2.1(2). By (1), we may assume that $|A|^{\mathbf{V}^{\mathbb{P}}} \geq \kappa$. Let $|A|^{\mathbf{V}^{\mathbb{P}}} = \lambda$ and enumerate $A = \{a_\xi : \xi < \lambda\}$. By [5, VII Lemma 6.8], there is a family of sets $\{B_\xi : \xi < \lambda\}$ in $\mathbf{V}$ such that for each $\xi < \lambda$, $a_\xi \in B_\xi$ and $|B_\xi|^{\mathbf{V}} \leq \kappa$. Let $B = \bigcup\{B_\xi : \xi < \lambda\}$. Then $B \in \mathbf{V}$, $A \subseteq B$ and $|B|^{\mathbf{V}} \leq \kappa \cdot \lambda = \lambda$. The set B is as required. □

In the proposition below, we consider three types of forcings; see [5, VII Definition 6.4(2)] for the definition of preserving cofinalities.

Proposition 2.4. *In the following, for a forcing $\mathbb{P}$, (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).*

(1) *$\mathbb{P}$ has the cardinal covering property.*
(2) *$\mathbb{P}$ preserves cofinalities.*
(3) *$\mathbb{P}$ preserves cardinals.*

Proof. (2) $\Longrightarrow$ (3) is [5, VII Lemma 5.8]. We will show that (1) implies (2). Suppose that $\mathbb{P}$ has the cardinal covering property. Let κ be a cardinal whose cofinality is λ, so $\lambda \leq \kappa$. Assume to the contrary that in $\mathbf{V}^{\mathbb{P}}$, the cofinaility of κ is less than λ. Let μ be the cofinality of κ in $\mathbf{V}^{\mathbb{P}}$, so $\mu < \lambda$. Then, in $\mathbf{V}^{\mathbb{P}}$, there is a strictly increasing function $f : \mu \to \kappa$ such that $f(\mu)$ is a cofinal subset of κ. Using the cardinal covering property of $\mathbb{P}$, we can find a set $B \in \mathbf{V}$ such that $f(\mu) \subseteq B$ and $|f(\mu)|^{\mathbf{V}^{\mathbb{P}}} = |B|^{\mathbf{V}^{\mathbb{P}}}$, and so $\mu = |B|^{\mathbf{V}^{\mathbb{P}}}$. We may assume that $B \subseteq \kappa$.

By Remark 2.2, we have $|B|^{\mathbf{V}^{\mathbb{P}}} = |B|^{\mathbf{V}}$. B is a cofinal subset of κ such that $|B|^{\mathbf{V}} < \lambda$, contradicting the fact that the cofinality of κ in $\mathbf{V}$ is λ. □

The implications in Proposition 2.4 do not reverse. The forcing in Fact 4.7 preserves cofinalities but does not satisfy the cardinal covering property. The Prikry forcing in Fact 3.6 preserves cardinals but does not preserve cofinalities. We prove in Proposition 2.6 that if there are no large cardinals, then (1), (2) and (3) in Proposition 2.4 are equivalent. To do so, we use the Covering Lemma for the Dodd-Jensen core model.

Theorem 2.5. ([6, Theorem 1.2]) *Assume that there is no inner model with a measurable cardinal. Let $\mathbf{K}$ be the Dodd-Jensen core model. Then for any set $A \subseteq \mathbf{V}$, there is a set $E \in \mathbf{K}$ such that $A \subseteq E$ and $|A| + \aleph_1 = |E|$.*

Proposition 2.6. *Assume that there is no inner model with a measurable cardinal. The following are equivalent:*

(1) $\mathbb{P}$ *has the cardinal covering property.*
(2) $\mathbb{P}$ *preserves cofinalities.*
(3) $\mathbb{P}$ *preserves cardinals.*

Proof. By Proposition 2.4, it suffices to show that (3) implies (1). Let $\mathbb{P}$ be a forcing which preserves cardinals. Then $(\aleph_1)^{\mathbf{V}} = (\aleph_1)^{\mathbf{V}^{\mathbb{P}}}$ so we just write $\aleph_1$. Fix $A \in \mathbf{V}^{\mathbb{P}}$ such that $A \subseteq \mathbf{V}$. We will find a set $B \in \mathbf{V}$ such that $A \subseteq B$ and $|A|^{\mathbf{V}^{\mathbb{P}}} = |B|^{\mathbf{V}^{\mathbb{P}}}$. Applying Theorem 2.5 in $\mathbf{V}^{\mathbb{P}}$, there is a set $E \in \mathbf{K}^{\mathbf{V}^{\mathbb{P}}}$ such that $A \subseteq E$ and

$$|A|^{\mathbf{V}^{\mathbb{P}}} + \aleph_1 = |E|^{\mathbf{V}^{\mathbb{P}}}.$$

Since $\mathbf{K}^{\mathbf{V}^{\mathbb{P}}} = \mathbf{K}$ by [4, Theorem 35.6(iv)], we have $E \in \mathbf{K}$ and so $E \in \mathbf{V}$.

First suppose that $|A|^{\mathbf{V}^{\mathbb{P}}} \geq \aleph_1$; then $|A|^{\mathbf{V}^{\mathbb{P}}} = |E|^{\mathbf{V}^{\mathbb{P}}}$. By letting $B = E$, we are done.

Next suppose that $|A|^{\mathbf{V}^{\mathbb{P}}} = \aleph_0$; then $|E|^{\mathbf{V}^{\mathbb{P}}} = \aleph_1$. Since $\mathbb{P}$ preserves cardinals, we have $|E|^{\mathbf{V}} = \aleph_1$. Let

$$\lambda = \min\{|B|^{\mathbf{V}} : B \in \mathbf{V} \text{ and } A \subseteq B\}.$$

Because of the set E, we have $\lambda \leq \aleph_1$. If $\lambda = \aleph_0$, then there is $B \in \mathbf{V}$ such that $A \subseteq B$ and $|B|^{\mathbf{V}} = \aleph_0$. Therefore, $|A|^{\mathbf{V}^{\mathbb{P}}} = |B|^{\mathbf{V}^{\mathbb{P}}}$ and we are done. Now we assume that $\lambda = \aleph_1$ and find a contradiction. Take $B \in \mathbf{V}$ such that $A \subseteq B$ and $|B|^{\mathbf{V}} = \aleph_1$. Enumerate $A = \{a_n : n < \omega\}$ and $B = \{b_\xi : \xi < \omega_1\}$. For each $n \in \omega$, choose $\xi_n < \omega_1$ such that $a_n = b_{\xi_n}$. Let $\gamma = \sup\{\xi_n : n < \omega\}$, and let $B' = \{b_\xi : \xi \leq \gamma\}$. Then $B' \in \mathbf{V}$, $A \subseteq B'$ and $|B'|^{\mathbf{V}} = \aleph_0$. This contradicts the fact that $\lambda = \aleph_1$. □

3. Case where $\varphi(a) = \kappa$ for all $a \in A$

We define a cardinal function which gives the minimum cardinality of a family of neighborhoods of a point whose intersection is not a neighborhood of the point.

Definition 3.1. Let $\langle X, \tau\rangle$ be a T_1 space. Define a cardinal function φ on X such that for a non-isolated point $x \in X$,

$$\varphi(x) = \min\left\{|\mathcal{U}| : \mathcal{U} \subseteq \mathcal{N}_\tau(x) \text{ and } \bigcap \mathcal{U} \notin \mathcal{N}_\tau(x)\right\};$$

for an isolated point $x \in X$, $\varphi(x)$ is undefined.

Remark 3.2. Let $\langle X, \tau\rangle$ be a T_1 space. Then for each $x \in X$, $\bigcap \mathcal{N}_\tau(x) = \{x\}$. Therefore, for a non-isolated point x, $\bigcap \mathcal{N}_\tau(x) \notin \mathcal{N}_\tau(x)$, and so $\varphi(x)$ is well-defined.

We study the preservation of a neighborhood base of a discrete set A such that for some cardinal κ, $\varphi(a) = \kappa$ for each non-isolated point $a \in A$. The main results are Proposition 3.5 and Proposition 3.11. It is straightforward to prove the lemma below.

Lemma 3.3. *Let $\langle X, \tau\rangle$ be a T_1 space. For each non-isolated point $x \in X$, $\varphi(x)$ is a regular cardinal.*

According to the lemma below, we do not have to consider isolated points when studying preservation of a neighborhood base.

Lemma 3.4. *Let $\langle X, \tau\rangle$ be a topological space and let $A \subseteq X$. Let I be the set of all isolated points in X. If a forcing $\mathbb{P}$ preserves a neighborhood base of $A \setminus I$, then $\mathbb{P}$ preserves a neighborhood base of A.*

Proof. Take $W \in \tau^{\mathbb{P}}$ such that $A \subseteq W$; we need to find $H \in \mathcal{N}_\tau(A)$ such that $H \subseteq W$. Since $\mathbb{P}$ preserves a neighborhood base of $A \setminus I$, there is $U \in \mathcal{N}_\tau(A \setminus I)$ such that $U \subseteq W$. Let $H = U \cup (A \cap I)$; then $H \in \mathcal{N}_\tau(A)$ and $H \subseteq W$. □

Let us look at a proposition below. We are not assuming that a set A is discrete in this proposition.

Proposition 3.5. *Let $\langle X, \tau\rangle$ be a T_1 space and let I be the set of all isolated points in X. Let $A \subseteq X$ be such that for some cardinal κ,*

(1) $|A \setminus I| < \kappa$ *and*
(2) $(\forall a \in A \setminus I)(\varphi(a) = \kappa)$.

Then any forcing that has the κ-cardinal covering property preserves a neighborhood base of A.

Proof. By Lemma 3.4, we may assume that A consists of non-isolated points of X. Let $\mathbb{P}$ be a forcing which satisfies the κ-cardinal covering property. Take $W \in \tau^{\mathbb{P}}$ such that $A \subseteq W$; we will find $H \in \mathcal{N}_\tau(A)$ such that $H \subseteq W$. Enumerate $A = \{\alpha_\xi : \xi < \lambda\}$; by the hypothesis, we have $\lambda < \kappa$. Since forcing preserves a neighborhood base of a point ([3, Proposition 1.5]), we can find $U_\xi \in \mathcal{N}_\tau(a_\xi)$ for each $\xi < \lambda$ such that $U_\xi \subseteq W$. Note that the family $\{U_\xi : \xi < \lambda\}$ may not be in $\mathbf{V}$. Therefore, we use the fact that $\mathbb{P}$ has the κ-cardinal covering property and find a set $\mathcal{H} \in \mathbf{V}$ such that $|\mathcal{H}|^{\mathbf{V}} = \lambda$ and

$$\{U_\xi : \xi < \lambda\} \subseteq \mathcal{H}.$$

For each $\xi < \lambda$, let

$$H_\xi = \bigcap(\mathcal{H} \cap \mathcal{N}_\tau(a_\xi)).$$

Since $\varphi(a_\xi) = \kappa > \lambda$ and $|\mathcal{H}|^{\mathbf{V}} = \lambda$, we have $H_\xi \in \mathcal{N}_\tau(a_\xi)$ for each $\xi < \lambda$. Because $U_\xi \in \mathcal{H} \cap \mathcal{N}_\tau(a_\xi)$, we have $H_\xi \subseteq U_\xi$ for each $\xi < \lambda$. Let

$$H = \bigcup\{H_\xi : \xi < \lambda\}.$$

The set H is in $\mathbf{V}$, and so $H \in \mathcal{N}_\tau(A)$. Since $H_\xi \subseteq U_\xi \subseteq W$ for each $\xi < \lambda$, we have $H \subseteq W$. Thus, $\mathbb{P}$ preserves a neighborhood base of A. $\square$

Proposition 3.5 leads us to a question: What if $|A \setminus I| \geq \kappa$? For the case where $|A \setminus I| = \kappa$, see Proposition 3.11, and for the case where $|A \setminus I| > \kappa$, see the paragraph after Proposition 3.11.

We give an example (Example 3.7) which shows that the condition that a forcing satisfies the κ-cardinal covering property in Proposition 3.5 cannot be weakened to the condition that a forcing preserves cardinals. This example is the reason why we focus on forcings which have the cardinal covering property instead of forcings which preserve cardinals. (Example 4.8 shows why we do not focus on forcings which preserve cofinalities.) Because of Proposition 2.6, we need to assume the existence of a measurable cardinal to construct such an example. We use the Prikry forcing.

Fact 3.6. Let κ be a measurable cardinal. The Prikry forcing $\mathbb{P}_r(\kappa)$ preserves cardinals and adds a countable set which is cofinal in κ ([4, Theorem 21.10]).

Example 3.7. Assume that there is a measurable cardinal κ. There are a space $\langle X, \tau \rangle$, a subset A of X, and a forcing $\mathbb{P}$ such that for the set I of all isolated points in X,

(1) $|A \setminus I| < \kappa$,
(2) $(\forall a \in A \setminus I)(\varphi(a) = \kappa)$, and
(3) $\mathbb{P}$ preserves cardinals and destroys a neighborhood base of A.

Proof. Let κ be a measurable cardinal. Let

- $X = \omega \times (\kappa + 1)$ and
- $A = \omega \times \{\kappa\}$.

Let the topology τ on X be the usual product topology, and we consider the order topology on ω, κ, and $\kappa + 1$. The set A is a countable subset of X and for each $a \in A$, $\varphi(a) = \kappa$; there is no isolated point in A. Let $\mathbb{P}_r(\kappa)$ be the Prikry forcing in Fact 3.6 and let $\{\alpha_n : n \in \omega\}$ be a countable set in $\mathbf{V}^{\mathbb{P}_r(\kappa)}$ which is cofinal in κ. Let $E = \{(n, \alpha_n) : n \in \omega\}$. It is easy to see that E is a closed subset of X. Let $W = X \setminus E$; then $W \in \tau^{\mathbb{P}_r(\kappa)}$ and $A \subseteq W$. In order to show that W witnesses the fact that $\mathbb{P}_r(\kappa)$ destroys a neighborhood base of A, take $H \in \mathcal{N}_\tau(A)$; we will show that $H \nsubseteq W$. In $\mathbf{V}$, for each $n \in \omega$, let

$$\beta_n = \min\{\beta < \kappa : (\forall \xi \geq \beta)((n, \xi) \in H)\}.$$

If, on the contrary, we had $H \subseteq W$, then $H \cap E = \emptyset$, and so $\alpha_n < \beta_n$ for all $n \in \omega$. This would imply that $\{\beta_n : n \in \omega\}$ is cofinal in κ, which contradicts the fact that $\{\beta_n : n \in \omega\}$ is in $\mathbf{V}$. Thus, we must have $H \nsubseteq W$. □

The lemma below says that we can expand a discrete set to a pairwise disjoint family of open sets under certain conditions.

Lemma 3.8. *Let $\langle X, \tau \rangle$ be a regular space. Let A be a discrete subset of X such that for some regular cardinal κ,*

(1) *$|A| = \kappa$ and*
(2) *$(\forall a \in A)(\varphi(a) = \kappa)$.*

Enumerate $A = \{a_\xi : \xi < \kappa\}$. Then there is a pairwise disjoint family of open sets $\{U_\xi : \xi < \kappa\}$ such that $a_\xi \in U_\xi$ for each $\xi < \kappa$.

Proof. By induction on $\gamma < \kappa$, we will define a pairwise disjoint family of open sets $\{U_\xi : \xi < \gamma\}$ such that for each $\xi < \gamma$, $\overline{U_\xi} \cap A = \{a_\xi\}$. Using the regularity of the space X, take an open set U_0 such that $\overline{U_0} \cap A = \{a_0\}$. Suppose that we have defined U_ξ for all $\xi < \gamma$. We will define U_γ. For each $\xi < \gamma$, take an open set V_ξ such that

- $\overline{V_\xi} \cap A = \{a_\gamma\}$ and
- $V_\xi \cap U_\xi = \emptyset$.

Let $H = \bigcap\{V_\xi : \xi < \gamma\}$; then $H \in \mathcal{N}_\tau(a_\gamma)$ because $\varphi(a_\gamma) = \kappa > \gamma$. We also have $H \cap U_\xi = \emptyset$ for all $\xi < \gamma$. Take an open set U_γ such that $a_\gamma \in U_\gamma \subseteq H$. Then U_γ is as required. □

We use the following forcing to prove Proposition 3.11.

Fact 3.9. Let κ be a regular cardinal. $Fn(\kappa,\kappa,\kappa)$ is the set of all partial functions from κ to κ whose cardinality is less than κ ([5, VII Definition 6.1]). For an $Fn(\kappa,\kappa,\kappa)$-generic filter G, let $g = \bigcup G$. Then for every $f \in {}^{\kappa}\kappa \cap \mathbf{V}$, $|\{\xi \in \kappa : f(\xi) < g(\xi)\}| = \kappa$.

Remark 3.10. $Fn(\kappa,\kappa,\kappa)$ is κ-closed so it has the κ-cardinal covering property by Lemma 2.3(1). If $2^{<\kappa} = \kappa$, then $|Fn(\kappa,\kappa,\kappa)| = \kappa$, so $Fn(\kappa,\kappa,\kappa)$ satisfies the κ^+-chain condition. Therefore, if $2^{<\kappa} = \kappa$, then $Fn(\kappa,\kappa,\kappa)$ has the cardinal covering property by Lemma 2.3(2).

Using Lemma 3.8, we prove the proposition below, where a forcing destroys a neighborhood base of a discrete set. We assume that a space is regular (see Example 3.12 for the reason for this assumption).

Proposition 3.11. *Let $\langle X, \tau\rangle$ be a regular space and let I be the set of all isolated points in X. Let A be a discrete subset of X such that for some regular cardinal κ,*

(1) $|A \setminus I| = \kappa$ *and*
(2) $(\forall a \in A \setminus I)(\varphi(a) = \kappa)$.

Then there is a forcing $\mathbb{P}$ such that $\mathbb{P}$ has the κ-cardinal covering property and destroys a neighborhood base of A. If $2^{<\kappa} = \kappa$, then $\mathbb{P}$ has the cardinal covering property.

Proof. By Lemma 3.4, we may assume that the set A consists of non-isolated points of X. Enumerate $A = \{a_\xi : \xi < \kappa\}$. For each $\xi < \kappa$, using the fact that $\varphi(a_\xi) = \kappa$, choose a family of open sets $\{U_{\xi,i} : i < \kappa\}$ such that

- for each $i < \kappa$, $a_\xi \in U_{\xi,i}$,
- if $i < j < \kappa$, then $U_{\xi,i} \supseteq U_{\xi,j}$, and
- $\bigcap\{U_{\xi,i} : i < \kappa\} \notin \mathcal{N}_\tau(a_\xi)$.

By Lemma 3.8, we may assume that the family $\{U_{\xi,0} : \xi < \kappa\}$ is pairwise disjoint. Let $\mathbb{P} = Fn(\kappa,\kappa,\kappa)$ as in Fact 3.9. By Remark 3.10, $\mathbb{P}$ has the κ-cardinal covering property and if $2^{<\kappa} = \kappa$, then $\mathbb{P}$ has the cardinal covering property. For a $\mathbb{P}$-generic filter G, let $g = \bigcup G$, and let

$$W = \bigcup\{U_{\xi,g(\xi)} : \xi < \kappa\}.$$

Then $A \subseteq W$ and $W \in \tau^{\mathbb{P}}$. In order to show that W witnesses the fact that $\mathbb{P}$ destroys a neighborhood base of A, take $H \in \mathcal{N}_\tau(A)$; we will show that $H \nsubseteq W$. Define $f \in {}^{\kappa}\kappa \cap \mathbf{V}$ such that

$$f(\xi) = \sup\{i < \kappa : H \cap U_{\xi,0} \subseteq U_{\xi,i}\}.$$

For some $i < \kappa$, we have $H \cap U_{\xi,0} \nsubseteq U_{\xi,i}$ because $\bigcap\{U_{\xi,i} : i < \kappa\} \notin \mathcal{N}_\tau(a_\xi)$. Therefore, $f(\xi) < \kappa$ for each $\xi < \kappa$. If, on the contrary, we had $H \subseteq W$, then $H \cap U_{\xi,0} \subseteq U_{\xi,g(\xi)}$ for all $\xi < \kappa$, and so $f(\xi) \geq g(\xi)$ for all $\xi < \kappa$.

This contradicts Fact 3.9. Therefore, we must have $H \nsubseteq W$. Thus, $\mathbb{P}$ destroys a neighborhood base of A. □

We do not know if there is a forcing which preserves the cardinal $|A \setminus I|$ and destroys a neighborhood base of A when $|A \setminus I| > \kappa$ (Question 4.17). We give an example which shows that the regularity of the space X in Proposition 3.11 cannot be weakened to Hausdorffness.

Example 3.12. There are a Hausdorff (non-regular) space $\langle X, \tau \rangle$ and a discrete subset A of X such that for the set I of all isolated points in X,

(1) $|A \setminus I| = \aleph_0$,
(2) $(\forall a \in A \setminus I)(\varphi(a) = \aleph_0)$, and
(3) every forcing preserves a neighborhood base of A.

Proof. See the proof of [3, Example 2.6]. The example where $X = \mathbb{R}$ and $A = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ satisfies the requirements. □

4. General case

In this section, we look at a case where A is any discrete subset of X. The main results are Theorem 4.4 and Theorem 4.15. It is straightforward to prove the lemma below.

Lemma 4.1. *Let $\langle X, \tau \rangle$ be a topological space, and let $A_0 \subseteq X$ and $A_1 \subseteq X$. If a forcing $\mathbb{P}$ preserves both a neighborhood base of A_0 and a neighborhood base of A_1, then $\mathbb{P}$ preserves a neighborhood base of $A_0 \cup A_1$.*

The following definition is crucial to our study. Using the cardinal function φ in Definition 3.1, we decompose a subset A of X.

Definition 4.2. Let $\langle X, \tau \rangle$ be a T_1 space, let $A \subseteq X$ and let I be the set of all isolated points in X. Define

- $\{A_\xi \subseteq A : \xi < \delta\}$ and
- $\{\kappa_\xi : \xi < \delta\}$

such that

(1) $A \setminus I = \bigcup \{A_\xi : \xi < \delta\}$,
(2) $A_\xi = \{a \in A : \varphi(a) = \kappa_\xi\}$ for each $\xi < \delta$,
(3) $(\forall \xi < \delta)(\exists a \in A)(\varphi(a) = \kappa_\xi)$, and
(4) if $\xi < \eta < \delta$, then $\kappa_\xi < \kappa_\eta$.

Remark 4.3. $\{A_\xi : \xi < \delta\}$ is a pairwise disjoint family of subsets of A. $\{\kappa_\xi : \xi < \delta\}$ is a set of regular cardinals by Lemma 3.3.

$\{A_\xi : \xi < \delta\}$ and $\{\kappa_\xi : \xi < \delta\}$ give sufficient conditions for a neighborhood base of A to be preserved by forcing in the theorem below. We are not assuming that a set A is discrete.

Theorem 4.4. *Let $\langle X, \tau \rangle$ be a T_1 space and let $A \subseteq X$. Let $\{A_\xi : \xi < \delta\}$ and $\{\kappa_\xi : \xi < \delta\}$ be as in Definition 4.2. Suppose that*

(1) *for each $\xi < \delta$, $|A_\xi| < \kappa_\xi$ and*
(2) *for each regular cardinal $\lambda \leq \delta$, $\sup\{\kappa_\xi : \xi < \lambda\} > \lambda$.*

Then any forcing that has the cardinal covering property preserves a neighborhood base of A.

Proof. By Lemma 3.4, we may assume that the set A consists of non-isolated points of X, so $A = \bigcup\{A_\xi : \xi < \delta\}$. For $\alpha < \beta \leq \delta$, define

$$A(\alpha, \beta) := \bigcup\{A_\xi : \alpha \leq \xi < \beta\}.$$

In this notation, $A = A(0, \delta)$. Let $\mathbb{P}$ be a forcing which satisfies the cardinal covering property. By induction on $\gamma \leq \delta$, we will show that for every $\alpha < \gamma$, $\mathbb{P}$ preserves a neighborhood base of $A(\alpha, \gamma)$. By Proposition 3.5, $\mathbb{P}$ preserves a neighborhood base of $A_0 = A(0, 1)$. Here is the setting for the inductive step:

- Fix $\gamma \leq \delta$.
- Fix $\alpha < \gamma$.
- Suppose that for every ξ and η with $\xi < \eta < \gamma$, $\mathbb{P}$ preserves a neighborhood base of $A(\xi, \eta)$.
- We will show that $\mathbb{P}$ preserves a neighborhood base of $A(\alpha, \gamma)$.

Case 1: $\sup\{\kappa_\xi : \xi < \gamma\} > \gamma$.
Let

$$\beta = \{\xi < \gamma : \kappa_\xi \leq \gamma\}.$$

We have that $\beta < \gamma$ and that if $\beta \leq \xi < \gamma$, then $\kappa_\xi > \gamma$.

Subcase 1.1: $\alpha \geq \beta$.
Take $W \in \tau^{\mathbb{P}}$ such that $A(\alpha, \gamma) \subseteq W$; we will find $H \in \mathcal{N}_\tau(A(\alpha, \gamma))$ such that $H \subseteq W$. By Proposition 3.5, $\mathbb{P}$ preserves a neighborhood base of A_ξ for each $\xi < \gamma$, so we can find $U_\xi \in \mathcal{N}_\tau(A_\xi)$ so that $U_\xi \subseteq W$ for each $\xi < \gamma$. Note that the family $\{U_\xi : \alpha \leq \xi < \gamma\}$ may not be in $\mathbf{V}$. Therefore, we use the fact that $\mathbb{P}$ satisfies the cardinal covering property and find a set $\mathcal{H} \in \mathbf{V}$ such that $|\mathcal{H}|^{\mathbf{V}} = |\gamma|^{\mathbf{V}}$ and

$$\{U_\xi : \alpha \leq \xi < \gamma\} \subseteq \mathcal{H}.$$

Write $[\alpha, \gamma) := \{\xi : \alpha \leq \xi < \gamma\}$. For each $\xi \in [\alpha, \gamma)$, let

$$H_\xi = \bigcap(\mathcal{H} \cap \mathcal{N}_\tau(A_\xi)).$$

For each $\xi \in [\alpha, \gamma)$, we have $\xi \geq \beta$, and so $\kappa_\xi > \gamma$ by the definition of β.

Therefore, $\kappa_\xi > |\mathcal{H}|^{\mathbf{V}}$ for each $\xi \in [\alpha, \gamma)$. Since $\varphi(a) = \kappa_\xi$ for each $a \in A_\xi$, we have $H_\xi \in \mathcal{N}_\tau(A_\xi)$ for each $\xi \in [\alpha, \gamma)$. Since $U_\xi \in \mathcal{H} \cap \mathcal{N}_\tau(A_\xi)$ for each $\xi \in [\alpha, \gamma)$, we have $H_\xi \subseteq U_\xi$ for each $\xi \in [\alpha, \gamma)$. Let

$$H = \bigcup \{H_\xi : \alpha \leq \xi < \gamma\}.$$

The set H is in $\mathbf{V}$, and so $H \in \mathcal{N}_\tau(A(\alpha, \gamma))$. Since $H_\xi \subseteq U_\xi \subseteq W$ for each $\xi \in [\alpha, \gamma)$, we have $H \subseteq W$. Thus, $\mathbb{P}$ preserves a neighborhood base of $A(\alpha, \gamma)$.

Subcase 1.2: $\alpha < \beta$.
By Subcase 1.1, $\mathbb{P}$ preserves a neighborhood base of $A(\beta, \gamma)$. By the induction hypothesis, $\mathbb{P}$ preserves a neighborhood base of $A(\alpha, \beta)$. Since $A(\alpha, \gamma) = A(\alpha, \beta) \cup A(\beta, \gamma)$, $\mathbb{P}$ preserves a neighborhood base of $A(\alpha, \gamma)$ by Lemma 4.1.

Case 2: $\sup\{\kappa_\xi : \xi < \gamma\} = \gamma$.
Since γ is the supremum of cardinals, γ is a cardinal. By the condition (2), γ is a singular cardinal. Let $cf(\gamma)$ denote the cofinality of γ; then $cf(\gamma) < \gamma$. For each $i < cf(\gamma)$, choose κ_{ξ_i} in strictly increasing order so that

$$\sup\{\kappa_{\xi_i} : i < cf(\gamma)\} = \gamma.$$

Take $i_0 < cf(\gamma)$ so that $\alpha < \kappa_{\xi_{i_0}}$. By the induction hypothesis, $\mathbb{P}$ preserves a neighborhood base of $A(\alpha, \kappa_{\xi_{i_0}})$. Since $A(\alpha, \gamma) = A(\alpha, \kappa_{\xi_{i_0}}) \cup A(\kappa_{\xi_{i_0}}, \gamma)$, it suffices to show by Lemma 4.1 that $\mathbb{P}$ preserves a neighborhood base of

$$A(\kappa_{\xi_{i_0}}, \gamma).$$

Let

$$j_0 = \{i < cf(\gamma) : \kappa_{\xi_i} \leq cf(\gamma)\}.$$

We have that $j_0 < cf(\gamma)$ and that if $j_0 \leq i < cf(\gamma)$, then $\kappa_{\xi_i} > cf(\gamma)$.

Subcase 2.1: $i_0 \geq j_0$. (This case is similar to Subcase 1.1.)
Take $W \in \tau^{\mathbb{P}}$ such that $A(\kappa_{\xi_{i_0}}, \gamma) \subseteq W$; we will find $H \in \mathcal{N}_\tau(A(\kappa_{\xi_{i_0}}, \gamma))$ such that $H \subseteq W$. For each $i < cf(\gamma)$, $\mathbb{P}$ preserves a neighborhood base of $A(\kappa_{\xi_i}, \kappa_{\xi_{i+1}})$ by the induction hypothesis. So, for each $i \geq i_0$, we can find $U_i \in \mathcal{N}_\tau(A(\kappa_{\xi_i}, \kappa_{\xi_{i+1}}))$ such that $U_i \subseteq W$. Note that the family $\{U_i : i_0 \leq i < cf(\gamma)\}$ may not be in $\mathbf{V}$. Therefore, we use the fact that $\mathbb{P}$ satisfies the cardinal covering property and find a set $\mathcal{H} \in \mathbf{V}$ such that $|\mathcal{H}|^{\mathbf{V}} = cf(\gamma)$ and

$$\{U_i : i_0 \leq i < cf(\gamma)\} \subseteq \mathcal{H}.$$

Write $[i_0, cf(\gamma)) := \{i : i_0 \leq i < cf(\gamma)\}$. For each $i \in [i_0, cf(\gamma))$, let

$$H_i = \bigcap (\mathcal{H} \cap \mathcal{N}_\tau(A(\kappa_{\xi_i}, \kappa_{\xi_{i+1}}))).$$

For each $i \in [i_0, cf(\gamma))$, we have $i \geq j_0$, and so $\kappa_{\xi_i} > cf(\gamma)$ by the definition of j_0. Therefore, $\kappa_{\xi_i} > |\mathcal{H}|^{\mathbf{V}}$ for each $i \in [i_0, cf(\gamma))$. Since $\varphi(a) \geq \kappa_{\xi_i}$ for each $a \in A(\kappa_{\xi_i}, \kappa_{\xi_{i+1}})$, we have $H_i \in \mathcal{N}_\tau(A(\kappa_{\xi_i}, \kappa_{\xi_{i+1}}))$ for each $i \in [i_0, cf(\gamma))$. Since $U_i \in \mathcal{H} \cap \mathcal{N}_\tau(A(\kappa_{\xi_i}, \kappa_{\xi_{i+1}}))$, we have $H_i \subseteq U_i$ for each $i \in [i_0, cf(\gamma))$. Let

$$H = \bigcup\{H_i : i_0 \leq i < cf(\gamma)\}.$$

The set H is in $\mathbf{V}$, and so $H \in \mathcal{N}_\tau(A(\kappa_{\xi_{i_0}}, \gamma))$. Since $H_i \subseteq U_i \subseteq W$ for each $i \in [i_0, cf(\gamma))$, we have $H \subseteq W$. Thus, $\mathbb{P}$ preserves a neighborhood base of $A(\kappa_{\xi_{i_0}}, \gamma)$.

Subcase 2.2: $i_0 < j_0$.
By Subcase 2.1, $\mathbb{P}$ preserves a neighborhood base of $A(\kappa_{\xi_{j_0}}, \gamma)$. By the induction hypothesis, $\mathbb{P}$ preserves a neighborhood base of $A(\kappa_{\xi_{i_0}}, \kappa_{\xi_{j_0}})$. Since $A(\kappa_{\xi_{i_0}}, \gamma) = A(\kappa_{\xi_{i_0}}, \kappa_{\xi_{j_0}}) \cup A(\kappa_{\xi_{j_0}}, \gamma)$, $\mathbb{P}$ preserves a neighborhood base of $A(\kappa_{\xi_{i_0}}, \gamma)$ by Lemma 4.1. □

Remark 4.5. Since $\kappa_\xi \geq \xi$ for all $\xi < \lambda$ by Definition 4.2(4), we have $\sup\{\kappa_\xi : \xi < \lambda\} \geq \lambda$. If $\sup\{\kappa_\xi : \xi < \lambda\} = \lambda$, then λ is a regular limit cardinal, and so it is a weakly inaccessible cardinal. Consequently, if there is no weakly inaccessible cardinal, then the condition (2) in Theorem 4.4 holds.

By Remark 4.5, we have a corollary.

Corollary 4.6. *Assume that there is no weakly inaccessible cardinal. Let $\langle X, \tau\rangle$ be a T_1 space and let $A \subseteq X$. Let $\{A_\xi : \xi < \delta\}$ and $\{\kappa_\xi : \xi < \delta\}$ be as in Definition 4.2. If $|A_\xi| < \kappa_\xi$ for each $\xi < \delta$, then any forcing that has the cardinal covering property preserves a neighborhood base of A.*

We give an example (Example 4.8) which shows that the condition that a forcing has the cardinal covering property in Theorem 4.4 cannot be weakened to the condition that a forcing preserves cofinalities (Proposition 2.4). Because of Proposition 2.6, we have to assume the existence of a measurable cardinal to construct such an example. We use a forcing defined by M. Gotik.

Fact 4.7. Let $\{\kappa_n : n \in \omega\}$ be an increasing sequence of measurable cardinals and let $\lambda = \sup\{\kappa_n : n \in \omega\}$. Let

$$\mathcal{F} = \{f \in {}^\omega\lambda : (\forall n \in \omega)(f(n) \in \kappa_n)\}.$$

Then there is a forcing $\mathbb{P}$ such that $\mathbb{P}$ preserves cofinalities and, in $\mathbf{V}^{\mathbb{P}}$, there is a function $g \in \mathcal{F} \cap \mathbf{V}^{\mathbb{P}}$ such that for each $f \in \mathcal{F} \cap \mathbf{V}$, $f(n) < g(n)$ for all but finitely many $n \in \omega$ ([2, Theorem 1.38]).

Example 4.8. Suppose that there are infinitely many measurable cardinals. Then there are a T_1 space $\langle X, \tau\rangle$, a subset A of X and a forcing $\mathbb{P}$ such that

(1) $\{A_\xi : \xi < \lambda\}$ and $\{\kappa_\xi : \xi < \lambda\}$ satisfy the hypothesis of Theorem 4.4 and
(2) $\mathbb{P}$ preserves cofinalities and destroys a neighborhood base of A.

Proof. Let $\{\kappa_n : n \in \omega\}$ be an increasing sequence of measurable cardinals. Let

- $X = \bigcup\{\{n\} \times (\kappa_n + 1) : n \in \omega\}$ and
- $A = \{(n, \kappa_n) : n \in \omega\}$.

Let us define the topology τ on X. For each $n < \omega$ and $\xi < \kappa_n$, (n, ξ) is an isolated point; for each $n < \omega$, a basic open neighborhood of (n, κ_n) has the form $\{(n, \xi) : \gamma < \xi \le \kappa_n\}$ for some $\gamma < \kappa_n$. Let φ be the cardinal function in Definition 3.1. For each $n \in \omega$, $\varphi((n, \kappa_n)) = \kappa_n$. For each $n \in \omega$, let $A_n = \{(n, \kappa_n)\}$. $\{A_n : n < \omega\}$ and $\{\kappa_n : n < \omega\}$ are as in Definition 4.2, and they satisfy the hypothesis of Theorem 4.4. Let $\mathcal{F}$, $\mathbb{P}$ and g be as in Fact 4.7. For each $f \in \mathcal{F} \cap \mathbf{V}$, define

$$U_f := A \cup \{(n, \xi) : n < \omega \text{ and } f(n) < \xi < \kappa_n\}.$$

Then $\mathcal{N}_\tau(A) = \{H \subseteq X : (\exists f \in \mathcal{F} \cap \mathbf{V})(U_f \subseteq H)\}$. Let $E = \{(n, g(n)) : n \in \omega\}$; then E is a closed subset of X, $E \cap A = \emptyset$ and $E \cap U_f \neq \emptyset$ for all $f \in \mathcal{F} \cap \mathbf{V}$. Let $W = X \setminus E$; then $W \in \tau^{\mathbb{P}}$, $A \subseteq W$ and $U_f \nsubseteq W$ for all $f \in \mathcal{F} \cap \mathbf{V}$. Therefore, $H \nsubseteq W$ for all $H \in \mathcal{N}_\tau(A)$. Thus, $\mathcal{N}_\tau(A)$ is not a neighborhood base of A in the space $\langle X, \tau^{\mathbb{P}}\rangle$. □

We use the following forcing in Theorem 4.15 to obtain a partial converse to Theorem 4.4.

Definition 4.9. Let λ be a regular cardinal and let $\{\kappa_\xi : \xi < \lambda\}$ be such that

(1) κ_ξ is a regular cardinal for each $\xi < \lambda$,
(2) if $\xi < \eta < \lambda$, then $\kappa_\xi < \kappa_\eta$, and
(3) $\sup\{\kappa_\xi : \xi < \lambda\} = \lambda$.

Define the forcing $\mathbb{Q}$ such that p is a member of $\mathbb{Q}$ if

- p is a function,
- $dom(p) \in \lambda$, where $dom(p)$ is the domain of p, and
- $p(\xi) \in \kappa_\xi$ for each $\xi \in dom(p)$.

Order $\mathbb{Q}$ by: $p \le q$ iff $p \supseteq q$ and $p \restriction dom(q) = q$.

Remark 4.10. The forcing $\mathbb{Q}$ is λ-closed. If $2^{<\lambda} = \lambda$, then $|\mathbb{Q}| = \lambda$, so $\mathbb{Q}$ has the λ^+-chain condition. Therefore, if $2^{<\lambda} = \lambda$, then $\mathbb{Q}$ has the cardinal covering property by Lemma 2.3(2).

One can prove the proposition below by a standard density argument.

Proposition 4.11. *Let $\mathbb{Q}$ be the forcing in Definition 4.9. Let*
$$\mathcal{F} = \{f \in {}^{\lambda}\lambda : (\forall \xi < \lambda)(f(\xi) < \kappa_\xi)\}.$$
For a $\mathbb{Q}$-generic filter G, let $g = \bigcup G$. Then $g \in \mathcal{F} \cap \mathbf{V}^{\mathbb{P}}$ and for every $f \in \mathcal{F} \cap \mathbf{V}$, $|\{\xi \in \lambda : f(\xi) < g(\xi)\}| = \lambda$.

We use the lemma below.

Lemma 4.12. *Let $\langle X, \tau \rangle$ be a hereditarily normal space and let A be a discrete subset of X. For every $B \subseteq A$, there are open sets U and V such that $B \subseteq U$, $A \setminus B \subseteq V$ and $U \cap V = \emptyset$.*

Proof. Since A is discrete, for every $B \subseteq A$, $\overline{B} \cap (A \setminus B) = \emptyset$ and $B \cap \overline{A \setminus B} = \emptyset$. Apply [1, Theorem 2.1.7]. □

If a space is hereditarily normal, then we can separate the family $\{A_\xi : \xi < \delta\}$ by disjoint open sets.

Lemma 4.13. *Let $\langle X, \tau \rangle$ be a hereditarily normal space and let A be a discrete subset of X. Let $\{A_\xi : \xi < \delta\}$ and $\{\kappa_\xi : \xi < \delta\}$ be as in Definition 4.2. Suppose that for each regular cardinal $\lambda < \delta$, $\sup\{\kappa_\xi : \xi < \lambda\} > \lambda$. Then there is a pairwise disjoint family of open sets $\{U_\xi : \xi < \delta\}$ such that $A_\xi \subseteq U_\xi$ for each $\xi < \delta$.*

Proof. By induction on $\gamma < \delta$, we will define
(1) a pairwise disjoint family of open sets $\{U_\xi : \xi < \gamma\}$ such that $A_\xi \subseteq U_\xi$ for each $\xi < \gamma$;
(2) a family of open sets $\{V_\xi : \xi < \gamma\}$ such that
- if $\xi < \eta < \delta$, then $V_\xi \supseteq V_\eta$,
- for each $\xi < \gamma$, $\bigcup_{i > \xi} A_i \subseteq V_\xi$, and
- for each $\xi < \gamma$, $V_\xi \cap U_\xi = \emptyset$.

Using Lemma 4.12, find open sets U_0 and V_0 such that $A_0 \subseteq U_0$, $\bigcup_{i>0} A_i \subseteq V_0$ and $U_0 \cap V_0 = \emptyset$. Fix $\gamma < \delta$ and suppose that we have defined $\{U_\xi : \xi < \gamma\}$ and $\{V_\xi : \xi < \gamma\}$ as above. We will define U_γ and V_γ. First let us prove the claim below.

Claim 4.14. $\kappa_\gamma > \gamma$.

Proof of Claim 4.14. By the condition (4) in Definition 4.2, we know that $\kappa_\xi \geq \xi$ for all $\xi < \delta$. If γ is a successor ordinal, then clearly $\kappa_\gamma > \gamma$ because κ_γ is a regular cardinal by Remark 4.3. Now, suppose that γ is a limit ordinal. Then $\gamma = \sup\{\xi : \xi < \gamma\} \leq \sup\{\kappa_\xi : \xi < \gamma\} \leq \kappa_\gamma$. If, on the contrary, we had $\kappa_\gamma = \gamma$, then γ would be a regular cardinal and $\sup\{\kappa_\xi : \xi < \gamma\} = \gamma$, which contradicts the hypothesis of this lemma. ⊣ (Claim 4.14)

By the second item of the condition (2), for each $\xi < \gamma$, we have $\bigcup_{i\geq\gamma} A_i \subseteq V_\xi$. Let

$$H = \bigcap\{V_\xi : \xi < \gamma\}.$$

For each $a \in \bigcup_{i\geq\gamma} A_i$, we have $\varphi(a) \geq \kappa_\gamma$. Since $\kappa_\gamma > \gamma$ by Claim 4.14, we have $\varphi(a) > \gamma$ for each $a \in \bigcup_{i\geq\gamma} A_i$, and so $H \in \mathcal{N}_\tau(\bigcup_{i\geq\gamma} A_i)$. By the third item of the condition (2), $H \cap U_\xi = \emptyset$ for all $\xi < \gamma$. Using Lemma 4.12, we can find open sets U_γ and V_γ such that $A_\gamma \subseteq U_\gamma \subseteq H$, $\bigcup_{i>\gamma} A_i \subseteq V_\gamma \subseteq H$ and $U_\gamma \cap V_\gamma = \emptyset$. U_γ and V_γ are as required. □

Here is a partial converse to Theorem 4.4. We assume that
- $\langle X, \tau\rangle$ is hereditarily normal,
- A is discrete, and
- the Generalized Continuum Hypothesis (GCH) holds.

These conditions are not in the hypothesis of Theorem 4.4.

Theorem 4.15. *Assume GCH. Let $\langle X, \tau\rangle$ be a hereditarily normal space and let A be a discrete subset of X. Let $\{A_\xi : \xi < \delta\}$ and $\{\kappa_\xi : \xi < \delta\}$ be as in Definition 4.2. Suppose that any forcing that has the cardinal covering property preserves a neighborhood base of A. Then*

(1) *for each $\xi < \delta$, $|A_\xi| < \kappa_\xi$ and*
(2) *for each regular cardinal $\lambda \leq \delta$, $\sup\{\kappa_\xi : \xi < \lambda\} > \lambda$.*

Proof. By Lemma 3.4, we may assume that the set A consists of non-isolated points of X, so $A = \bigcup\{A_\xi : \xi < \delta\}$. We prove the contrapositive by considering two cases.

Case 1. The condition (1) does not hold.
In this case, there is $\alpha < \delta$ such that $|A_\alpha| \geq \kappa_\alpha$. Take $B_\alpha \subseteq A_\alpha$ such that $|B_\alpha| = \kappa_\alpha$. By the proof of Proposition 3.11, $\mathbb{P} := Fn(\kappa_\alpha, \kappa_\alpha, \kappa_\alpha)$ destroys a neighborhood base of B_α. So we can find $W \in \tau^{\mathbb{P}}$ such that $B_\alpha \subseteq W$ and for every $H \in \mathcal{N}_\tau(B_\alpha)$, we have $H \not\subseteq W$. GCH implies $2^{<\kappa_\alpha} = \kappa_\alpha$, so $\mathbb{P}$ has the cardinal covering property by Remark 3.10. By Lemma 4.13, we can find a pairwise disjoint family of open sets $\{U_\xi : \xi < \delta\}$ such that $A_\xi \subseteq U_\xi$ for each $\xi < \delta$. Using Lemma 4.12, take open sets $U_{\alpha,0}$ and $U_{\alpha,1}$ such that $B_\alpha \subseteq U_{\alpha,0}$, $A_\alpha \setminus B_\alpha \subseteq U_{\alpha,1}$, and $U_{\alpha,0} \cap U_{\alpha,1} = \emptyset$. We may assume that $U_{\alpha,0} \cup U_{\alpha,1} \subseteq U_\alpha$. Let

$$W^\dagger = (W \cap U_{\alpha,0}) \cup U_{\alpha,1} \cup \bigcup\{U_\xi : \xi < \delta \text{ and } \xi \neq \alpha\}.$$

The family $\{U_{\alpha,0}, U_{\alpha,1}\} \cup \{U_\xi : \xi < \delta \text{ and } \xi \neq \alpha\}$ is a pairwise disjoint colletion of open sets. We have $A \subseteq W^\dagger$ and $W^\dagger \in \tau^{\mathbb{P}}$. To show that $W^\dagger$ witnesses the fact that $\mathbb{P}$ destroys a neighborhood base of A, take $H \in N_\tau(A)$; we will show that $H \not\subseteq W^\dagger$. If, on the contrary, we had $H \subseteq W^\dagger$, then $H \cap U_{\alpha,0} \subseteq W^\dagger \cap U_{\alpha,0} = W \cap U_{\alpha,0} \subseteq W$. This is a contradiction because $H \cap U_{\alpha,0} \in \mathcal{N}_\tau(B_\alpha)$ and so $H \cap U_{\alpha,0} \nsubseteq W$. Therefore, we have $H \nsubseteq W^\dagger$. Thus, $\mathbb{P}$ destroys a neighborhood base of A.

Case 2. The condition (2) does not hold.
In this case, there is a regular cardinal $\lambda \leq \delta$ such that $\sup\{\kappa_\xi : \xi < \lambda\} = \lambda$. Let λ be the smallest such regular cardinal. If $\lambda < \delta$, then, using Lemma 4.12, take open sets J_0 and J_1 such that $\bigcup\{A_\xi : \xi < \lambda\} \subseteq J_0$, $\bigcup\{A_\xi : \xi \geq \lambda\} \subseteq J_1$, and $J_0 \cap J_1 = \emptyset$. If $\lambda = \delta$, then let $J_0 = X$ and $J_1 = \emptyset$. By Lemma 4.13, there is a pairwise disjoint family of open sets $\{U_\xi : \xi < \lambda\}$ such that $A_\xi \subseteq U_\xi$ for each $\xi < \lambda$. We may assume that $U_\xi \subseteq J_0$ for all $\xi < \lambda$. For each $\xi < \lambda$, pick $a_\xi \in A_\xi$. Using the regularity of the space X, take open sets $U_{\xi,0}$ and $U_{\xi,1}$ for each $\xi < \lambda$ such that

- $a_\xi \in U_{\xi,0} \subseteq U_\xi$,
- $A_\xi \setminus \{a_\xi\} \subseteq U_{\xi,1} \subseteq U_\xi$, and
- $U_{\xi,0} \cap U_{\xi,1} = \emptyset$.

Using the fact that $\varphi(a_\xi) = \kappa_\xi$, choose for each $\xi < \lambda$, a family of open sets $\{V_{\xi,i} : i < \kappa_\xi\}$ such that

- $a_\xi \in V_{\xi,i} \subseteq U_{\xi,0}$ for each $i < \kappa_\xi$ and
- $\bigcap\{V_{\xi,i} : i < \kappa_\xi\} \notin \mathcal{N}_\tau(a_\xi)$.

Let $\mathbb{Q}$ be the forcing in Definition 4.9. GCH implies $2^{<\lambda} = \lambda$, so $\mathbb{Q}$ has the cardinal covering property by Remark 4.10. For a $\mathbb{Q}$-generic filter G, let $g = \bigcup G$. Let

$$W = \bigcup\{V_{\xi,g(\xi)} : \xi < \lambda\} \cup \bigcup\{U_{\xi,1} : \xi < \lambda\} \cup J_1.$$

The family $\{V_{\xi,g(\xi)} : \xi < \lambda\} \cup \{U_{\xi,1} : \xi < \lambda\} \cup \{J_1\}$ is a pairwise disjoint collection of open sets. We have $A \subseteq W$ and $W \in \tau^{\mathbb{Q}}$. To show that W witnesses the fact that $\mathbb{Q}$ destroys a neighborhood base of A, take $H \in \mathcal{N}_\tau(A)$; we will show that $H \nsubseteq W$. Define $f \in {}^\lambda\lambda \cap \mathbf{V}$ such that

$$f(\xi) = \sup\{i < \kappa_\xi : H \cap V_{\xi,0} \subseteq V_{\xi,i}\}.$$

Since $\bigcap\{V_{\xi,i} : i < \kappa_\xi\} \notin \mathcal{N}_\tau(a_\xi)$, $H \cap V_{\xi,0} \nsubseteq V_{\xi,j}$ for some $j < \kappa_\xi$. Therefore, $f(\xi) < \kappa_\xi$ for each $\xi < \lambda$, and so $f \in \mathcal{F} \cap \mathbf{V}$, where $\mathcal{F}$ is as in Proposition 4.11. If, on the contrary, we had $H \subseteq W$, then $H \cap V_{\xi,0} \subseteq V_{\xi,g(\xi)}$ for each $\xi < \lambda$, and so $f(\xi) \geq g(\xi)$ for all $\xi < \lambda$. This contradicts Proposition 4.11. Thus, we have $H \nsubseteq W$. Hence, $\mathbb{Q}$ destroys a neighborhood base of A. $\square$

Combining Theorem 4.4 and Theorem 4.15, we obtain the corollary below.

Corollary 4.16. *Assume GCH. Let $\langle X, \tau\rangle$ be a hereditarily normal space and let A be a discrete subset of X. Let $\{A_\xi : \xi < \delta\}$ and $\{\kappa_\xi : \xi < \delta\}$ be as in Definition 4.2. Then the following are equivalent:*

(1) *Any forcing that has the cardinal covering property preserves a neighborhood base of* A.
(2) • *For each* $\xi < \delta$, $|A_\xi| < \kappa_\xi$ *and*
• *for each regular cardinal* $\lambda \leq \delta$, $\sup\{\kappa_\xi : \xi < \lambda\} > \lambda$.

Proof. (1) $\Longrightarrow$ (2) is by Theorem 4.15.
(2) $\Longrightarrow$ (1) is by Theorem 4.4. (In this direction, GCH is not needed, X can be just a T_1 space, and a set A does not have to be discrete.) □

Before concluding this note, let us ask a question which we previously raised in [7, Question 5 on page 359] in a slightly different way.

Question 4.17. Let $\langle X, \tau \rangle$ be a regular (or normal) space and let I be the set of all isolated points in X. Let A be a discrete subset of X such that

(1) $|A \setminus I| > \aleph_0$ and
(2) $(\forall a \in A \setminus I)(\varphi(a) = \aleph_0)$.

Is there a forcing that preserves the cardinal $|A \setminus I|$ and destroys a neighborhood base of A?

If $\langle X, \tau \rangle$ is hereditarily normal, then the answer would be yes by Theorem 4.15 (without assuming GCH).

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