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CLOSED DISCRETE SELECTION IN THE COMPACT OPEN TOPOLOGY

CHRISTOPHER CARUVANA AND JARED HOLSHOUSER

Abstract. V. V. Tkachuk [*Closed discrete selections for sequences of open sets in function spaces*, Acta Math. Hungar. **154** (2018), no. 1, 56–68] isolates the closed discrete selection property while working on problems related to function spaces. In this paper, we study the closed discrete selection property and the related games and strategies on $C_k(X)$. Steven Clontz and Jared Holshouser [*Limited information strategies and discrete selectivity* Preprint. Available at arXiv:1806.06001v1 [math.GN]] show that the closed discrete selection game on $C_p(X)$ is equivalent to a modification of the point-open game on X. In this paper, we show that the closed discrete selection game on $C_k(X)$ is equivalent to a modification of the compact-open game on X. We also connect discrete selection properties on $C_k(X)$ to a variety of other properties on X, $C_k(X)$, and hyperspaces of X.

1. Introduction

In a 2018 paper [20], V. V. Tkachuk isolates the closed discrete selection property while working on problems related to function spaces. He connects this property for $C_p(X)$ and $C_p(X, [0, 1])$ to topological properties of X. He also studies the game version of the closed discrete selection property and connects strategies in that game on $C_p(X)$ to strategies in Gruenhage's W-game on $C_p(X)$ and the point open game on X [21]. Steven Clontz and Jared Holshouser [5] strengthen this relationship, showing that strategies for the discrete selection game on $C_p(X)$ are equivalent to strategies in a non-trivial modification of the point-open game on X.

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They also relate limited information strategies in the closed discrete selection game on $C_p(X)$ to topological properties of both $C_p(X)$ and X.

In this paper, we study the closed discrete selection property and the related games and strategies on $C_k(X)$, the continuous functions from X to $\mathbb{R}$ endowed with the compact-open topology. There is a history of connecting properties of $C_k(X)$ to properties of both X and its hyperspaces. We will be referencing and modifying results from Richard F. Arens [1], Marion Scheepers [15], and Ljubiša Kočinac [9]. We combine these techniques with classical methods, the ideas in Tkachuk's work, and a new approach to game duality currently in development by Clontz [4].

We have striven to be as general as possible in our methods and have produced robust lists of equivalences for the closed discrete selection principle on $C_k(X)$, strategies for the corresponding closed discrete game, and limited information strategies for the same game. As in [5], the closed discrete selection game on $C_k(X)$ is shown to be equivalent to a non-trivial modification of the compact-open game on X.

2. Definitions

Definition 2.1. For a topological space X, we let $C_p(X)$ denote the set of all continuous functions $X \to \mathbb{R}$ endowed with the topology of point-wise convergence. We also let $\mathbf{0}$ be the function which identically zero.

Definition 2.2. For a topological space X, we let $C_k(X)$ denote the set of all continuous functions $X \to \mathbb{R}$ endowed with the topology of uniform convergence on compact subsets of X. We will write

$$[f; K, \varepsilon] = \{g \in C_k(X) : \sup\{|f(x) - g(x)| : x \in K\} < \varepsilon\}$$

for $f \in C_k(X)$, $K \subseteq X$ compact, and $\varepsilon > 0$.

Definition 2.3. For a topological space X, we let $K(X)$ denote the family of all non-empty compact subsets of X and $\mathbb{K}(X)$ be the set $K(X)$ endowed with the *Vietoris topology* which is the topology generated by sets of the form

- $\mathrm{below}(U) := \{K \in K(X) : K \subseteq U\}$ and
- $\mathrm{touch}(U) := \{K \in K(X) : K \cap U \neq \emptyset\}$

for each open $U \subseteq X$. We will write

$$[U_0; U_1, U_2, \ldots, U_n] = \mathrm{below}(U_0) \cap \bigcap_{j=1}^{n} \mathrm{touch}(U_j),$$

for $U_0, U_1, U_2, \ldots, U_n \subseteq X$ open. Without loss of generality, we will impose the restrictions that $U_j \subseteq U_0$ for $1 \leqslant j \leqslant n$ and that $U_1, U_2, \ldots, U_n$ are pair-wise disjoint. For a deeper discussion of this topology, the authors recommend [11].

In this paper, we will be concerned with selection principles and related games. For classical results, basic tools, and notation, the authors recommend [10] and [13].

Definition 2.4. Given a set $\mathcal{A}$ and another set $\mathcal{B}$, we define the *finite selection principle* $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ to be the assertion that, given any sequence $\{A_n : n \in \omega\} \subseteq \mathcal{A}$, there exists a sequence $\{\mathcal{F}_n : n \in \omega\}$ so that, for each $n \in \omega$, $\mathcal{F}_n$ is a finite subset of A_n (denoted as $\mathcal{F}_n \in [A_n]^{<\omega}$ hereinafter) and $\bigcup\{\mathcal{F}_n : n \in \omega\} \in \mathcal{B}$.

Definition 2.5. Given a set $\mathcal{A}$ and another set $\mathcal{B}$, we define the *single selection principle* $S_1(\mathcal{A}, \mathcal{B})$ to be the assertion that, given any sequence $\{A_n : n \in \omega\} \subseteq \mathcal{A}$, there exists a sequence $\{B_n : n \in \omega\}$ so that, for each $n \in \omega$, $B_n \in A_n$ and $\{B_n : n \in \omega\} \in \mathcal{B}$.

Definition 2.6. Let X be a topological space. We say that $\mathscr{U}$ is an *ω-cover* of X provided that $\mathscr{U}$ is an open cover X, $X \notin \mathscr{U}$, and given any finite subset F of X, there exists some $U \in \mathscr{U}$ so that $F \subseteq U$. An infinite ω-cover $\mathscr{U}$ is said to be a *γ-cover* if, for every finite subset $F \subseteq X$, $\{U \in \mathscr{U} : F \not\subseteq U\}$ is finite. If $\mathscr{U} = \{U_n : n \in \omega\}$, then $\mathscr{U}$ is a γ-cover if and only if every cofinal sequence of the U_n forms an ω-cover.

Definition 2.7. Let X be a topological space. We say that$\mathscr{U}$ is a *k-cover* of X provided that $\mathscr{U}$ is an open cover X, $X \notin \mathscr{U}$, and, given any compact subset K of X, there exists some $U \in \mathscr{U}$ so that $K \subseteq U$. An infinite k-cover $\mathscr{U}$ is said to be a *γ_k-cover* if, for every compact $K \subseteq X$, $\{U \in \mathscr{U} : K \not\subseteq U\}$ is finite. If $\mathscr{U} = \{U_n : n \in \omega\}$, then $\mathscr{U}$ is a γ_k-cover if and only if every cofinal sequence of the U_n forms a k-cover.

Notation. We let

- $\neg A$ denote the complement of $\mathcal{A}$ for any collection $\mathcal{A}$;
- $\mathscr{T}_X$ denote the set of all non-empty subsets of X;
- $\Omega_{X,x}$ denote the set of all $A \subseteq X$ with $x \in \text{cl}_X(A)$; We also call $A \in \Omega_{X,x}$ a *blade* of x.
- $\Gamma_{X,x}$ denote the set of all sequences $\{x_n : n \in \omega\} \subseteq X$ with $x_n \to x$;
- $\mathcal{D}_X$ denote the collection of all dense subsets of X;
- CD_X denote the collection of all closed and discrete subsets of X;
- $\mathcal{O}_X$ denote the collection of all open covers of X;
- Ω_X denote the collection of all ω-covers of X;
- Γ_X denote the collection of all γ-covers of X;
- $\mathcal{K}_X$ denote the collection of all k-covers of X;
- $\Gamma_k(X)$ denote the collection of all γ_k-covers of X;

- for a family of sets $\mathcal{A}$, $\mathcal{O}(X, \mathcal{A})$ be all open covers $\mathscr{U}$ so that for every $A \in \mathcal{A}$, there is an open set $U \in \mathscr{U}$ which contains A; i.e., $\mathcal{O}(X, \mathcal{A}) = \{\mathscr{U} \in \mathcal{O}_X : (\forall A \in \mathcal{A})(\exists U \in \mathscr{U})[A \subseteq U]\}$;
- for a family of sets $\mathcal{A}$, $\Gamma(X, \mathcal{A})$ be all infinite open covers $\mathscr{U}$ so that for every $A \in \mathcal{A}$, $\{U \in \mathscr{U} : A \not\subseteq U\}$ is finite.

Definition 2.8. The following selection principles are known by the following names.

- $S_{\text{fin}}(\Omega_{X,x}, \Omega_{X,x})$ is known as the *countable fan tightness* property for X at $x \in X$.
- $S_1(\Omega_{X,x}, \Omega_{X,x})$ is known as the *strong countable fan tightness* property for X at $x \in X$.
- $S_{\text{fin}}(\mathcal{D}_X, \Omega_{X,x})$ is known as the *countable dense fan tightness* property for X at $x \in X$.
- $S_1(\mathcal{D}_X, \Omega_{X,x})$ is known as the *strong countable dense fan tightness* property for X at $x \in X$.

Definition 2.9. A space is said to be a *Fréchet-Urysohn* space if, for any $A \subseteq X$ and $x \in \mathrm{cl}_X(A)$, there exists a sequence $\{x_n : n \in \omega\} \subseteq A$ so that $x_n \to x$. A space is said to be a *strong Fréchet-Urysohn* space if, for any sequence $\{A_n : n \in \omega\}$ and

$$x \in \bigcap_{n \in \omega} \mathrm{cl}_X(A),$$

there exists a sequence $\{x_n : n \in \omega\}$ so that, for each $n \in \omega$, $x_n \in A_n$ and $x_n \to x$.

One easily sees that if X has countable fan tightness at each of its points, then X has countable dense fan tightness at each of its points. Similarly, if X has countable dense fan tightness at each of its points, then X is strong Fréchet-Urysohn.

Games of length ω with two players are useful for characterizing and calibrating a variety of topological properties. In full generality, these games are played as follows.

(1) The game is played in rounds indexed by the natural numbers. In each round, both players play elements from some set $\mathcal{E}$.
(2) The result of a play of the game is a sequence $A_0, B_0, A_1, B_1, \cdots$.
(3) Which player wins is decided by a set $\mathcal{X} \subseteq \mathcal{E}^\omega$. Player One wins if $(A_0, B_0, A_1, B_1, \cdots) \in \mathcal{X}$. Otherwise, Player Two wins. Frequently, part of the win condition will be that Player One must play elements of some set $\mathcal{A}$ and Player Two must play elements of a different set $\mathcal{B}$.

This can be written in tabular form:

I	A_0	A_1	$A_2 \cdots$
II	B_0	B_1	$B_2 \cdots$

A *strategy* is a function $\sigma : \mathcal{E}^{<\omega} \to \mathcal{E}$. Frequently, σ is used to refer to a strategy for Player One and τ is reserved for Player Two. A strategy for Player One will produce A_n from the previous $n-1$ rounds. A strategy for Player Two will produce B_n from the previous $n-1$ rounds and A_n.

- Player One has a winning strategy if there is a strategy σ for One so that no matter what Player Two plays in response, One wins the resulting play of the game. We write $\mathrm{I} \uparrow \mathcal{G}$.
- Player Two has a winning strategy if there is a strategy τ for Two so that no matter what Player One plays in response, Two wins the resulting play of the game. We write $\mathrm{II} \uparrow \mathcal{G}$.

If Player One has a winning strategy, then Player Two does not and vice versa. It is possible that neither Player One nor Player Two has a winning strategy.

The strategies just discussed are strategies of perfect information. It is also possible to have limited information strategies.

- A *tactic* for One is a strategy which only considers the most recent move from Player Two. Formally, it is a function $\sigma : \mathcal{E} \to \mathcal{E}$. If One has a winning tactic, we write $\mathrm{I} \underset{\text{tact}}{\uparrow} \mathcal{G}$.
- A *Markov strategy* for Two is a strategy which only considers the most recent move of Player One and the current turn number. Formally, it is a function $\tau : \mathcal{E} \times \omega \to \mathcal{E}$. If Two has a winning Markov strategy, we write $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathcal{G}$.
- A *predetermined strategy* for One is a strategy which only considers the current turn number. We call this kind of strategy predetermined because One is not reacting to Two's moves; each is just running through a pre-planned script. Formally, it is a function $\sigma : \omega \to \mathcal{E}$. If One has a winning predetermined strategy, we write $\mathrm{I} \underset{\text{pre}}{\uparrow} \mathcal{G}$.
- All of these strategy types can be defined symmetrically for the other player.

Definition 2.10. Two games $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be *strategically dual* provided that the following two hold:

- $\mathrm{I} \uparrow \mathcal{G}_1$ iff $\mathrm{II} \uparrow \mathcal{G}_2$
- $\mathrm{I} \uparrow \mathcal{G}_2$ iff $\mathrm{II} \uparrow \mathcal{G}_1$

Two games $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be *Markov dual* provided that the following two hold:

- I $\underset{\text{pre}}{\uparrow}$ $\mathcal{G}_1$ iff II $\underset{\text{mark}}{\uparrow}$ $\mathcal{G}_2$
- I $\underset{\text{pre}}{\uparrow}$ $\mathcal{G}_2$ iff II $\underset{\text{mark}}{\uparrow}$ $\mathcal{G}_1$

Two games $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be *dual* provided that they are both strategically dual and Markov dual.

Of particular interest are games derived from selection principles, i.e., selection games.

Definition 2.11. Given a set $\mathcal{A}$ and another set $\mathcal{B}$, we define the *finite selection game* $G_{\text{fin}}(\mathcal{A}, \mathcal{B})$ for $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$\begin{array}{c|ccc} \text{I} & A_0 & A_1 & A_2 \cdots \\ \hline \text{II} & \mathcal{F}_0 & \mathcal{F}_1 & \mathcal{F}_2 \cdots \end{array}$$

where each $A_n \in \mathcal{A}$ and $\mathcal{F}_n \in [A_n]^{<\omega}$. We declare Two the winner if $\bigcup\{\mathcal{F}_n : n \in \omega\} \in \mathcal{B}$. Otherwise, One wins.

Definition 2.12. Similarly, we define the *single selection game* $G_1(\mathcal{A}, \mathcal{B})$ as follows:

$$\begin{array}{c|ccc} \text{I} & A_0 & A_1 & A_2 \cdots \\ \hline \text{II} & B_0 & B_1 & B_2 \cdots \end{array}$$

where each $A_n \in \mathcal{A}$ and $B_n \in A_n$. We declare Two the winner if $\{B_n : n \in \omega\} \in \mathcal{B}$. Otherwise, One wins.

Remark 2.13. In general, $S_1(\mathcal{A}, \mathcal{B})$ is true if and only if I $\underset{\text{pre}}{\not\uparrow}$ $G_1(\mathcal{A}, \mathcal{B})$ (see [5, Proposition 13]).

Remark 2.14. The game $G_{\text{fin}}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known Menger game and the game $G_1(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known Rothberger game.

Notation. For $A \subseteq X$, let $\mathscr{N}(A)$ be all open sets U so that $A \subseteq U$. We will write $\mathscr{N}_x$ in place of $\mathscr{N}(\{x\})$. Set $\mathscr{N}[X] = \{\mathscr{N}_x : x \in X\}$, and, in general, if $\mathcal{A}$ is a collection of subsets of X, then $\mathscr{N}[\mathcal{A}] = \{\mathscr{N}(A) : A \in \mathcal{A}\}$.

Remark 2.15. Oftentimes, when $\mathscr{N}[\mathcal{A}]$ is being used in a game, we will use the identification of A with $\mathscr{N}(A)$ to simplify notation, particularly when One picks $A \in \mathcal{A}$ and Two's response is an open set U so that $A \subseteq U$.

Remark 2.16. The game $G_1(\mathscr{N}[X], \neg\mathcal{O}_X)$ is the well-known point-open game first appearing in [6]: Player One is trying to build an open cover and Player Two is trying to avoid building an open cover. The game $G_1(\mathscr{N}[K(X)], \neg\mathcal{O}_X)$ is the compact-open game.

Definition 2.17. A topological space X is called *discretely selective* if, for any sequence $\{U_n : n \in \omega\}$ of non-empty open sets, there exists a closed discrete set $\{x_n : n \in \omega\} \subseteq X$ so that $x_n \in U_n$ for each $n \in \omega$; i.e., $S_1(\mathscr{T}_X, CD_X)$ holds. This notion is first isolated by Tkachuk in [20].

Definition 2.18. For a topological space X, the *closed discrete selection game* on X, is $G_1(\mathscr{T}_X, CD_X)$. Tkachuk studies this game in [21].

Note that X is discretely selective if and only if $\mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} G_1(\mathscr{T}_X, CD_X)$.

Definition 2.19. For a topological space X and a point $x \in X$, the *closure game* for X at $x \in X$ is $G_1(\mathscr{T}_X, \neg\Omega_{X,x})$. Tkachuk studies this game in [21] as well.

Definition 2.20. For a topological space X and $x \in X$, *Gruenhage's W-game* for X at x is $G_1(\mathscr{N}_x, \neg\Gamma_{X,x})$.

Proposition 2.21. *For a topological space X, the following are equivalent:*

(a) *X is strong Fréchet-Urysohn;*
(b) *X has strong countable fan tightness at each of its points;*
(c) *for all $x \in X$, $S_1(\Omega_{X,x}, \Omega_{X,x})$;*
(d) *for all $x \in X$, $\mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} G_1(\Omega_{X,x}, \Omega_{X,x})$.*

3. Results

Definition 3.1 ([4])**.** For a set X, let

$$\mathbf{C}(X) = \left\{ f : X \to \bigcup X : f(x) \in x \right\}.$$

For a set $\mathcal{A}$, we say that $\mathcal{B} \subseteq \mathcal{A}$ is a *selection basis* provided that

$$(\forall A \in \mathcal{A})(\exists B \in \mathcal{B})(B \subseteq A).$$

A set $\mathcal{R}$ is said to be a *reflection* of a set $\mathcal{A}$ if

$$\{f[\mathcal{R}] : f \in \mathbf{C}(\mathcal{R})\}$$

is a selection basis for $\mathcal{A}$.

Theorem 3.2 ([4, Corollary 17])**.** *If $\mathcal{R}$ is a reflection of $\mathcal{A}$, then for a set $\mathcal{B}$, the two games $G_1(\mathcal{A}, \mathcal{B})$ and $G_1(\mathcal{R}, \neg\mathcal{B})$ are dual.*

Proposition 3.3. *Given a topological space X and a collection $\mathcal{A}$ of subsets of X, $\mathscr{N}[\mathcal{A}]$ is a reflection of $\mathcal{O}(X, \mathcal{A})$.*

Proof. Let $\mathscr{U} \in \mathcal{O}(X, \mathcal{A})$. Define $f : \mathcal{A} \to \mathscr{U}$ by choosing $f(A) \in \mathscr{U}$ so that $A \subseteq f(A)$. Now define $F : \mathscr{N}[\mathcal{A}] \to \bigcup \mathscr{N}[\mathcal{A}]$ by

$$F(\mathscr{N}(A)) = f(A).$$

Since $f(A) \in \mathscr{U}$ for all $A \in \mathcal{A}$, we see that

$$F[\mathscr{N}[\mathcal{A}]] \subseteq \mathscr{U}.$$

Hence, $\mathscr{N}[\mathcal{A}]$ is a reflection of $\mathcal{O}(X, \mathcal{A})$. $\square$

Corollary 3.4. *Let X be a topological space, $\mathcal{A}$ be a collection of subsets of X, and $\mathcal{B}$ be a family of open covers of X. Then the games $G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{B})$ and $G_1(\mathcal{O}(X, \mathcal{A}), \mathcal{B})$ are dual.*

Proof. Apply Theorem 3.2 and Proposition 3.3. $\square$

This general duality theorem extends the known results that the point-open game and the Rothberger game are dual (see [6]), and that the compact-open game and $G_1(\mathcal{K}_X, \mathcal{O}_X)$ are dual (see [17] and [2]). We also obtain the following as corollaries.

Corollary 3.5. *The games $G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$ and $G_1(\mathcal{K}_X, \mathcal{K}_X)$ are dual.*

Corollary 3.6. *The games $G_1(\mathscr{N}[K(X)], \neg\mathcal{O}_X)$ and $G_1(\mathcal{K}_X, \mathcal{O}_X)$ are dual.*

Proposition 3.7 ([2])**.** $\mathrm{I} \uparrow G_1(\mathscr{N}[K(X)], \neg\mathcal{O}_X) \iff \mathrm{II} \uparrow G_{\mathrm{fin}}(\mathcal{O}_X, \mathcal{O}_X)$.

Lemma 3.8. $\mathrm{I} \uparrow G_{\mathrm{fin}}(\mathcal{O}_X, \mathcal{O}_X) \implies \mathrm{II} \uparrow G_1(\mathscr{N}[K(X)], \neg\mathcal{O}_X)$.

Proof. Suppose $\mathrm{I} \uparrow G_{\mathrm{fin}}(\mathcal{O}_X, \mathcal{O}_X)$ and let σ be a winning strategy. It suffices to assume that One is playing compact sets in $G_1(\mathscr{N}[K(X)], \neg\mathcal{O}_X)$. Let $\mathscr{U}_0 = \sigma(\emptyset)$ and $K_0 \subseteq X$ be compact. Since K_0 is compact, we let

$$\tau(K_0) = \mathcal{F}_0 \in [\mathscr{U}_0]^{<\omega}$$

be so that $K_0 \subseteq \bigcup \mathcal{F}_0$.

Now, for $n \in \omega$, suppose we have open covers $\mathscr{U}_0, \mathscr{U}_1, \ldots, \mathscr{U}_{n+1}$, compact sets $K_0, K_1, \ldots, K_n$, and $\mathcal{F}_j \in [\mathscr{U}_j]^{<\omega}$ for each $j \leqslant n$ so that

$$\langle \mathscr{U}_0, \mathcal{F}_0, \mathscr{U}_1, \mathcal{F}_1, \ldots, \mathscr{U}_n, \mathcal{F}_n, \mathscr{U}_{n+1} \rangle \in \sigma$$

and

$$\left\langle K_0, \bigcup \mathcal{F}_0, K_1, \bigcup \mathcal{F}_1, \ldots, K_n, \bigcup \mathcal{F}_n \right\rangle \in \tau.$$

Now, for any compact $K_{n+1} \subseteq X$, we can let $\mathcal{F}_{n+1} \in [\mathscr{U}_{n+1}]^{<\omega}$ be so that $K_{n+1} \subseteq \bigcup \mathcal{F}_{n+1}$ and define

$$\tau\left(K_0, \bigcup \mathcal{F}_0, K_1, \bigcup \mathcal{F}_1, \ldots, K_n, \bigcup \mathcal{F}_n, K_{n+1}\right) = \bigcup \mathcal{F}_{n+1}.$$

This completes the definition of τ.

Now, since One wins $G_{\mathrm{fin}}(\mathcal{O}_X, \mathcal{O}_X)$ with σ, it must be the case that $\bigcup\{\mathcal{F}_n : n \in \omega\}$ fails to be a cover of X. That is, if Two plays $G_1(\mathscr{N}[K(X)], \neg\mathcal{O}_X)$ according to τ, Two wins. $\square$

In [2], Leandro F. Aurichi and Rodrigo R. Dias show that it is consistent with ZFC that, for a space X, $S_{\text{fin}}(\mathcal{O}_X, \mathcal{O}_X)$ holds, but $S_1(\mathscr{N}[K(X)], \neg\mathcal{O}_X)$ fails. From this example, they ask if it is consistent with ZFC that the games $G_1(\mathscr{N}[K(X)], \neg\mathcal{O}_X)$ and $G_{\text{fin}}(\mathcal{O}_X, \mathcal{O}_X)$ are dual. This is still open.

3.1. Covering Properties.

The following is a generalization of a result from Radislav Telgársky [16].

Theorem 3.9. *Let $\mathcal{A}$ be a collection of G_δ subsets of X. Then the following are equivalent:*

(a) $\mathrm{I} \uparrow G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{O}_X)$
(b) *there is a sequence* $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ *so that* $X = \bigcup_n A_n$
(c) $\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{O}_X)$.

Proof. (a) $\Rightarrow$ (b) For each $A \in \mathcal{A}$, let $\mathcal{U}_A$ be a countable family of descending open sets so that $A = \bigcap \mathcal{U}_A$. Let σ be a winning strategy for One. We recursively define collections of partial plays of $G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{O}_X)$ which are being played according to σ:

- $T_0 = \emptyset$.
- Given $n \in \omega$, define
$$T_{n+1} = \{w^\frown\langle\sigma(w), U\rangle : w \in T_n \text{ and } U \in \mathcal{U}_{\sigma(w)}\}.$$

Notice that each T_n is countable so
$$W := \bigcup_{n\in\omega} \bigcup_{w \in T_n} \sigma(w)$$
is a countable union of sets from $\mathcal{A}$.

To finish, we show that $X = W$. Toward this end, suppose that, by way of contradiction, there exists $x \in X \setminus W$. Then

- $x \notin A_0 := \sigma(\emptyset)$ and we can find $U_0 \in \mathcal{U}_{A_0}$ so that $x \notin U_0$.
- Suppose we have
$$w = \langle A_0, U_0, A_1, U_1, \ldots, A_n, U_n\rangle \in T_{n+1}$$
defined. Then $x \notin A_{n+1} := \sigma(w)$ whence we can find $U_{n+1} \in \mathcal{U}_{A_{n+1}}$ so that $x \notin U_{n+1}$.

Continuing in this way, we produce a run of the game which contradicts that σ is winning for One. Therefore, $X = W$.

(b) $\Rightarrow$ (c) Suppose $X = \bigcup_n A_n$. Define σ so that at round n, σ will play $\mathscr{N}(A_n)$. This is obviously a pre-determined winning strategy.

(c) $\Rightarrow$ (a) Obvious. $\square$

The following was originally proved by Fred Galvin [6].

Corollary 3.10. *Suppose X is so that each singleton set is a G_δ. Then* $\mathrm{I} \uparrow G_1(\mathscr{N}[X], \neg\mathcal{O}_X)$ *(the point-open game) if and only if X is countable.*

The following, combined with Proposition 3.7, generalizes [18, Corollary 4].

Corollary 3.11. *Suppose X is so that each compact set is a G_δ. Then* $\mathrm{I} \uparrow G_1(\mathscr{N}[K(X)], \neg\mathcal{O}_X)$ *(the compact-open game) if and only if X is σ-compact.*

Theorem 3.12. *Let $\mathcal{A}$ be a collection of G_δ subsets of X. Then the following are equivalent:*

(a) $\mathrm{I} \uparrow G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{O}(X, \mathcal{A}))$*;*
(b) *there is a sequence $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ with the property that for each $A \in \mathcal{A}$, there exists an n so that $A \subseteq A_n$ and $X = \bigcup_n A_n$;*
(c) $\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{O}(X, \mathcal{A}))$.

Proof. (a) $\Rightarrow$ (b) For each $A \in \mathcal{A}$, let $\mathcal{U}_A$ be a countable family of descending open sets so that $A = \bigcap \mathcal{U}_A$. Let σ be a winning strategy for One and, without loss of generality, suppose σ is producing elements of $\mathcal{A}$. We recursively define collections of partial plays of $G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{O}(X, \mathcal{A}))$ which are being played according to σ:

- $T_0 = \emptyset$.
- Given $n \in \omega$, define
$$T_{n+1} = \{w \frown \langle \sigma(w), U\rangle : w \in T_n \text{ and } U \in \mathcal{U}_{\sigma(w)}\}.$$

Notice that each T_n is countable so
$$\mathscr{F} := \bigcup_{n\in\omega} \{\sigma(w) : w \in T_n\}$$
is a countable family of sets from $\mathcal{A}$.

Suppose, toward a contradiction, that $A \in \mathcal{A}$ is so that
$$(\forall F \in \mathscr{F})(A \not\subseteq F).$$
Then

- for $A_0 = \sigma(\emptyset)$, we can find $x_0 \in A \setminus A_0$ and $U_0 \in \mathcal{U}_{K_0}$ so that $x_0 \notin U_0$.
- Suppose we have
$$w = \langle A_0, U_0, A_1, U_1, \ldots, A_n, U_n\rangle \in T_{n+1}$$
defined. Then, for $A_{n+1} = \sigma(w)$, we can find $x_{n+1} \in A \setminus A_{n+1}$ and $U_{n+1} \in \mathcal{U}_{K_{n+1}}$ so that $x_{n+1} \notin U_{n+1}$.

Continuing in this way, we obtain a full play

$$A_0, U_0, A_1, U_1, \ldots$$

of $G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{O}(X,\mathcal{A}))$ according to σ. That is, there must be some $n \in \omega$ so that $A \subseteq U_n$, contradicting $x_n \notin U_n$.

(b) $\Rightarrow$ (c) Suppose $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ has the property that for each $A \in \mathcal{A}$, there exists an n so that $A \subseteq A_n$ and $X = \bigcup_n A_n$. Define σ so that at round n, σ will play $\mathscr{N}(A_n)$. This is obviously a pre-determined winning strategy.

(c) $\Rightarrow$ (a) Obvious. $\square$

Corollary 3.13. *Suppose X is so that each compact set is a G_δ. Then* $\mathrm{I} \uparrow G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$ *if and only if X is hemicompact.*

Example 3.14. Corollaries 3.11 and 3.13 demonstrate that $G_1(\mathscr{N}[K(X)], \neg\mathcal{O}_X)$ and $G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$ are different games since $\mathbb{Q}$ is σ-compact but not hemicompact. Explicitly, $\mathrm{I} \uparrow G_1(\mathscr{N}[K(\mathbb{Q})], \neg\mathcal{O}_{\mathbb{Q}})$ but

$$\mathrm{I} \not\uparrow G_1(\mathscr{N}[K(\mathbb{Q})], \neg\mathcal{K}_{\mathbb{Q}}).$$

The following is a generalization of [19, p. 460, V. 416]. Moreover, if we replace $\mathcal{A}$ with the space of singletons X, we obtain [7, Theorem 1].

Lemma 3.15. *Let $\mathcal{A}$ be a family of sets closed under finite unions. Then* $\mathrm{I} \uparrow G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{O}(X,\mathcal{A}))$ *if and only if* $\mathrm{I} \uparrow G_1(\mathscr{N}[\mathcal{A}], \neg\Gamma(X,\mathcal{A}))$.

Proof. Let s be a winning strategy for One in $G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{O}(X,\mathcal{A}))$. We call a finite tuple $\langle U_0, U_1, \ldots, U_n\rangle$ of open subsets of X *adequate* if there exist $A_j \in \mathcal{A}$, $0 \leqslant j \leqslant n$, so that

$$\langle A_0, U_0, A_1, U_1, \ldots, A_n\rangle \in s$$

and $A_n \subseteq U_n$. Also, we will call a sequence $\{U_n : n \in \omega\}$ of open sets *adequate* if, for each $n \in \omega$, $\langle U_0, U_1, \ldots, U_n\rangle$ is adequate.

Now, suppose $\{U_n : n \in \omega\}$ is adequate and let $n \in \omega$. Then, let $A_0, A_1, \ldots, A_n \in \mathcal{A}$ so that

$$\langle A_0, U_0, A_1, U_1, \ldots, A_n\rangle \in s$$

and $A_n \subseteq U_n$. Notice that $A_0 = s(\emptyset)$. Consider

$$A_{n+1} := s(A_0, U_0, A_1, U_1, \ldots, A_n, U_n)$$

and let $A_0', A_1', \ldots, A_{n+1}' \in \mathcal{A}$ be so that

$$\langle A_0', U_0, A_1', U_1, \ldots, A_{n+1}'\rangle \in s$$

and $A_{n+1}' \subseteq U_{n+1}$. Again, observe that $A_0' = s(\emptyset) = A_0$.

To see that $A_j = A'_j$ for all $j \leqslant n+1$, fix $\ell \geqslant 0$ and suppose that we have shown $A_j = A'_j$ for all $j \leqslant \ell$. Then, since we have been playing according to s, we see that

$$\begin{aligned} A_{\ell+1} &= s(A_0, U_0, A_1, U_1, \ldots, A_\ell, U_\ell) \\ &= s(A'_0, U_0, A'_1, U_1, \ldots, A'_\ell, U_\ell) \\ &= A'_{\ell+1}. \end{aligned}$$

It follows that $\{U_n : n \in \omega\}$ arises as the sequence of Two's plays in a full run of the game $G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{O}(X, \mathcal{A}))$ according to s. In particular, if $\{U_n : n \in \omega\}$ is an adequate family, $\{U_n : n \in \omega\} \in \mathcal{A}$.

For any adequate sequence $\langle U_0, U_1, \ldots, U_n\rangle$, let $w(U_0, U_1, \ldots, U_n) = \langle A_0, A_1, \ldots, A_n\rangle$ be so that

$$\langle A_0, U_0, \ldots, A_n, U_n\rangle$$

is a play according to s and define

$$\gamma(U_0, U_1, \ldots, U_n) = s(A_0, U_0, \ldots, A_n, U_n) \cup \bigcup_{j=0}^{n} A_j.$$

Observe that $\gamma(U_0, U_1, \ldots, U_n) \in \mathcal{A}$ as it is a finite union of sets from $\mathcal{A}$.

Now we will define a new strategy σ and we start with $\sigma(\emptyset) = A_0 = s(\emptyset)$. Suppose that we have defined

$$\langle A_0, U_0, A_1, U_1, \ldots, A_{n-1}, U_{n-1}, A_n\rangle \in \sigma$$

for $n \geqslant 0$ so that, for a fixed open set U_n with $A_n \subseteq U_n$, we have

(i) for any $0 \leqslant j_0 < j_1 < \cdots < j_k \leqslant n$, $\langle U_{j_0}, U_{j_1}, \ldots, U_{j_k}\rangle$ is an adequate sequence and

(ii) for any $0 \leqslant j_0 < j_1 < \cdots < j_k \leqslant \ell < n$, $\gamma(U_{j_0}, U_{j_1}, \ldots, U_{j_k}) \subseteq A_{\ell+1}$.

Define

$$A_{n+1} = \bigcup \{\gamma(U_{j_0}, U_{j_1}, \ldots, U_{j_k}) : 0 \leqslant j_0 < j_1 < \cdots < j_k \leqslant n\}$$

which is a set in $\mathcal{A}$, let

$$\langle A_0, U_0, A_1, U_1, \ldots, A_{n-1}, U_{n-1}, A_n, U_n, A_{n+1}\rangle \in \sigma,$$

and fix an open set U_{n+1} with $A_{n+1} \subseteq U_{n+1}$.

To address (i), let $0 \leqslant j_0 < j_1 < \cdots < j_{k-1} < j_k = n+1$. Notice that $\langle U_{j_0}, U_{j_1}, \ldots, U_{j_{k-1}}\rangle$ is adequate by the inductive hypothesis so let

$$\langle B_{j_0}, B_{j_1}, \ldots, B_{j_{k-1}}\rangle = w(U_{j_0}, U_{j_1}, \ldots, U_{j_{k-1}})$$

and

$$B_{j_k} = s(B_{j_0}, U_{j_0}, B_{j_1}, U_{j_1}, \ldots, B_{j_{k-1}}, U_{j_{k-1}}).$$

It follows that

$$B_{j_k} \subseteq \gamma(U_{j_0}, U_{j_1}, \ldots, U_{j_{k-1}}) \subseteq A_{n+1} \subseteq U_{n+1}.$$

Hence,

$$B_{j_0}, U_{j_0}, B_{j_1}, U_{j_1}, \ldots, B_{j_{k-1}}, U_{j_{k-1}}, B_{j_k}, U_{j_k}$$

is a play according to s; i.e., $\langle U_{j_0}, U_{j_1}, \ldots, U_{j_k}\rangle$ is adequate.

The fact that (ii) holds for $0 \leqslant j_0 < j_1 < \cdots < j_k \leqslant \ell < n+1$ follows immediately from the definition of A_{n+1}. So this finishes the construction of σ.

Suppose $A_0, U_0, A_1, U_1, \ldots$ is a full run of the game $G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{O}(X, \mathcal{A}))$ played according to σ, and suppose, by way of contradiction, that there is $A \in \mathcal{A}$ so that

$$(\forall n \in \omega)(\exists m \geqslant n)(A \not\subseteq U_m).$$

Then we can build a co-final sequence $j_0 < j_1 < \cdots$ so that $A \not\subseteq U_{j_n}$ for each $n \in \omega$.

By construction of σ, $\langle U_{j_0}, U_{j_1}, \ldots, U_{j_n}\rangle$ is adequate for each $n \in \omega$ which means that $\{U_{j_n} : n \in \omega\}$ is adequate. By above, $\{U_{j_n} : n \in \omega\} \in \mathcal{O}(X, \mathcal{A})$. In particular, there must be some $n \in \omega$ so that $A \subseteq U_{j_n}$, a contradiction. Therefore, $\{U_n : n \in \omega\} \in \Gamma(X, \mathcal{A})$, and σ is a winning strategy for One in $G_1(\mathscr{N}[\mathcal{A}], \neg\Gamma(X, \mathcal{A}))$. $\square$

Corollary 3.16. *We have the following strengthenings of strategies.*

(i) $\mathrm{I} \uparrow G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$ *iff* $\mathrm{I} \uparrow G_1(\mathscr{N}[K(X)], \neg\Gamma_k(X))$.

(ii) $\mathrm{II} \uparrow G_1(\mathcal{K}_X, \mathcal{K}_X)$ *iff* $\mathrm{II} \uparrow G_1(\mathcal{K}_X, \Gamma_k(X))$.

(iii) $\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$ *iff* $\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathscr{N}[K(X)], \neg\Gamma_k(X))$.

(iv) $\mathrm{II} \underset{\text{mark}}{\uparrow} G_1(\mathcal{K}_X, \mathcal{K}_X)$ *iff* $\mathrm{II} \underset{\text{mark}}{\uparrow} G_1(\mathcal{K}_X, \Gamma_k(X))$.

Proof. (i) This is immediate from Lemma 3.15.

(ii) Apply(i) and corollaries 3.4 and 3.5.

(iii) Note that if s is a pre-determined winning strategy for One in $G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$, then

$$\sigma(n) = \bigcup_{i \leqslant n} s(i)$$

is a pre-determined winning strategy for One in $G_1(\mathscr{N}[K(X)], \neg\Gamma_k(X))$. To see this, note that any cofinal play of the game according to σ can be unraveled into a play of the game according to s by having Player One play each $s(i)$ and having Player Two play her response to $\sigma(n)$ repeatedly against the $s(i)$.

(iv) Apply (iii) and Corollary 3.4. $\square$

Lemma 3.17. $\mathrm{I} \uparrow G_{\text{fin}}(\mathcal{O}_X, \mathcal{O}_X) \implies \mathrm{I} \uparrow G_1(\mathcal{K}_X, \mathcal{K}_X)$.

Proof. Let s be a winning strategy for One in $G_{\text{fin}}(\mathcal{O}_X, \mathcal{O}_X)$. For any open cover $\mathscr{U}$, let

$$\mathscr{U}^+ = \left\{ \bigcup \mathcal{F} : \mathcal{F} \in [\mathscr{U}]^{<\omega} \right\}.$$

Now we define σ: Let $\sigma(\emptyset) = \mathscr{U}_0 := s(\emptyset)^+$. Since $s(\emptyset)$ is an open cover of X, $\mathscr{U}_0$ is a k-cover of X.

For $n \in \omega$, suppose we have

$$\langle \mathscr{V}_0, \mathcal{F}_0, \ldots, \mathscr{V}_{n-1}, \mathcal{F}_{n-1}, \mathscr{V}_n \rangle \in s$$

and

$$\langle \mathscr{U}_0, U_0, \ldots, \mathscr{U}_{n-1}, U_{n-1}, \mathscr{U}_n \rangle \in \sigma$$

where $\mathscr{U}_j = \mathscr{V}_j^+$ for each $j \leqslant n$ and $U_j = \bigcup \mathcal{F}_j$ for each $j < n$. For any choice $U_n \in \mathscr{U}_n$, there exists $\mathcal{F}_n \in [\mathscr{V}_n]^{<\omega}$ so that $U_n = \bigcup \mathcal{F}_n$. Then we obtain some open cover $\mathscr{V}_{n+1}$ with

$$\langle \mathscr{V}_0, \mathcal{F}_0, \ldots, \mathscr{V}_{n-1}, \mathcal{F}_{n-1}, \mathscr{V}_n, \mathcal{F}_n, \mathscr{V}_{n+1} \rangle \in s$$

and we let $\mathscr{U}_{n+1} = \mathscr{V}_{n+1}^+$ and

$$\langle \mathscr{U}_0, U_0, \ldots, \mathscr{U}_{n-1}, U_{n-1}, \mathscr{U}_n, U_n, \mathscr{U}_{n+1} \rangle \in \sigma.$$

This finishes the definition of σ.

Since s is winning for One in $G_{\text{fin}}(\mathcal{O}_X, \mathcal{O}_X)$, $\bigcup\{\mathcal{F}_n : n \in \omega\}$ is not a cover of X. It follows that $\{U_n : n \in \omega\}$ is not a cover of X and, therefore, σ is a winning strategy for One in $G_1(\mathcal{K}_X, \mathcal{K}_X)$. □

3.2. Topological characterizations and equivalent games.

Theorem 3.18. *Let $\mathcal{A}$ be a collection of closed subsets of X so that each point of X can be separated from each $A \in \mathcal{A}$; i.e., for each $A \in \mathcal{A}$ and $x \in X \setminus A$, there exists an open set U so that $A \subseteq U$ and $x \notin U$. Then $\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{O}_X)$ if and only if there is a sequence $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ so that $X = \bigcup_n A_n$.*

Proof. ($\Rightarrow$) Let $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ so that $X = \bigcup\{A_n : n \in \omega\}$. Then let One play A_n in the n^{th} inning.

($\Leftarrow$) Suppose One has a predetermined winning strategy σ and let $A_n = \sigma(n)$ for each $n \in \omega$. By way of contradiction, suppose there is some $x \in X \setminus \bigcup\{A_n : n \in \omega\}$. For each $n \in \omega$, we can find an open set U_n so that $A_n \subseteq U_n$ and $x \notin U_n$. Now we have a contradiction to the fact that σ is a winning predetermined strategy since $x \notin \bigcup\{U_n : n \in \omega\}$. □

Corollary 3.19. *A T_1 space X is countable if and only if*

$$\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathscr{N}[X], \neg\mathcal{O}_X).$$

Corollary 3.20. *A Hausdorff space X is σ-compact if and only if* $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} G_1(\mathscr{N}[K(X)], \neg\mathcal{O}_X)$.

Theorem 3.21. *Let $\mathcal{A}$ be a collection of closed subsets of X so that each point of X can be separated from each $A \in \mathcal{A}$; i.e., for each $A \in \mathcal{A}$ and $x \in X \setminus A$, there exists an open set U so that $A \subseteq U$ and $x \notin U$. Then* $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{O}(X, \mathcal{A}))$ *if and only if there is a sequence $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ so that $X = \bigcup_n A_n$, and for each $A \in \mathcal{A}$, there exists an n so that $A \subseteq A_n$.*

Proof. ($\Rightarrow$) Let $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ be so that $X = \bigcup\{A_n : n \in \omega\}$, and for every $A \in \mathcal{A}$, there exists $n \in \omega$ so that $A \subseteq A_n$. Then let One play A_n in the n^{th} inning.

($\Leftarrow$) Suppose One has a predetermined winning strategy σ and let $A_n = \sigma(n)$ for each $n \in \omega$. By way of contradiction, suppose there is some $A \in \mathcal{A}$ so that, for each $n \in \omega$, $A \not\subseteq A_n$. Then let $x_n \in A \setminus A_n$ for each $n \in \omega$. For each $n \in \omega$, we can find an open set U_n so that $A_n \subseteq U_n$ and $x_n \notin U_n$. But then $A_0, U_0, A_1, U_1, \ldots$ is a full run of $G_1(\mathscr{N}[\mathcal{A}], \neg\mathcal{O}(X, \mathcal{A}))$ played according to σ, so there must be some $n \in \omega$ for which $x_n \in A \subseteq U_n$, a contradiction. $\square$

In Theorem 3.22, the implications

$$\text{(a)} \Rightarrow \text{(d)} \Rightarrow \text{(e)} \Rightarrow \text{(a)}$$

are adapted from [1], and we have included the proof for the sake of clarity.

Theorem 3.22. *For a Tychonoff space X, the following are equivalent.*

(a) *X is hemicompact.*
(b) *$\mathbb{K}(X)$ is hemicompact.*
(c) *$\mathbb{K}(X)$ is σ-compact.*
(d) *$C_k(X)$ is metrizable.*
(e) *$C_k(X)$ is first-countable.*
(f) $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$, *a variant of the compact-open game.*
(g) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} G_1(\mathcal{K}_X, \mathcal{K}_X)$, *a variant of the Rothberger game.*
(h) $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} G_1(\mathscr{N}_{\mathbf{0}}, \neg\Gamma_{C_k(X),\mathbf{0}})$, *Gruenhage's W-game.*
(i) $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} G_1(\mathscr{N}_{\mathbf{0}}, \neg\Omega_{C_k(X),\mathbf{0}})$, *the closure game.*
(j) $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} G_1(\mathscr{T}_{C_k(x)}, CD_{C_k(X)})$, *the closed discrete selection game.*
(k) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} G_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$, *the countable fan tightness game.*
(l) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} G_1(\mathcal{D}_{C_k(X)}, \Omega_{C_k(X),\mathbf{0}})$, *the countable dense fan tightness game.*

Proof. (a) $\Rightarrow$ (b) Let $\{K_n : n \in \omega\}$ be a collection of compact sets so that, for every compact $K \subseteq X$, there exists $n \in \omega$ so that $K \subseteq K_n$. Notice that

$$\mathscr{F}_n := \{K \in \mathbb{K}(X) : K \subseteq K_n\}$$

is compact for each $n \in \omega$ and that

$$\mathbb{K}(X) = \bigcup_{n\in\omega} \mathscr{F}_n.$$

To see that $\mathbb{K}(X)$ is actually hemicompact, let $\mathscr{F} \subseteq \mathbb{K}(X)$ be compact. Then $\bigcup \mathscr{F} \subseteq X$ is compact which means there exists $n \in \omega$ so that $\bigcup \mathscr{F} \subseteq K_n$. It follows that $\mathscr{F} \subseteq \mathscr{F}_n$, establishing the hemicompactness of $\mathbb{K}(X)$.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (a) Let $\{\mathscr{F}_n : n \in \omega\}$ be a collection of compact subsets of $\mathbb{K}(X)$ so that

$$\mathbb{K}(X) = \bigcup_{n\in\omega} \mathscr{F}_n.$$

Then notice that

$$K_n := \bigcup \mathscr{F}_n$$

is compact in X. Let $K \subseteq X$ be compact. Then there exists $n \in \omega$ so that $K \in \mathscr{F}_n$ which provides that $K \subseteq K_n$.

(a) $\Rightarrow$ (d) Let $\{K_n : n \in \omega\}$ be an ascending sequence of compact subsets of X so that, for each compact $K \subseteq X$, there exists $n \in \omega$ so that $K \subseteq K_n$. For each $n \in \omega$ and $f, g \in C_k(X)$, define

$$\|f - g\|_n = \min\{\sup\{|f(x) - g(x)| : x \in K_n\}, 1\}$$

and define $d : C_k(X)^2 \to [0, 1]$ by

$$d(f, g) = \sum_{n=0}^{\infty} \frac{\|f - g\|_n}{2^{n+1}}.$$

It is straightforward to verify that d is indeed a metric.

To see that this metric is compatible with the topology on $C_k(X)$, first consider a basic neighborhood $[f; K, \varepsilon]$ about f. Then choose $n \in \omega$ so large that $2^{-(n+1)} < \varepsilon$ and $K \subseteq K_n$. Let $g \in B_d(f, \delta)$ where

$$\delta = \frac{\varepsilon - 2^{-(n+1)}}{n + 1} > 0.$$

Behold that

$$\frac{\|f - g\|_n}{2^{n+1}} \leqslant d(f, g) < \frac{\varepsilon - 2^{-(n+1)}}{n + 1}$$

which provides

$$(n+1)\cdot\frac{\|f-g\|_n}{2^{n+1}} < \varepsilon - 2^{-(n+1)} \implies \sum_{j=0}^{n}\frac{\|f-g\|_n}{2^{j+1}} < \varepsilon - 2^{-(n+1)}.$$

Moreover,

$$\begin{aligned}\|f-g\|_n &= \sum_{j=0}^{n}\frac{\|f-g\|_n}{2^{j+1}} + \sum_{j=n+1}^{\infty}\frac{\|f-g\|_n}{2^{j+1}}\\ &< \varepsilon - 2^{-(n+1)} + \sum_{j=n+1}^{\infty}\frac{\|f-g\|_n}{2^{j+1}}\\ &\leqslant \varepsilon - 2^{-(n+1)} + \sum_{j=n+1}^{\infty}\frac{1}{2^{j+1}}\\ &= \varepsilon.\end{aligned}$$

In particular, $|f(x)-g(x)|<\varepsilon$ for every $x\in K_n$, and since $K\subseteq K_n$, we see that this necessitates

$$B_d(f;\delta)\subseteq [f;K,\varepsilon].$$

Now consider a basic d-ball $B_d(f,\varepsilon)$ centered at f and choose $n\in\omega$ large enough so that

$$\sum_{j=n+1}^{\infty}\frac{1}{2^{j+1}} < \frac{\varepsilon}{2}.$$

Let $g\in[f;K_n,\varepsilon/2]$ and notice that, since the $\{K_n : n\in\omega\}$ were chosen to be ascending, $\|f-g\|_n \leqslant \|f-g\|_{n+1}$ for all $n\in\omega$. It follows that

$$\begin{aligned}d(f,g) &= \sum_{j=0}^{\infty}\frac{\|f-g\|_j}{2^{j+1}}\\ &= \sum_{j=0}^{n}\frac{\|f-g\|_j}{2^{j+1}} + \sum_{j=n+1}^{\infty}\frac{\|f-g\|_j}{2^{j+1}}\\ &\leqslant \sum_{j=0}^{n}\frac{\|f-g\|_n}{2^{j+1}} + \sum_{j=n+1}^{\infty}\frac{1}{2^{j+1}}\\ &< \|f-g\|_n + \frac{\varepsilon}{2}\\ &\leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

It follows that $[f;K_n,\varepsilon/2]\subseteq B_d(f;\varepsilon)$.

(d) $\Rightarrow$ (e) Obvious.

(e) ⇒ (a) Suppose $C_k(X)$ is first-countable. Let $\mathcal{B} = \{U_n : n \in \omega\}$ be a descending neighborhood basis at $\mathbf{0}$. Then, for each $n \in \omega$, let $K_n \subseteq X$ be compact and $q_n > 0$ be a rational number so that

$$[\mathbf{0}; K_n, q_n] \subseteq U_n.$$

Suppose, toward a contradiction, that there is some compact $K \subseteq X$ so that, for each $n \in \omega$, $K \not\subseteq K_n$. Then, for $n \in \omega$, we can choose $x_n \in K \setminus K_n$ and define $f_n : X \to [0,1]$ so that $f_n(x_n) = 1$ and $f_n[K_n] \equiv 0$. Since $f_n \in [\mathbf{0}; K_n, q_n]$ for each $n \in \omega$, $f_n \to \mathbf{0}$. In particular, for any $\varepsilon \in (0,1)$, there must be some $n \in \omega$ so that

$$f_n \in [\mathbf{0}; K, \varepsilon] \implies 1 = f_n(x_n) < \varepsilon,$$

an absurdity. Hence, X is hemicompact.

(a) ⇔ (f) This follows from Theorem 3.21.

(f) ⇔ (g) This follows from Corollary 3.5.

(g) ⇒ (k) Let t be a winning mark for Two in $G_1(\mathcal{K}_X, \mathcal{K}_X)$. By Corollary 3.16, we can assume t is a winning mark for Two in $G_1(\mathcal{K}_X, \Gamma_k(X))$. For any $A \in \Omega_{C_k(X),\mathbf{0}}$, let

$$U_{A,n} = t(\mathscr{U}(A,n), n),$$

where

$$\mathscr{U}(A,n) := \{f^{-1}[(-2^{-n}, 2^{-n})] : f \in A\}.$$

Note that $\mathscr{U}(A,n)$ is a k-cover, so this definition makes sense. Say $U_{A,n} = f_{A,n}^{-1}[(-2^{-n}, 2^{-n})]$ and set

$$\tau(A,n) = f_{A,n}.$$

We claim that τ is a winning mark for Two in $G_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$. Suppose $A_0, f_{A_0,0}, \cdots$ is a play of this game according to τ. Set $f_n = f_{A_n,n}$ and $U_n = U_{A_n,n}$. We need to show that $\mathbf{0} \in \overline{\{f_n : n \in \omega\}}$. Let $K \subseteq X$ be compact and $\varepsilon > 0$. Then $K \subseteq U_n$ for some n with $2^{-n} < \varepsilon$. So $f_n[K] \subseteq (-2^{-n}, 2^{-n}) \subseteq (-\varepsilon, \varepsilon)$ and thus $f_n \in [\mathbf{0}; K, \varepsilon]$. Therefore, $\mathbf{0} \in \overline{\{f_n : n \in \omega\}}$ and τ is a winning mark for Two in $G_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$.

(k) ⇒ (l) This follows immediately from the fact that $\mathcal{D}_{C_k(X)} \subseteq \Omega_{C_k(X),\mathbf{0}}$.

(l) ⇒ (g) Let t be a winning mark for Two in $G_1(\mathcal{D}_{C_k(X)}, \Omega_{C_k(X),\mathbf{0}})$. Let

$$D(\mathscr{U}) = \{g \in C_k(X) : (\exists U \in \mathscr{U})(g[X \setminus U] \equiv 1)\}.$$

We claim that $D(\mathscr{U})$ is dense in $C_k(X)$. Indeed, let $f \in C_k(X)$, $K \in K(X)$, and $\varepsilon > 0$. Choose $U \in \mathscr{U}$ so that $K \subseteq U$, and then let $g \in C_k(X)$ be so that $g|_K \equiv f|_K$ and $g[X \setminus U] \equiv 1$. Notice that $g \in D(\mathscr{U}) \cap [f; K, \varepsilon]$. Consequently, $D(\mathscr{U})$ is dense in $C_k(X)$.

Define $\tau : \mathcal{K}_X \times \omega \to \mathscr{T}_X$ by letting $\tau(\mathscr{U}, n)$ be $U \in \mathscr{U}$ so that $f[X \setminus U] \equiv 1$ where $f = t(D(\mathscr{U}), n)$. Let $\mathscr{U}_1$ and $\mathscr{U}_2$ be k-covers played by One and

$$U_n = \tau(\mathscr{U}_n, n).$$

Let K be compact. Then there is some $n \in \omega$ so that $f_n \in [\mathbf{0}; K, 1/2]$. Since $f_n[X \setminus U_n] \equiv 1$, it must be the case that $K \cap (X \setminus U_n) = \emptyset$. Thus, $K \subseteq U_n$.

(e) $\Rightarrow$ (h) Let $\{U_n : n \in \omega\}$ be a descending sequence of open sets which form a base for the topology of $C_k(X)$ at $\mathbf{0}$. Then let One play U_n in the n^{th} inning and notice that this defines a predetermined winning strategy for One in $G_1(\mathscr{N}_{\mathbf{0}}, \neg\Gamma_{C_k(X),\mathbf{0}})$.

(h) $\Rightarrow$ (i) $\Rightarrow$ (j) Obvious.

(j) $\Rightarrow$ (a) Let $U_n = \sigma(n)$. Say K_n is the support of U_n. We claim that $\{K_n : n \in \omega\}$ witnesses that X is hemicompact. Towards a contradiction, suppose otherwise. Then there is a compact $K \subseteq X$ so that $K \not\subseteq K_n$ for any n. Define $f_n \in U_n$ with the property that $f(x_n) = n$ where $x_n \in K \setminus K_n$. Then $U_0, f_0, \cdots$ is a play of $G_1(\mathscr{T}_{C_k(x)}, CD_{C_k(X)})$ according to σ, but $\{f_n : n \in \omega\}$ has no limit points. Therefore, Two has won the game, which is a contradiction. $\square$

Theorem 3.23. *For a Tychonoff space X, the following are equivalent.*

(a) $\mathrm{I} \uparrow G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$*, a variant of the compact-open game.*
(b) $\mathrm{I} \uparrow G_1(\mathscr{N}_{\mathbf{0}}, \neg\Gamma_{C_k(X),\mathbf{0}})$*, Gruenhage's W-game.*
(c) $\mathrm{I} \uparrow G_1(\mathscr{N}_{\mathbf{0}}, \neg\Omega_{C_k(X),\mathbf{0}})$*, the closure game.*
(d) $\mathrm{I} \uparrow G_1(\mathscr{T}_{C_k(x)}, CD_{C_k(X)})$*, the closed discrete selection game.*
(e) $\mathrm{II} \uparrow G_1(\mathcal{K}_X, \mathcal{K}_X)$*, a variant of the Rothberger game.*
(f) $\mathrm{II} \uparrow G_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$*, the countable fan tightness game.*
(g) $\mathrm{II} \uparrow G_1(\mathcal{D}_{C_k(X)}, \Omega_{C_k(X),\mathbf{0}})$*, the countable dense fan tightness game.*

Proof. (a) $\Rightarrow$ (b) Suppose One wins $G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$. By Lemma 3.15, we know that One wins $G_1(\mathscr{N}[K(X)], \neg\Gamma_k(X))$ so let s be a winning strategy for One in $G_1(\mathscr{N}[K(X)], \neg\Gamma_k(X))$ and let $K_0 = s(\emptyset)$. We define

$$\sigma(\emptyset) = U_0 = [\mathbf{0}; K_0, 1].$$

For Two's response $f_0 \in U_0$, let $V_0 = f_0^{-1}[(-1, 1)]$ and notice that $K_0 \subseteq V_0$.

For $n \in \omega$, suppose we have a partial run

$$\langle K_0, V_0, \ldots, K_n, V_n \rangle$$

of $G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$ and a partial run

$$\langle U_0, f_0, \ldots, U_n, f_n \rangle$$

of $G_1(\mathscr{N}_{\mathbf{0}}, \neg\Gamma_{C_k(X),\mathbf{0}})$ and let

$$K_{n+1} = s(K_0, V_0, \ldots, K_n, V_n).$$

Define

$$\sigma(U_0, f_0, \ldots, U_n, f_n) = U_{n+1} = \left[\mathbf{0}; \bigcup_{j=0}^{n+1} K_j, \frac{1}{2^n}\right].$$

For Two's choice of $f_{n+1} \in U_{n+1}$, define $V_{n+1} = f_{n+1}^{-1}[(-2^{-n}, 2^{-n})]$ and notice that $K_{n+1} \subseteq V_{n+1}$. This completes the definition of σ.

Now, we will show that $f_n \to \mathbf{0}$. Let $K \subseteq X$ be an arbitrary compact set $\varepsilon > 0$, and consider $[\mathbf{0}; K, \varepsilon]$. Choose $n_0 \in \omega$ so that $2^{-n_0} < \varepsilon$. By our choice of strategy, $\{V_n : n \geqslant n_0\}$ is a k-cover of X so there must be some $n_1 \geqslant n_0$ so that $K \subseteq V_n$ for $n \geqslant n_1$. That is, for any $n \geqslant n_1$,

$$\begin{aligned} K \subseteq V_{n+1} &\implies (\forall x \in K)\left(|f_{n+1}(x)| < \frac{1}{2^n} \leqslant \frac{1}{2^{n_0}} < \varepsilon\right) \\ &\implies f_{n+1} \in [\mathbf{0}; K, \varepsilon]. \end{aligned}$$

Therefore, $f_n \to \mathbf{0}$.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) Obvious.

(d) $\Rightarrow$ (a) Let s be a winning strategy for One in $G_1(\mathscr{T}_{C_k(x)}, CD_{C_k(X)})$. For every non-empty open subset $W \subseteq C_k(X)$, pick $f \in W$, a compact $K \subseteq X$, and $\varepsilon \in (0,1)$ so that $[f; K, \varepsilon] \subseteq W$ and define $\phi(W) = f$, $\mathrm{supp}(W) = K$, and $\gamma(W) = [f; K, \varepsilon]$.

Now we define a strategy σ for One as follows. To start, let $W_0 = s(\emptyset)$ and $\sigma(\emptyset) = K_0 = \mathrm{supp}(W_0)$. For $n \in \omega$, suppose we have

$$\langle K_0, U_0, K_1, U_1, \ldots, K_n\rangle \in \sigma$$

defined along with

$$\langle W_0, f_0, W_1, f_1, \ldots, W_n\rangle \in s$$

so that

- $(\forall j \leqslant n)(\mathrm{supp}(W_j) = K_j)$,
- $(\forall j < n)(f_j \in \gamma(W_j))$, and
- $(\forall j < n)(f_j[X \setminus U_j] \equiv j)$.

Let $U_n \subseteq X$ be open so that $K_n \subseteq U_n$. As K_n is compact, we can find continuous $g_n : X \to [0,1]$ so that $g_n[K_n] \equiv 1$ and $g_n[X \setminus U_n] \equiv 0$. Then let

$$f_n = \phi(W_n) \cdot g_n + (1 - g_n) \cdot n$$

and observe that

- $f_n|_{K_n} \equiv \phi(W_n)|_{K_n} \implies f_n \in \gamma(W_n)$ and
- $f_n[X \setminus U_n] \equiv n$.

Then let

$$W_{n+1} = s(W_0, f_0, W_1, f_1, \ldots, W_n, f_n)$$

and

$$\sigma(K_0, U_0, K_1, U_1, \ldots, K_n, U_n) = K_{n+1} := \mathrm{supp}(W_{n+1}).$$

This completes the definition of σ.

Now, we will show that σ is a winning strategy for One in $G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$. Toward this end, suppose $K_0, U_0, K_1, U_1, \ldots$ is a play of $G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$ according to σ, along with the corresponding play $W_0, f_0, W_1, f_1, \ldots$ of $G_1(\mathscr{T}_{C_k(x)}, CD_{C_k(X)})$ according to s. Since s is winning in $G_1(\mathscr{T}_{C_k(x)}, CD_{C_k(X)})$, let $f \in C_k(X)$ be an accumulation point of $\{f_n : n \in \omega\}$.

By way of contradiction, assume there is a compact $K \subseteq X$ so that $K \not\subseteq U_n$ for all $n \in \omega$ and consider the neighborhood $[f; K, 1]$ of f. As f is continuous, $f[K]$ is bounded, so choose $n \in \omega$ so that $|f(x)| < n$ for all $x \in K$. Now, for any $m > n$, we can find $x \in K \setminus U_m$ which necessitates

$$1 \leqslant m - n = f_m(x) - n < f_m(x) - f(x) = |f_m(x) - f(x)|.$$

In other words, $f_m \notin [f; K, 1]$ for all $m > n$, which is a contradiction. Therefore, One wins $G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$.

(a) $\Leftrightarrow$ (e) This follows from Corollary 3.5.

(e) $\Rightarrow$ (f) For $A \in \Omega_{C_k(X), \mathbf{0}}$, recall that

$$\mathscr{U}(A, n) := \{f^{-1}[(-2^{-n}, 2^{-n})] : f \in A\}$$

is a k-cover of X. From Lemma 3.15, we know that $\mathrm{I} \uparrow G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$ if and only if $\mathrm{I} \uparrow G_1(\mathscr{N}[K(X)], \neg\Gamma_k(X))$. By Corollary 3.4, the games $G_1(\mathscr{N}[K(X)], \neg\Gamma_k(X))$ and $G_1(\mathcal{K}_X, \Gamma_k(X))$ are dual. Hence, we let t be a winning strategy for Two in $G_1(\mathcal{K}_X, \Gamma_k(X))$.

We now define τ. Given $A_0 \in \Omega_{C_k(X), \mathbf{0}}$, let $U_0 = t(\mathscr{U}(A_0, 0))$. Choose $f_0 \in A_0$ so that $U_0 = f_0^{-1}[(-1, 1)]$ and let $\tau(A_0) = f_0$.

Suppose we have

- $A_0, A_1, \ldots, A_n \in \Omega_{C_k(X), \mathbf{0}}$,
- $U_{j+1} = t(\mathscr{U}(A_0, 0), U_0, \mathscr{U}(A_1, 1), U_1, \ldots, \mathscr{U}(A_j, j))$, and
- $f_j \in A_j$ so that $U_j = f_j^{-1}[(-2^{-j}, 2^{-j})]$.

Further suppose that $\tau(A_0, \cdots, A_j) = f_j$. Given $A_{n+1} \in \Omega_{C_k(X), \mathbf{0}}$, let

$$U_{n+1} = t(\mathscr{U}(A_0, 0), U_0, \mathscr{U}(A_1, 1), U_1, \ldots, \mathscr{U}(A_n, n), U_n, \mathscr{U}(A_{n+1}, n+1))$$

Then choose $f_{n+1} \in A_{n+1}$ so that $U_{n+1} = f_{n+1}^{-1}[(-2^{-(n+1)}, 2^{-(n+1)})]$ and let

$$\tau(A_0, f_0, A_1, f_1, \ldots A_n, f_n, A_{n+1}) = f_{n+1}.$$

This finishes the definition of τ.

To see that τ is winning for Two, consider a full run

$$A_0, f_0, A_1, f_1, \ldots$$

of the game $G_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$ played according to τ and consider the corresponding run

$$\mathscr{U}(A_0, 0), U_0, \mathscr{U}(A_1, 1), U_1, \ldots$$

of the game $G_1(\mathcal{K}_X, \Gamma_k(X))$ played according to t. We have that $\{U_n : n \in \omega\} \in \Gamma_k(X)$.

We wish to show that $\{f_n : n \in \omega\} \in \Omega_{C_k(X),\mathbf{0}}$. So let $K \in K(X)$ and $\varepsilon > 0$ be arbitrary. Then pick $n \in \omega$ so that $2^{-n} < \varepsilon$. Observe that there exists $m \geqslant n$ so that

$$K \subseteq U_m = f_m^{-1}[(-2^{-m}, 2^{-m})].$$

Then $f_m \in [\mathbf{0}; K, \varepsilon]$, establishing that $\{f_n : n \in \omega\} \in \Omega_{C_k(X),\mathbf{0}}$.

(f) $\Rightarrow$ (g) This follows from the fact that $\mathcal{D}_{C_k(X)} \subseteq \Omega_{C_k(X),\mathbf{0}}$.

(g) $\Rightarrow$ (e) Let t be a winning strategy for Two in $G_1(\mathcal{D}_{C_k(X)}, \Omega_{C_k(X),\mathbf{0}})$. For $\mathscr{U} \in \mathcal{K}_X$, recall that

$$D(\mathscr{U}) = \{g \in C_k(X) : (\exists U \in \mathscr{U})(g[X \setminus U] \equiv 1)\}$$

is dense in $C_k(X)$.

We define τ. Given $\mathscr{U}_0 \in \mathcal{K}_X$, let $f_0 = t(D(\mathscr{U}_0))$. Then choose $U_0 \in \mathscr{U}_0$ so that $f_0[X \setminus U_0] \equiv 1$ and define $\tau(\mathscr{U}_0) = U_0$. Inductively, given $\mathscr{U}_{n+1} \in \mathcal{K}_X$, let

$$f_{n+1} = t(D(\mathscr{U}_0), f_0, D(\mathscr{U}_1), f_1, \ldots, D(\mathscr{U}_n))$$

and choose $U_{n+1} \in \mathscr{U}_{n+1}$ so that $f_{n+1}[X \setminus U_{n+1}] \equiv 1$. Set

$$\tau(\mathscr{U}_0, U_0, \mathscr{U}_1, U_1, \ldots, \mathscr{U}_{n+1}) = U_{n+1}.$$

To see that $\{U_n : n \in \omega\}$ is a k-cover of X, let $K \in K(X)$. Then there must exist $n \in \omega$ so that $f_n \in [\mathbf{0}; K, 1/2]$. Since $f_n[X \setminus U_n] \equiv 1$, it must be the case that $K \cap (X \setminus U_n) = \emptyset$; i.e., $K \subseteq U_n$. Therefore, τ is a winning strategy. $\square$

Example 3.24. Give ω_1 the discrete topology and let $L(\omega_1)$ be the one-point Lindelöfication of ω_1. Then $L(\omega_1)$ is not countable, and so Player One does not have a pre-determined strategy in the finite-open game on $L(\omega_1)$. However, Player One does have a winning strategy in the finite-open game on $L(\omega_1)$. One opens with the point at infinity, $K_0 = \{\omega_1\}$. Now, let $\ell \in \omega$. Two responds with $L(\omega_1) \setminus A_\ell$ where $A_\ell = \{x_{\ell,n} : n \in \omega\}$ with $A_\ell \cap K_\ell = \emptyset$. One plays $K_{\ell+1} = \{\omega_1\} \cup \{x_{j,k} : j, k < \ell + 1\}$.

Since every compact subset of $L(\omega_1)$ must be finite, $L(\omega_1)$ is an example of a space where $\mathrm{I} \uparrow G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$, but $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$.

We have not yet been able to find a space X where

- $\mathrm{I} \not\uparrow G_1(\mathscr{N}[X^{<\omega}], \neg\mathcal{O}_X)$ (the finite-open game),
- $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$, and
- $\mathrm{I} \uparrow G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$.

3.3. **Selection principles.**

Propositions 3.25 and 3.27 are generalizations of Janusz Pawlikowski's results [12]. Proposition 3.25 also generalizes a result of W. Hurewicz ([8, Theorem 10]). Hurewicz's result was also proved by Scheepers in [14, Lemma 12 and Theorem 13].

Proposition 3.25. *Suppose* $\mathcal{A} \subseteq \mathcal{B}$ *and* $S_{\text{fin}}(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{B}))$. *Then* $\mathrm{I} \not\uparrow G_{\text{fin}}(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{B}))$. *Thus,*

$$\mathrm{I} \underset{\text{pre}}{\uparrow} G_{\text{fin}}(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{B})) \Leftrightarrow \mathrm{I} \uparrow G_{\text{fin}}(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{B})).$$

Proof. By way of contradiction, suppose that $S_{\text{fin}}(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{B}))$ holds and that One has a winning strategy for $G_{\text{fin}}(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{B}))$. Since $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{O}(X, \mathcal{B}) \subseteq \mathcal{O}(X, \mathcal{A})$. Note that every cover $\mathscr{U} \in \mathcal{O}(X, \mathcal{A})$ contains a countable subcover $\{U_n : n \in \omega\} \in \mathcal{O}(X, \mathcal{B})$. So without loss of generality, we can assume that Player One is playing covers of the form $\{U_n : n \in \omega\} \in \mathcal{O}(X, \mathcal{B})$ with the property that $U_n \subseteq U_{n+1}$, and that Player Two is choosing a single set from each cover. Using this, we can identify a strategy for Player One with a family $\{U_s : s \in \omega^{<\omega}\}$ satisfying

- for each s, $\{U_{s\frown k} : k \in \omega\} \in \mathcal{O}(X, \mathcal{B})$ and
- for each s and i, $U_{s\frown k} \subseteq U_{\sigma\frown(k+1)}$.

We can also identify the response by Player Two as a function $f : \omega \to \omega$. It suffices, then, to show the following. Given a strategy σ for One, we can find a function $f : \omega \to \omega$ with the property that whenever $B \in \mathcal{B}$, there are infinitely many $n \in \omega$ for which $B \subseteq U_{f|_n}$.

We will use $S_{\text{fin}}(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{B}))$ to define this f, so, first, we need to create a suitable sequence of covers in order to apply the selection principle. For integers $j > 0$ and $k, m \geqslant 0$, define

$$V_k(m, j) = \bigcap \{U_{s\frown k} : s \in j^m\}.$$

We claim that for all m and j, $\{V_k(m, j) : k \in \omega\} \in \mathcal{O}(X, \mathcal{B})$ and $V_k(m, j) \subseteq V_{k+1}(m, j)$. Fix $j > 0$, $m \geqslant 0$, and $B \in \mathcal{B}$. For any $s \in j^m$, $\{U_{s\frown k} : k \in \omega\} \in \mathcal{O}(X, \mathcal{B})$, so we can find $k_s \in \omega$ so that $B \subseteq U_{s\frown k_s}$. Since $\{U_{s\frown k} : k \in \omega\}$ are monotonic in k, this means $B \subseteq U_{s\frown k}$ for all $k \geqslant k_s$. Set $k^* = \max\{k_s : s \in j^m\}$. Then $B \subseteq U_{s\frown k^*}$ for all $s \in j^m$ and, therefore, $B \subseteq V_{k^*}(m, j)$.

For integers $j > 0$, $k \geqslant j$, and $m, n \geqslant 0$, define

$$W_k^n(m,j) = \bigcup \left\{ \bigcap_{i=0}^{n} V_{k_{i+1}}(m+i, k_i) : j = k_0 \leqslant k_1 \leqslant \cdots \leqslant k_{n+1} = k \right\}.$$

We claim that for all m, n, and j, $\{W_k^n(m,j) : k \geqslant j\} \in \mathcal{O}(X,\mathcal{B})$ and that $W_k^n(m,j) \subseteq W_{k+1}^n(m,j)$. Fix $j > 0$, $m, n \geqslant 0$, and $B \in \mathcal{B}$. Then we can find $k_1 \geqslant j$ so that $B \subseteq V_{k_1}(m,j)$. We can next find $k_2 \geqslant k_1$ so that $B \subseteq V_{k_2}(m+1,k_1)$. Continue recursively in this way, defining k_i so that $B \subseteq V_{k_i}(m+i,k_i)$ until we get to k_{n+1} and set $k = k_{n+1}$. Then $B \subseteq W_k^n(m,j)$, as witnessed by $j, k_1, k_2, \cdots, k_n, k$.

We can now apply $S_{\text{fin}}(\mathcal{O}(X,\mathcal{A}), \mathcal{O}(X,\mathcal{B}))$ to $\{\{W_k^n(m,j) : k \geqslant j\} : n \in \omega\}$ for each m and j. From this we get functions $f_{m,j} : \omega \to \omega$ with the property that when $B \in \mathcal{B}$, there is an $n \in \omega$ so that $B \subseteq W^n_{f_{m,j}(n)}(m,j)$. In fact, by partitioning ω into disjoint infinite subsets $A_0, A_1, \cdots$, we can apply $S_{\text{fin}}(\mathcal{O}(X,\mathcal{A}), \mathcal{O}(X,\mathcal{B}))$ to $\{\{W_k^n(m,j) : k \geqslant j\} : n \in A_\ell\}$ for each m and j and ℓ. In this way, we get functions $f_{m,j} : \omega \to \omega$ with the property that when $B \in \mathcal{B}$, there are infinitely many $n \in \omega$ so that $B \subseteq W^n_{f_{m,j}(n)}(m,j)$. Let $f : \omega \to \omega$ be a strictly increasing function so that for any m and j, there is an $N_{m,j}$ so that $f(m+n) \geqslant f_{m,j}(n)$ for all $n \geqslant N_{m,j}$. Then for every $B \in \mathcal{B}$ and every pair m and j, there is an n so that $B \subseteq W^n_{f(m+n)}(m,j)$.

We claim that f is as desired. Towards a contradiction, suppose there is a $B \in \mathcal{B}$ and an m so that $B \not\subseteq U_{f|_{m+n}}$ for all n. We know, however, that there is an n so that $B \subseteq W^n_{f(m+n)}(m, f(m))$. So there is a sequence

$$f(m) = k_0 \leqslant k_1 \leqslant \cdots \leqslant k_{n+1} = f(m+n)$$

so that $B \subseteq \bigcap_{i=0}^n V_{k_{i+1}}(m+i,k_i)$. Now $f|_m \in k_0^m$, $B \subseteq V_{k_1}(m,k_0)$, and $B \not\subseteq U_{(f|_m)^\frown f(m)}$. But $f|_{m+1} \in k_1^{m+1}$, $B \subseteq V_{k_2}(m+1,k_1)$, and $B \not\subseteq U_{(f|_{m+1})^\frown f(m+1)}$. Thus, $k_2 > f(m+1)$. Proceeding in this way, $k_i > f(m+i-1)$ and eventually $k_{n+1} > f(m+n)$. This contradicts the choice of the k_i. $\square$

Note that all we really use about $\mathcal{O}(X,\mathcal{A})$ is that $\mathcal{O}(X,\mathcal{B}) \subseteq \mathcal{O}(X,\mathcal{A})$ and whenever $\{U_n : n \in \omega\} \in \mathcal{O}(X,\mathcal{A})$, then $\{\bigcup_{i \leqslant n} U_i : n \in \omega\} \in \mathcal{O}(X,\mathcal{A})$ as well.

Corollary 3.26. *Suppose $S_{\text{fin}}(\mathcal{K}_X,\mathcal{K}_X)$. Then One has no winning strategy in $G_{\text{fin}}(\mathcal{K}_X,\mathcal{K}_X)$. Also,*

$$\mathrm{I} \uparrow G_{\text{fin}}(\mathcal{K}_X,\mathcal{K}_X) \Leftrightarrow \mathrm{I} \underset{\text{pre}}{\uparrow} G_{\text{fin}}(\mathcal{K}_X,\mathcal{K}_X).$$

Proposition 3.27. *Suppose $\mathcal{A} \subseteq \mathcal{B}$ and $S_1(\mathcal{O}(X,\mathcal{A}),\mathcal{O}(X,\mathcal{B}))$. Then* $\mathrm{I} \not\uparrow G_1(\mathcal{O}(X,\mathcal{A}),\mathcal{O}(X,\mathcal{B}))$. *Thus,*

$$\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathcal{O}(X,\mathcal{A}),\mathcal{O}(X,\mathcal{B})) \Leftrightarrow \mathrm{I} \uparrow G_1(\mathcal{O}(X,\mathcal{A}),\mathcal{O}(X,\mathcal{B})).$$

Proof. Since $S_1(\mathcal{O}(X,\mathcal{A}),\mathcal{O}(X,\mathcal{B}))$ is true, every cover $\mathscr{U} \in \mathcal{O}(X,\mathcal{A})$ contains a countable subcollection $\{U_n : n \in \omega\} \in \mathcal{O}(X,\mathcal{B})$. We can, therefore, identify a strategy for Player One in $G_1(\mathcal{O}(X,\mathcal{A}),\mathcal{O}(X,\mathcal{B}))$ with a collection $\{U_s : s \in \omega^{<\omega}\}$ with the property that for each $s \in \omega^{<\omega}$, $\{U_{s^\frown k} : k \in \omega\} \in \mathcal{O}(X,\mathcal{B})$. We can also identify the response by Player Two as a function $f : \omega \to \omega$. It suffices, then, to show the following. Given a strategy σ for One, we can find a function $f : \omega \to \omega$ with the property that whenever $B \in \mathcal{B}$, there are infinitely many $n \in \omega$ for which $B \subseteq U_{f|_n}$.

For $j > 0$, $m \geqslant 0$, and $s : |j^m| \to \omega$, define

$$U_s(m,j) = \bigcap \left\{ \bigcup \{U_{t^\frown (s|_k)} : 0 < k \leqslant |j^m|\} : t \in j^m \right\}.$$

Notice that if Player Two has played numbers $< j$ on rounds prior to round m and she plays according to s afterwards, then she is guaranteed to cover $U_s(m,j)$ at round $m + |j^m|$.

We claim that for any integers $j > 0$ and $m \geqslant 0$, $\{U_s(m,j) : s : |j^m| \to \omega\} \in \mathcal{O}(X,\mathcal{B})$. Suppose $j > 0$, $m \geqslant 0$, and $B \in \mathcal{B}$. Enumerate j^m as $\{t_k : k < |j^m|\}$. We recursively define an $s : |j^m| \to \omega$ so that $B \subseteq U_s(m,j)$. First, choose $s(0)$ so that $B \subseteq U_{t_0{}^\frown s(0)}$. Then choose $s(1)$ so that $B \subseteq U_{t_1{}^\frown s(0)^\frown s(1)}$. Continuing in this way, we can define $s(k)$ for all $k < |j^m|$. Then $B \subseteq U_s(m,j)$.

We next claim that there are increasing functions $g, h : \omega \to \omega$ so that whenever $B \in \mathcal{B}$, there are infinitely many n for which there is an $s : (g(n+1) - g(n)) \to h(n+1)$ so that $B \subseteq U_s(g(n),h(n))$. That is,

$$(\forall B \in \mathcal{B})(\exists^\infty n)(\exists s : (g(n+1) - g(n)) \to h(n+1))[B \subseteq U_s(g(n),h(n))].$$

We define g and h by playing a game. First, set $g(0) = 1$ and $h(0) = 0$. In the n^{th} round of this game, One plays a cover $\{U_s(g(n),h(n)) : s : |h(n)^{g(n)}| \to \omega\} \in \mathcal{O}(X,\mathcal{A})$ and Two responds by setting $h(n+1) \geqslant h(n)$. Two also sets $g(n+1) = g(n) + |h(n)^{g(n)}|$. Player Two is effectively playing

$$\bigcup \{U_s(g(n),h(n)) : s : |h(n)^{g(n)}| \to h(n+1)\}$$

as a response in the game $G_{\text{fin}}(\mathcal{O}(X,\mathcal{A}),\mathcal{O}(X,\mathcal{B}))$. We know from Proposition 3.25 that One does not have a winning strategy for $G_{\text{fin}}(\mathcal{O}(X,\mathcal{A}), \mathcal{O}(X,\mathcal{B}))$.

So given a strategy $\{U_s : s \in \omega^{<\omega}\}$ for One in this defined game, we can view it as a strategy for One in $G_{\text{fin}}(\mathcal{O}(X,\mathcal{A}),\mathcal{O}(X,\mathcal{B}))$ and find a play by Two against this strategy which is a winning one for Two. This counterplay by Two in $G_{\mathrm{fin}}(\mathcal{O}(X,\mathcal{A}),\mathcal{O}(X,\mathcal{B}))$ amounts to choosing $h(n)$ so that

$$\left\{\bigcup\{U_s(g(n),h(n)) : s : |h(n)^{g(n)}| \to h(n+1)\} : n \in \omega\right\} \in \mathcal{O}(X,\mathcal{B}),$$

and each $B \in \mathcal{B}$ is contained in cofinally many of these open sets. Any such h and subsequently defined g will satisfy the claim.

Take g and h to be as desired. Now for $k_1 < \cdots < k_n$ and $s_i : (g(k_{i+1}) - g(k_i)) \to h(k_{i+1})$, define

$$W_n(k_1,\cdots,k_n;s_1,\cdots,s_n) = W_n(\vec{k};\vec{s}) = \bigcap_{i=1}^{n} U_{s_i}(g(k_i),h(k_i)).$$

Note that g and h have been chosen so that

$$\{W_n(\vec{k};\vec{s}) : (k_1 < \cdots < k_n) \wedge (\forall i < n)[s_i : (g(k_{i+1}) - g(k_i)) \to h(k_{i+1})]\}$$

is in $\mathcal{O}(X,\mathcal{B})$ for every n. So as $S_1(\mathcal{O}(X,\mathcal{A}),\mathcal{O}(X,\mathcal{B}))$ is true, we can find $k_{i,n}$ and $s_{i,n}$ for $n \in \omega$ and $i \leqslant |2^n|$ so that

$$\{W_n(k_{1,n},\cdots,k_{|2^n|,n};s_{1,n},\cdots,s_{|2^n|,n}) : n \in \omega\} \in \mathcal{O}(X,\mathcal{B}),$$

and, moreover, for any $B \in \mathcal{B}$, we can find infinitely many n so that

$$B \subseteq W_n(k_{1,n},\cdots,k_{|2^n|,n};s_{1,n},\cdots,s_{|2^n|,n}).$$

Now take $\ell_n \in \{k_{1,n},\cdots,k_{|2^n|,n}\} \setminus \bigcup_{j<n}\{k_{1,j},\cdots,k_{|2^j|,j}\}$, say $\ell_n = k_{i,n}$, and set $t_n = s_{i,n}$. Notice that the ℓ_n are all distinct and that whenever $B \in \mathcal{B}$, we can find infinitely many n so that $B \subseteq U_{t_n}(g(\ell_n),h(\ell_n))$. Define $f : \omega \to \omega$ by

$$f(g(\ell_n) + i) = t_n(i)$$

for $n \in \omega$ and $i \in \mathrm{dom}(t_n)$. (Note that the sets $\{g(\ell_n) + i : i \in \mathrm{dom}(t_n)\}$ are pairwise disjoint.) For other n, set $f(n) = 0$.

We claim that f is our desired function. Let $B \in \mathcal{B}$. Then there are infinitely many n so that $B \subseteq U_{t_n}(g(\ell_n),h(\ell_n))$. But if $B \subseteq U_{t_n}(g(\ell_n),h(\ell_n))$, then

$$B \subseteq \bigcup\{U_{(f|_{g(\ell_n)})^\frown(t_n|_i)} : 0 \leqslant i \leqslant g(\ell_n+1) - g(\ell_n)\}$$

as $f|_{g(\ell_n)} : g(\ell_n) \to h(\ell_n)$. Now $(f|_{g(\ell_n)})^\frown(t_n|_i) = f|_{g(\ell_n)+i}$. So $B \subseteq U_{f|_{g(\ell_n)+i}}$ for some i with $0 \leqslant i \leqslant g(\ell_n+1) - g(\ell_n)$. Since this happened for infinitely many n, we have that $B \subseteq U_{f|_i}$ for infinitely many i, as desired. $\square$

Corollary 3.28. *Suppose $S_1(\mathcal{K}_X,\mathcal{K}_X)$. Then One has no winning strategy in $G_1(\mathcal{K}_X,\mathcal{K}_X)$. Also, $\mathrm{I} \uparrow G_1(\mathcal{K}_X,\mathcal{K}_X)$ if and only if $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} G_1(\mathcal{K}_X,\mathcal{K}_X)$.*

Theorem 3.29. *The following are equivalent for all Tychonoff spaces.*

(a) $\mathrm{II} \uparrow G_1(\mathscr{N}[K(X)],\neg\mathcal{K}_X)$, *a variation of the compact-open game.*

(b) II $\underset{\text{mark}}{\uparrow}$ $G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$.
(c) I $\uparrow G_1(\mathcal{K}_X, \mathcal{K}_X)$, *a variation of the Rothberger game.*
(d) *X fails $S_1(\mathcal{K}_X, \mathcal{K}_X)$; that is,* I $\underset{\text{pre}}{\uparrow}$ $G_1(\mathcal{K}_X, \mathcal{K}_X)$.
(e) I $\uparrow G_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$, *the countable fan tightness game.*
(f) *$C_k(X)$ is not sCFT; that is,* I $\underset{\text{pre}}{\uparrow}$ $G_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$.
(g) I $\uparrow G_1(\mathcal{D}_{C_k(X)}, \Omega_{C_k(X),\mathbf{0}})$, *the countable dense fan tightness game.*
(h) *$C_k(X)$ is not sCDFT; that is,* I $\underset{\text{pre}}{\uparrow}$ $G_1(\mathcal{D}_{C_k(X)}, \Omega_{C_k(X),\mathbf{0}})$.
(i) II $\uparrow G_1(\mathscr{N}_{\mathbf{0}}, \Gamma_{C_k(X),\mathbf{0}})$, *Gruenhage's W-game.*
(j) II $\underset{\text{mark}}{\uparrow}$ $G_1(\mathscr{N}_{\mathbf{0}}, \Gamma_{C_k(X),\mathbf{0}})$.
(k) II $\uparrow G_1(\mathscr{N}_{\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$, *the closure game.*
(l) II $\underset{\text{mark}}{\uparrow}$ $G_1(\mathscr{N}_{\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$.
(m) II $\uparrow G_1(\mathscr{T}_{C_k(X)}, CD_{C_k(X)})$, *the closed discrete selection game.*
(n) II $\underset{\text{mark}}{\uparrow}$ $G_1(\mathscr{T}_{C_k(X)}, CD_{C_k(X)})$.

Proof. The equivalences (a) $\Leftrightarrow$ (c) and (b) $\Leftrightarrow$ (d) are established by Corollary 3.5. Corollary 3.28 establishes the equivalence of (d) and (c). Thus, (a), (c), (b), and (d) are equivalent.

The equivalence of (d) and (f) is [9, Theorem 2.2].

(d) $\Rightarrow$ (h) This is an adaptation of [3] and is done by contrapositive. That is, suppose $S_1(\mathcal{D}_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$. We will show $S_1(\mathcal{K}_X, \mathcal{K}_X)$. For $\mathscr{U} \in \mathcal{K}_X$, recall that

$$D(\mathscr{U}) = \{g \in C_k(X) : (\exists U \in \mathscr{U})(g[X \setminus U] \equiv 1)\}$$

is dense in $C_k(X)$ as seen in the proof of Theorem 3.22.

Now let $\{\mathscr{U}_n : n \in \omega\}$ be a sequence of k-covers of X. It follows that $\{D(\mathscr{U}_n) : n \in \omega\}$ is a sequence of dense subsets of $C_k(X)$, so we can apply $S_1(\mathcal{D}_{C_k(X)}, \Omega_{C_k(X),\mathbf{0}})$ to obtain $\{f_n : n \in \omega\} \in \Omega_{C_k(X),\mathbf{0}}$ so that $f_n \in D(\mathscr{U}_n)$ for each $n \in \omega$. For each $n \in \omega$, let $U_n \in \mathscr{U}_n$ be so that $f_n[X \setminus U_n] \equiv 1$.

We now show that $\{U_n : n \in \omega\}$ is a k-cover of X. Indeed, let $K \in K(X)$ and let $n \in \omega$ be so that

$$f_n \in [\mathbf{0}; K, 1/2].$$

Since $f_n[X \setminus U_n] \equiv 1$, $K \cap (X \setminus U_n) = \emptyset$. Hence, $K \subseteq U_n$, establishing that $\{U_n : n \in \omega\}$ is a k-cover of X.

(h) $\Rightarrow$ (f) This follows from $\mathcal{D}_{C_k(X)} \subseteq \Omega_{C_k(X),\mathbf{0}}$. Thus, (d), (f), and (h) are equivalent.

(f) $\Rightarrow$ (e) Obvious.

(e) $\Rightarrow$ (f) We do this direction by the contrapositive. This is similar to the proof of Theorem 13 in [15]. Suppose $S_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$ and let σ be a strategy for One in $G_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$. By $S_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$, any blade at $\mathbf{0}$ yields a countable subset which is also a blade at $\mathbf{0}$, so assume One is playing countable blades. If One ever plays a blade which contains $\mathbf{0}$, Two can choose $\mathbf{0}$ and thus win the game. So we assume that One only plays sets which do not contain $\mathbf{0}$.

We will translate σ into a strategy σ^* for One in $G_1(\mathcal{K}_X, \mathcal{K}_X)$. Suppose $\sigma(\emptyset) = \{f_{\emptyset,n} : n \in \omega\}$. For each n, let

$$U_{\emptyset,n} = f_{\emptyset,n}^{-1}\left[\left(-\frac{1}{2}, \frac{1}{2}\right)\right].$$

Set $\sigma^*(\emptyset) = \{U_{\emptyset,n} : n \in \omega\}$. We claim that this is a k-cover of X. Indeed, let $K \subseteq X$ be compact. Since the $\{f_{\emptyset,n} : n \in \omega\}$ is a blade at $\mathbf{0}$, there exists N so that $f_{\emptyset,N} \in [\mathbf{0}; K, 1/3]$. Hence, $K \subseteq U_{\emptyset,N}$.

Now Two will play some $U_{\emptyset,n_0}$ in response, and we can translate this play back to $C_k(X)$ by having Two play $f_{\emptyset,n_0}$ in that game. Suppose $\sigma(f_{\emptyset,n_0}) = \{f_{n_0,n} : n \in \omega\}$. Set

$$U_{n_0,n} = f_{n_0,n}^{-1}\left[\left(-\frac{1}{2^2}, \frac{1}{2^2}\right)\right].$$

Set $\sigma^*(U_{\emptyset,n_0}) = \{U_{n_0,n} : n \in \omega\}$. We can continue defining σ^* recursively in this way.

We will show that σ is not a winning strategy for One. By (d) $\Rightarrow$ (f), we have that

$$S_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}}) \implies S_1(\mathcal{K}_X, \mathcal{K}_X)$$

which, in turn, implies that One does not have a winning strategy in $G_1(\mathcal{K}_X, \mathcal{K}_X)$. Thus, there is some play $U_{\emptyset,n_0}, U_{n_0,n_1}, \cdots$ by Two which forms a k-cover. Consider the corresponding play $f_{\emptyset,n_0}, f_{n_0,n_1}$ by Two against σ. Set $f_0 = f_{\emptyset,n_0}$, $f_1 = f_{n_0,n_1}$ and so on, and likewise define U_n. Let $K \subseteq X$ be compact and $\varepsilon > 0$. Then $K \subseteq U_i$ for infinitely many i. Thus,

$$K \subseteq f_i^{-1}\left[\left(-\frac{1}{2^{i+1}}, \frac{1}{2^{i+1}}\right)\right].$$

Therefore, $f_i \in [\mathbf{0}; K, \varepsilon]$ for infinitely many i. Since K and ε are arbitrary, this shows that $\mathbf{0}$ is in the closure of $\{f_i : i \in \omega\}$. Thus, Two has won this run of the game and σ is not a winning strategy. This shows that (e) implies (f).

(h) $\Rightarrow$ (g) Obvious.

Proposition 21 of [5] shows that (g) $\Rightarrow$ (k) $\Rightarrow$ (i) $\Rightarrow$ (e).

Proposition 22 of [5] shows that (h) $\Rightarrow$ (l) $\Rightarrow$ (j) $\Rightarrow$ (f).

(b) $\Rightarrow$ (n) Suppose τ is a winning Markov strategy for Two in $G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$. Let $[f; K, \varepsilon]$ be a subset of Player One's play in the n^{th} inning in $G_1(\mathscr{T}_X, CD_{C_k(X)})$. Then let $V = \tau(K, n)$. We can find a continuous $g : X \to \mathbb{R}$ with the property that $g|_K = f|_K$ and $g[X \setminus V] = \{n\}$. We define $\tau^*([f; K, \varepsilon], n) = g$. Then τ^* is a Markov strategy for Two in $G_1(\mathscr{T}_X, CD_{C_k(X)})$.

Suppose $U_n = [f_n; K_n, \varepsilon_n]$, $n \in \omega$, is a play of $G_1(\mathscr{T}_X, CD_{C_k(X)})$ by Player One. We claim that $\{g_n = \tau^*(U_n, n) : n \in \omega\}$ is closed discrete. Let $f \in C_k(X)$. Set $V_n = \tau(K_n, n)$ and notice that $\{V_n : n \in \omega\}$ is not a k-cover. Thus, we can find a compact $K \subseteq X$ so that $K \not\subseteq V_n$ for all n. Hence, there is some $x_n \in K \setminus V_n$ so that $g_n(x_n) = n$. Since f is continuous, $f[K]$ is bounded. Hence, we can find an N so that $f(x) + 1 < N$ for all $x \in K$. Then $g_n \notin [f; K, 1]$ for all $n > N$. Thus, f is not a limit point of $\{g_n : n \in \omega\}$. Since f was arbitrary, $\{g_n : n \in \omega\}$ is closed discrete. Therefore, τ^* is a winning mark for Two in $G_1(\mathscr{T}_X, CD_{C_k(X)})$.

Finally, (n) $\Rightarrow$ (m) $\Rightarrow$ (k). □

Corollary 3.30. *For a Tychonoff space X, the following are equivalent.*

(a) $S_1(\mathcal{K}_X, \mathcal{K}_X)$*; that is,* $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_1(\mathcal{K}_X, \mathcal{K}_X)$.

(b) $S_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$*; that is,* $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$.

(c) $S_1(\mathcal{D}_{C_k(X)}, \Omega_{C_k(X),\mathbf{0}})$*; that is,* $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_1(\mathcal{D}_{C_k(X)}, \Omega_{C_k(X),\mathbf{0}})$.

(d) $C_k(X)$ *has strong countable fan tightness.*

(e) $C_k(X)$ *has strong countable dense fan tightness.*

(f) $\mathrm{II} \underset{\text{mark}}{\not\uparrow} G_1(\mathscr{T}_{C_k(X)}, CD_{C_k(X)})$.

(g) $\mathrm{II} \underset{\text{mark}}{\not\uparrow} G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$.

Proof. All of these equivalences follow from Theorem 3.29 except (b) $\Leftrightarrow$ (d) and (c) $\Leftrightarrow$ (e). These follow from the fact that $C_k(X)$ is a homogeneous space. □

4. Further Questions

1. How much of this theory can be recovered for $C_k(X, [0,1])$?
2. To what extent can propositions 3.25 and 3.27 be generalized?
3. Does there exist a space X so that $\mathrm{I} \uparrow G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$ but One does not have a winning strategy in the finite-open game nor a pre-determined strategy in $G_1(\mathscr{N}[K(X)], \neg\mathcal{K}_X)$?
4. How much of this theory carries over to games of longer length?

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