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THE HYPERSPACE OF SEQUENCES OF A DENDRITE IS CONTRACTIBLE

ALEJANDRO ILLANES

Abstract. Given a Hausdorff space X, let $S_c(X)$ be the hyperspace of nontrivial convergent sequences. It is an open problem to determine if the contractibility of X implies the contractibility of $S_c(X)$. In this paper we prove that if X is a dendrite (a locally connected continuum without simple closed curves), then $S_c(X)$ is contractible. This answers a question by Javier Camargo, David Maya, and Patricia Pellicer-Covarrubias [*Path connectedness, local path connectedness and contractibility of* $S_c(X)$. Available at arXiv:1802.00725v1 [math.GN].]

1. Introduction

The *harmonic sequence* is the subspace $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ of the real line. Given a Hausdorff space X, the *hyperspace of nontrivial convergent sequences* $S_c(X)$ is defined as

$$S_c(X) = \{A \subset X : A \text{ is homeomorphic to the harmonic sequence}\}.$$

This hyperspace is considered with the Vietoris topology. In the case that X is a metric space, the Vietoris topology in $S_c(X)$ is induced by the Hausdorff metric H.

The hyperspace $S_c(X)$ is introduced by S. García-Ferreira and Y. F. Ortiz-Castillo in [3]. Several authors study different aspects related to these hyperspaces in [2]–[10]. A very natural problem is the following.

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Problem 1.1. *Suppose that X is a Hausdorff contractible space. Does this imply that the hyperspace $S_c(X)$ is contractible?*

A general solution for Problem 1.1 seems to be very difficult. So the following problem is also interesting.

Problem 1.2. *Find families of contractible spaces X for which $S_c(X)$ is contractible.*

The most general result related to Problem 1.2 is [2, Theorem 4.3]. This result says that for every non-degenerate connected subset X of a tree (a *tree* is a connected union of finitely many arcs, such that it does not contain simple closed curves), we have that $S_c(X)$ is contractible.

In particular, Problem 1.2 is interesting for the family of cones over compact Hausdorff spaces [2, Question 4.6] (still unsolved).

A *continuum* is a compact connected metric space with more than one point. A *dendrite* is a locally connected continuum containing no simple closed curves. The *harmonic fan* is the cone over the harmonic sequence.

In [7], it is shown that if X is the harmonic fan, then $S_c(X)$ is contractible.

As can be seen in [7, §3] and in the main result of this paper, even for families of very simple contractible spaces X, it is difficult to prove the contractibility of $S_c(X)$.

The following interesting problems remain open.

Problem 1.3. *Is the hyperspace $S_c([0,1]^2)$ contractible?*

Problem 1.4. *Let X be the cone over the Cantor set. Is $S_c(X)$ contractible?*

Problem 1.5. *Does there exist a non-contractible continuum X for which $S_c(X)$ is contractible? If S^1 is a simple closed curve, is $S_c(S^1)$ contractible* [2, Question 4.7]*?*

In this paper we prove that if X is a dendrite, then $S_c(X)$ is contractible.

A *mapping* is a continuous function.

Given a continuum X, let

$$2^X = \{A \subset X : A \text{ is nonempty and closed}\}.$$

The hyperspace 2^X is considered with the Hausdorff metric H. Let $\mathcal{F}(X) = \{A \in 2^X : A \text{ is finite}\}$.

If d is a metric for X and nonempty sets $A, B \subset X$, let $\operatorname{dist}(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\}$. Given $p \in X$, $A \subset X$, and $\varepsilon > 0$, let $B(p, \varepsilon)$ be the ε-neighborhood in X around p and $N(A, \varepsilon) = \bigcup\{B(a, \varepsilon) : a \in A\}$. We extend the definition of diameter by defining $\operatorname{diameter}(\emptyset) = 0$. The cardinality of a set A is denoted by $|A|$.

A *dendroid* is an arcwise connected continuum such that the intersection of any two of its subcontinua is connected. By [12, 8.23 and 10.35], a continuum X is a dendrite if and only if X is a locally connected dendroid. If X is a dendroid, for any two points $p, q \in X$, pq denotes the unique arc in X joining p and q, if $p \neq q$; and $pq = \{p\}$, if $p = q$.

2. **Dendrites**

A metric d for a continuum X is *convex* if for every pair of points $p, q \in X$, there exists an isometry $\phi : [0, d(p,q)] \to X$ such that $\phi(0) = p$ and $\phi(d(p,q)) = q$. Working independently, R. H. Bing [1] and Edward E. Moise [11] prove that each locally connected continuum X admits a convex metric.

Throughout this section, X will denote a dendrite with a convex metric d such that diameter$(X) \leq 1$. The convexity of the metric d implies that pq is isometric to the interval $[0, d(p,q)]$.

This section is devoted to proving that $S_c(X)$ is contractible. We fix a point $p_0 \in X$.

Given $S \in S_c(X)$, S is homeomorphic to the harmonic sequence. Let $\lim S$ be the point that corresponds to the number 0. Notice that for each neighborhood U of $\lim S$ in X, we have that $S \setminus U$ is finite.

Define $K : 2^X \times [0,1] \to 2^X$ by

$$K(A,t) = \{p \in X : \text{there exists } a \in A \text{ such that } d(a,p) \leq t\}.$$

In the case that $K = \{q\}$ for some $q \in X$, we write $K(q,t)$ instead of $K(\{q\},t)$.

2.1. **Properties of** K.

Let $A \in 2^X$. Then

(1) K is continuous [12, 0.65.3],
(2) $K(A,0) = A$,
(3) $K(A,1) = X$,
(4) if $t \in [0,1]$, then each component of $K(A,t)$ intersects A,
(5) if $t > 0$, then $K(A,t)$ has a finite number of components.

We prove (4). Let D be a component of $K(A,t)$. Take a point $p \in D$. Let $a \in A$ be such that $d(a,p) \leq t$. Since ap is isometric to the interval $[0, d(a,p)]$, we have $ap \subset K(A,t)$ and $ap \subset D$.

We prove (5). By compactness of A, there exist $r \in \mathbb{N}$ and $a_1, \ldots, a_r \in A$ such that $A \subset B(a_1,t) \cup \cdots \cup B(a_r,t)$. Since d is convex, each $B(a_i,t)$ is connected. Let D be a component of $K(A,t)$. By (4), there exists a point $a \in A \cap D$. Let $i \in \{1, \ldots, r\}$ be such that $a \in B(a_i,t)$. Since $B(a_i,t) \subset K(A,t)$ and $B(a_i,t)$ is connected, we have $B(a_i,t) \subset D$. We have shown that each component of $K(A,t)$ contains a point a_i. Therefore, $K(A,t)$ has a finite number of components.

Given $p \in X$ and $A \in C(X)$, since X is a dendroid, there exists a unique point $\sigma(p,A) \in X$ such that $p\sigma(p,A) \cap A = \{\sigma(p,A)\}$.

2.2. **Properties of** $\sigma(p,A)$.

Let $A \in C(X)$. Then

(6) if $q \in A$, then $\sigma(p,A) \in pq$;
(7) the function $\sigma|_{X\times\{A\}} : X \times \{A\} \to A$ is continuous.

The proof of (7) can be found in [13, Lemma 10.25].

Lemma 2.1. *Let $(S,t) \in S_c(X) \times (0,1]$ and $p \in X$. Then for each component D of $K(S,t) \cap p_0p$, there exists a finite subset J of S such that $D = K(J,t) \cap p_0p$.*

Proof. Let $R = \{q \in S : K(q,t) \cap D \neq \emptyset\}$. Given $q \in R$, since X is a dendroid, $K(q,t) \cap p_0p$ is connected, so $\emptyset \neq D \cap K(q,t) \cap p_0p \subset K(S,t) \cap p_0p$, then $K(q,t) \cap p_0p \subset D$. This implies that $D = K(R,t) \cap p_0p$.

If R is finite, we are done. Suppose then that R is infinite.

Since K is continuous, we have R is compact.

We consider the function $g : R \to D$ given by $g(q) = \sigma(q,D)$. By (7), g is continuous.

Given $x \in \operatorname{Im} g$, $g^{-1}(x)$ is a compact subset of R, so we can fix a point $h(x) \in g^{-1}(x)$ such that $d(x,h(x)) = \min\{d(x,q) : q \in g^{-1}(x)\}$.

We claim that $K(h(x),t) \cap p_0p = K(g^{-1}(x),t) \cap p_0p$. In order to prove this, let $q \in g^{-1}(x)$ and $y \in K(q,t) \cap p_0p \subset D$. Since $g(h(x)) = x = g(q)$ and $y \in D$, by (6), $x \in yh(x) \cap yq$. Since d is convex, $d(y,h(x)) = d(y,x)+d(x,h(x)) \leq d(y,x)+d(x,q) = d(y,q) \leq t$. Thus, $y \in K(h(x),t) \cap p_0p$. This proves that $K(h(x),t) \cap p_0p = K(g^{-1}(x),t) \cap p_0p$. Since $R = \bigcup\{g^{-1}(x) : x \in \operatorname{Im} g\}$, we have

$$D = K(R,t) \cap p_0p = K(\operatorname{Im} h,t) \cap p_0p.$$

In the case that $\operatorname{Im} h$ is finite, set $J = \operatorname{Im} h$ and we are done. Hence, we suppose that $\operatorname{Im} h$ is infinite. Then $\operatorname{Im} g$ is also infinite.

Let $s_0 = \lim S$. Since $\operatorname{Im} h \subset R \subset S$, we have $s_0 \in R$. Let $q_0 = g(s_0)$ and $\operatorname{Im} g = \{q_0, q_1, q_2, \ldots\}$ where $q_0, q_1, q_2, \ldots$ are pairwise distinct.

Since $g(h(q_n)) = q_n$ for all $n \geq 0$, the points $h(q_0), h(q_1), h(q_2), \ldots$ are pairwise distinct. Since the points $h(q_n)$ belong to S, $\lim_{n\to\infty} h(q_n) = s_0$.

In the case that $s_0 \notin D$, let M be a compact connected neighborhood of s_0 such that $M \cap D = \emptyset$. Then there exists $N \in \mathbb{N}$ such that for each $n \geq N$, $h(q_n) \in M$. Since $M \cup s_0g(s_0)$ is a subcontinuum of X, such that $(M \cup s_0g(s_0)) \cap D = \{g(s_0)\}$, we have that for each $n \geq N$, $q_n = g(h(q_n)) = g(s_0)$. This contradicts the choice of the points q_n. Therefore, $s_0 \in D$ and

$s_0 = g(s_0) = q_0$. Since g is continuous, $\lim_{n\to\infty} q_n = \lim_{n\to\infty} g(h(q_n)) = s_0$. Since $s_0 \in D$, $g(s_0) = s_0$, so $h(s_0) = h(g(s_0)) = s_0$ and $h(q_0) = q_0$.

Since d is convex, we may identify p_0p with the interval $[0, d(p, p_0)]$ with all its structures as a subset of the real line, identifying the point p_0 with 0 and p with $d(p, p_0)$.

For each $n \in \mathbb{N} \cup \{0\}$, let $[a_n, b_n] = K(h(q_n), t) \cap p_0p$.

Since d is convex, for each $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} a_n &= \max\{p_0, q_n - (t - d(h(q_n), q_n))\} \text{ and} \\ b_n &= \min\{p, q_n + (t - d(h(q_n), q_n))\}. \end{aligned}$$

In particular, $a_0 = \max\{p_0, q_0 - t\}$ and $b_0 = \min\{p, q_0 + t\}$.

Notice that $D = K(\operatorname{Im} h, t) \cap p_0p = \bigcup\{K(h(q_n), t) \cap p_0p : n \in \mathbb{N} \cup \{0\}\} = \bigcup\{[a_n, b_n] : n \in \mathbb{N} \cup \{0\}\}$.

Since $\lim_{n\to\infty} q_n = q_0$, there exists $N_0 \in \mathbb{N}$ such that $q_n \in [a_0, b_0]$ for all $n \geq N_0$.

We are going to show that there exist $n_1, n_2, N \in \mathbb{N} \cup \{0\}$ such that $N \geq N_0$ and for each $n \geq N$, $[a_n, b_n] \subset [a_{n_1}, b_{n_1}] \cup [a_0, b_0] \cup [a_{n_2}, b_{n_2}]$.

First, we consider the case that there exists $n_1 \geq N_0$ such that $p_0 \leq a_{n_1} < a_0$. Since $\lim_{n\to\infty} q_n = \lim_{n\to\infty} h(q_n) = q_0$, we have $\lim_{n\to\infty} a_n = a_0$. Then there exists $N \in \mathbb{N}$ such that $N \geq N_0$ and $a_{n_1} < a_n$ for all $n \geq N$. Moreover, $q_{n_1} \in [a_0, b_0]$.

In the case that $a_0 \leq a_n$ for all $n \geq N_0$, we can take $n_1 = 0$ and $N = N_0$. In both cases, there exist $N \geq N_0$ and $n_1 \in \mathbb{N} \cup \{0\}$ such that for all $n \geq N$, $a_{n_1} \leq a_n$ and $q_{n_1} \in [a_0, b_0]$.

Proceeding in a similar way, and taking a larger N if necessary, we have that there exist $N \geq N_0$ and $n_2 \in \mathbb{N} \cup \{0\}$ such that $b_n \leq b_{n_2}$ for all $n \geq N$ and $q_{n_2} \in [a_0, b_0]$.

Notice that $[a_n, b_n] \subset [a_{n_1}, b_{n_1}] \cup [a_0, b_0] \cup [a_{n_2}, b_{n_2}]$.

This completes the proof of the existence of n_1, n_2, and N.

Let $L = \{0, 1, \ldots, N\} \cup \{n_1, n_2\}$ and $J = \{h(q_n) : n \in L\}$. Then

$$\begin{aligned} D &= K(\operatorname{Im} h, t) \cap p_0p \\ &= \bigcup\{K(h(q_n), t) \cap p_0p : n \in \mathbb{N} \cup \{0\}\} \\ &= \bigcup\{[a_n, b_n] : n \in \mathbb{N} \cup \{0\}\} \\ &= \bigcup\{[a_n, b_n] : n \in L\} \\ &= \bigcup\{K(h(q_n), t) \cap p_0p : n \in L\} \\ &= K(J, t) \cap p_0p. \end{aligned}$$

□

Given $A \subset \mathbb{R}$ such that A is closed and has m components $C_1, \ldots, C_m$, define

$$\lambda(A) = \textstyle\sum_{i=1}^{m} \text{diameter}(C_i).$$

Lemma 2.2. *Let $e > 0$, $A, B \in 2^{[0,e]}$, and $\delta > 0$ be such that A has m components $C_1, \dots, C_m$; B has k components; $H(A,B) < \delta$; and dist$(C_i, C_j) \geq 4\delta$ if $i \neq j$. Then $|\lambda(A) - \lambda(B)| < 4k\delta$.*

Proof. We suppose that for each $i \in \{1, \dots, m\}$, $C_i = [a_i, b_i]$ and $b_i + 4\delta \leq a_{i+1}$, if $i < m$. Then the sets $(a_1 - \delta, b_1 + \delta), \dots, (a_m - \delta, b_m + \delta)$ are pairwise disjoint. Since $H(A,B) < \delta$, we have that B is contained in the union of these sets and intersects each of them. This implies that B has at least m components, so $k \geq m$.

Given $i \in \{1, \dots, m\}$, let $D_1^{(i)}, \dots, D_{r(i)}^{(i)}$ be the components of B contained in $(a_i - \delta, b_i + \delta)$; we suppose that for each $j \in \{1, \dots, r(i)\}$, $D_j^{(i)} = [c_j^{(i)}, e_j^{(i)}]$ with $e_j^{(i)} < c_{j+1}^{(i)}$ if $j < r(i)$.

Let $1 \leq j < r(i)$. If $e_j^{(i)} + \delta < c_{j+1}^{(i)} - \delta$, then fix a point $u \in (e_j^{(i)} + \delta, c_{j+1}^{(i)} - \delta)$. Since $c_{j+1}^{(i)} \leq b_i + \delta$, we have $u \leq b_i$. Similarly, $a_i \leq u$. Thus, $u \in [a_i, b_i] \subset A$. However, $\operatorname{dist}(\{u\}, B) > \delta$. This contradicts the fact that $H(A,B) < \delta$. We have shown that $c_{j+1}^{(i)} - \delta \leq e_j^{(i)} + \delta$. Since $H(A,B) < \delta$, for each $i \in \{1, \dots, m\}$, $c_1^{(i)} - \delta < a_i \leq b_i < e_{r(i)}^{(i)} + \delta$. Therefore,

$$\begin{aligned} \lambda(B) &= \textstyle\sum_{i=1}^m \sum_{j=1}^{r(i)} (e_j^{(i)} - c_j^{(i)}) \leq \sum_{i=1}^m (e_{r(i)}^{(i)} - c_1^{(i)}) \\ &\leq \textstyle\sum_{i=1}^m (b_i + \delta - (a_i - \delta)) = \lambda(A) + 2m\delta \leq \lambda(A) + 2k\delta, \end{aligned}$$

and

$$\begin{aligned} \lambda(A) = \sum_{i=1}^m (b_i - a_i) &\leq \sum_{i=1}^m (e_{r(i)}^{(i)} + \delta - (c_1^{(i)} - \delta)) = 2m\delta + \sum_{i=1}^m (e_{r(i)}^{(i)} - c_1^{(i)}) \\ &= 2m\delta + \textstyle\sum_{i=1}^m [(e_{r(i)}^{(i)} - c_{r(i)}^{(i)}) + (c_{r(i)}^{(i)} - e_{r(i)-1}^{(i)}) + (e_{r(i)-1}^{(i)} - c_{r(i)-1}^{(i)}) + \\ &\qquad \cdots + (e_2^{(i)} - c_2^{(i)}) + (c_2^{(i)} - e_1^{(i)}) + (e_1^{(i)} - c_1^{(i)})] \\ &= 2m\delta + \textstyle\sum_{i=1}^m (\sum_{j=1}^{r(i)} (e_j^{(i)} - c_j^{(i)}) + \sum_{i=1}^m \sum_{j=1}^{r(i)-1} (c_{j+1}^{(i)} - e_j^{(i)}) \\ &\leq 2m\delta + \lambda(B) + 2k\delta \leq \lambda(B) + 4k\delta. \end{aligned}$$

□

By (5), for every $(S,t) \in S_c(X) \times (0,1]$, we can put $K(S,t)$ as the union of its components in the following way:

$$K(S,t) = D_1(S,t) \cup \cdots \cup D_{m(S,t)}(S,t).$$

Define $L : S_c(X) \times (0,1] \to \mathcal{F}(X)$ by

$$L(S,t) = \{\sigma(p_0, D_1(S,t)), \dots, \sigma(p_0, D_{m(S,t)}(S,t))\}.$$

2.3. **Properties of** L.

Let $S \in S_c(X)$ and $s, t \in (0,1]$. Then

(8) $|L(S,t)| = m(S,t)$,
(9) L is not necessarily continuous,
(10) $L(S,1) = \{p_0\}$,
(11) if $0 < s < t < 1$, then $|L(S,t)| \leq |L(S,s)|$.

Property (10) follows from (3).

We prove (11). Take $0 < s < t < 1$. Notice that $K(S,s) \subset K(S,t)$. Given a component D of $K(S,t)$, by (4), we have $\emptyset \neq D \cap S \subset D \cap K(S,s)$. This implies that the number of components of $K(S,t)$ is less than or equal to the number of components of $K(S,s)$. By (8), $|L(S,t)| \leq |L(S,s)|$.

Let $(S,t) \in S_c(X) \times (0,1]$. Given $p \in X$, since X is a dendrite, each of the sets $D_1(S,t) \cap p_0p, \ldots, D_{m(S,t)}(S,t) \cap p_0p$ is either the empty set or a (possibly degenerate) closed subarc of p_0p. Then we can define the number

$$\chi_{(S,t)}(p) = \textstyle\sum_{i=1}^{m(S,t)} \text{diameter}(D_i(S,t) \cap p_0p).$$

Since d is convex, $\chi_{(S,t)}(p) \leq d(p_0, p)$.

Define

$$\begin{aligned} \beta(p) = {} & \text{the unique point in } p_0p \text{ such that} \\ & d(p_0, \beta(p)) = d(p_0,p) - \chi_{(S,t)}(p). \end{aligned}$$

Define $G : S_c(X) \times [0,1] \to \mathcal{F}(X) \cup \{S\}$ by

$$G(S,t) = \begin{cases} \{\beta(p) : p \in L(S,t)\}, & \text{if } t > 0, \\ S, & \text{if } t = 0. \end{cases}$$

2.4. **Properties of** G.

Let $S \in S_c(X)$. Then

(12) $G(S,t) \in \mathcal{F}(X)$, for all $t > 0$;
(13) $G(S,1) = \{p_0\}$ and $G(S,0) = S$;
(14) let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X converging to a point $x \in X$. Let $\{(S_n,t_n)\}_{n=1}^{\infty}$ be a sequence in $S_c(X) \times (0,1]$ converging to the element $(S,t) \in S_c(X) \times (0,1]$. Then $\lim_{n\to\infty} \chi_{(S_n,t_n)}(x_n) = \chi_{(S,t)}(x)$;
(15) if $0 < t$, then $G(S,t) = \{\beta(q) : q \in K(S,t)\}$.
(16) G is continuous.

Property (12) follows from (8) and (13) follows from (10).

We prove (14). Since d is convex, we identify the arc p_0x with the interval $[0, d(p_0,x)]$, with p_0 corresponding with 0. We analyze two cases.

Case 1. $K(S,t) \cap p_0x \subset \{x\}$.

Given $\varepsilon > 0$, since $\lim_{n\to\infty} x_n = x$, there exists $N \in \mathbb{N}$ such that for each $n \geq N$, $x_n \in K(x,\varepsilon)$. Since $K(x,\varepsilon)$ is connected, $p_0x_n \subset p_0x \cup K(x,\varepsilon)$. In the case that $K(S_n,t_n)\cap(p_0x\setminus B(x,\varepsilon)) \neq \emptyset$ for infinitely many numbers n, since $\lim_{n\to\infty} K(S_n,t_n) = K(S,t)$, we obtain that $K(S,t) \cap (p_0x \setminus B(x,\varepsilon)) \neq \emptyset$, a contradiction. Thus, we may assume that for each $n \geq N$, $K(S_n,t_n)\cap p_0x \subset B(x,\varepsilon) \subset K(x,\varepsilon)$. Let $a_n, b_n \in K(S_n,t_n)\cap p_0x_n$ be such that $K(S_n,t_n) \cap p_0x_n \subset a_nb_n$. Since d is convex, $\chi_{(S_n,t_n)}(x_n) \leq d(a_n,b_n) \leq 2\varepsilon$. Therefore, $\lim_{n\to\infty} \chi_{(S_n,t_n)}(x_n) = 0 = \chi_{(S,t)}(x)$.

Case 2. $K(S,t) \cap p_0x$ is not contained in $\{x\}$.

Let $\varepsilon > 0$. Fix $\delta_0 > 0$ such that $4\delta_0 < \varepsilon$, $\delta_0 < \frac{d(p_0,x)}{2}$, and $(K(S,t) \cap p_0x) \setminus K(x,\delta_0) \neq \emptyset$.

Let $w_0 = \sigma(p_0, K(x,\delta_0))$.

By (6), $w_0 \in p_0x$. Notice that $w_0 \notin \{p_0,x\}$ and $K(S,t) \cap p_0w_0 \neq \emptyset$. We write $K(S,t) \cap p_0w_0$ as the union of its components:

$$K(S,t) \cap p_0w_0 = (D_1(S,t) \cap p_0w_0) \cup \cdots \cup (D_m(S,t) \cap p_0w_0).$$

Note that $m \leq m(S,t)$.

Given $i \in \{1,\ldots,m\}$, by Lemma 2.1, there exists a finite set J_i of S such that $D_i(S,t) \cap p_0w_0 = K(J_i,t) \cap p_0w_0$.

Note that we can assume that for each $p \in J_i$, $K(p,t)\cap p_0w_0 \neq \emptyset$. With this assumption, we have that the sets $J_1,\ldots,J_m$ are pairwise disjoint.

Set $J = J_1\cup\cdots\cup J_m$. Then $|J| \geq m$ and $K(S,t)\cap p_0w_0 = K(J,t)\cap p_0w_0$.

Take $\delta > 0$ such that

$$\begin{aligned} 8\delta < \min(&\{\operatorname{dist}(D_i(S,t) \cap p_0w_0, D_j(S,t) \cap p_0w_0) : i \neq j\} \\ &\cup\{\tfrac{\varepsilon}{2|J|}, d(p_0,w_0), \delta_0\}). \end{aligned}$$

Let $N \in \mathbb{N}$ be such that for each $n \geq N$, $H(S_n,S) < \delta$, $|t_n - t| < \delta$, and $d(x_n,x) < \delta$.

Take $n \geq N$.

For each $u \in J \subset S$, fix a point $v(u) \in S_n$ such that $d(u,v(u)) < \delta$ and let $w(u) = \sigma(v(u), p_0w_0)$.

2.5. **Property of** $v(u)$.

(17) Given $u \in J$, we have that
- (a) either $K(v(u),t_n) \cap p_0w_0 \neq \emptyset$ and $H(K(u,t) \cap p_0w_0, K(v(u),t_n) \cap p_0w_0) < 2\delta$
- (b) or $K(v(u),t_n) \cap p_0w_0 = \emptyset$ and $K(u,t) \cap p_0w_0 \subset K(w(u),2\delta)$.

Before continuing the proof of (14), we prove (17).

Notice that for each point $y \in K(v(u),t_n)$, we have $d(u,y) \leq d(u,v(u)) + d(v(u),y) < t_n + \delta < t + 2\delta$.

This proves that for each $y \in K(v(u),t_n)$, $d(u,y) < t + 2\delta$. Similarly, for each $z \in K(u,t)$, $d(v(u),z) < t_n + 2\delta$.

Let $w = \sigma(u, p_0w_0)$.

Since $K(u,t) \cap p_0w_0 \neq \emptyset$, there exists a point $y_1 \in p_0w_0$ such that $d(u,y_1) \leq t$. By (6), $w \in y_1u$. Since d is convex, we have $d(u,w) \leq t$. Given $y \in K(v(u),t_n) \cap p_0w_0$, by (6), $w \in yu$. As we saw before, $d(u,y) < t + 2\delta$. Let y_0 be the first point in the arc yu, going from y to u, such that $d(u,y_0) \leq t$. Since d is convex, $d(y,y_0) \leq 2\delta$. Since $d(u,w) \leq t$, $y_0 \in yw$. This implies that $y_0 \in p_0w_0$. We have shown that there exists $y_0 \in K(u,t) \cap p_0w_0$ such that $d(y,y_0) < 2\delta$. Thus, $K(v(u),t_n) \cap p_0w_0 \subset N(K(u,t) \cap p_0w_0, 2\delta)$.

Reasoning as in the last paragraph, in the case that $K(v(u),t_n) \cap p_0w_0 \neq \emptyset$, we have $K(u,t) \cap p_0w_0 \subset N(K(v(u),t_n) \cap p_0w_0, 2\delta)$. Therefore, $H(K(v(u),t_n) \cap p_0w_0, K(u,t) \cap p_0w_0) < 2\delta$.

Now, suppose that $K(v(u),t_n) \cap p_0w_0 = \emptyset$. Then $d(v(u),w(u)) > t_n$. Given $z \in K(u,t) \cap p_0w_0$, we know that $d(v(u),z) < t_n + 2\delta$. Since $w(u) \in v(u)z$ and d is convex, we have $d(v(u),z) = d(v(u),w(u)) + d(w(u),z)$. Thus, $d(w(u),z) < 2\delta$. This proves that $K(u,t) \cap p_0w_0 \subset K(w(u),2\delta)$, and the proof of (17) is finished.

Now we continue proving (14).

Let $e = d(p_0,w_0)$. Since d is convex, we identify the arc p_0w_0 with the interval $[0,e]$, with p_0 corresponding with 0.

Let $A = K(S,t) \cap p_0w_0$.

Define $B \subset p_0w_0$ in the following way. Given $u \in J$, if $K(v(u),t_n) \cap p_0w_0 \neq \emptyset$, let $B(u) = K(v(u),t_n) \cap p_0w_0$, and in the case that $K(v(u),t_n) \cap p_0w_0 = \emptyset$, let $B(u) = \{w(u)\}$. Then we define $B = \bigcup\{B(u) : u \in J\}$. Notice that each $B(u)$ is connected, so the number of components of B is finite; let us say that B has k components. Then $k \leq |J|$.

By (17), we have that for each $u \in J$, $H(B(u), K(u,t) \cap p_0w_0) \leq 2\delta$. Then $H(B, K(J,t) \cap p_0w_0) = H(B, \bigcup\{K(u,t) \cap p_0w_0) : u \in J\} \leq 2\delta$. Therefore, $H(B,A) \leq 2\delta$.

By Lemma 2.2, $|\lambda(A) - \lambda(B)| < 8\,|J|\,\delta$. Then $\lambda(A) < \lambda(B) + 8\,|J|\,\delta$.

Since $x_n \in K(x,\delta_0)$ and $K(x,\delta_0)$ is connected, we have $p_0x_n \subset p_0w_0 \cup K(x,\delta_0)$ and $w_0 \in p_0x_n$. Then $w_0x_n \subset K(x,\delta_0)$ and $d(w_0,x_n) < 2\delta_0 < \frac{\varepsilon}{2}$.

Notice that $B \subset (K(S_n,t_n) \cap p_0w_0) \cup \{w(u) : u \in J\}$. Since J is finite, $\lambda(B) \leq \lambda(K(S_n,t_n) \cap p_0w_0) \leq \lambda(K(S_n,t_n) \cap p_0x_n) = \chi_{(S_n,t_n)}(x_n)$.

Since $K(S,t) \cap p_0x = (K(S,t) \cap p_0w_0) \cup (K(S,t) \cap w_0x)$, we have $\lambda(K(S,t) \cap p_0x) = \lambda(K(S,t) \cap p_0w_0) + \lambda(K(S,t) \cap w_0x)$. Then

$$\begin{aligned}\chi_{(S,t)}(x) &= \lambda(A) + \lambda(K(S,t) \cap w_0x) \leq \lambda(A) + d(w_0,x)\\ &\leq \lambda(A) + \delta_0 < \lambda(B) + 8\,|J|\,\delta + \delta_0\\ &< \chi_{(S_n,t_n)}(x_n) + 8\,|J|\,\delta + \delta_0 < \chi_{(S_n,t_n)}(x_n) + \varepsilon.\end{aligned}$$

We have shown that for each $n \geq N$, $\chi_{(S,t)}(x) - \varepsilon < \chi_{(S_n,t_n)}(x_n)$.

Now we will prove that there exists $N_1 \in \mathbb{N}$ such that for each $n \geq N_1$, $\chi_{(S_n,t_n)}(x_n) < \chi_{(S,t)}(x) + \varepsilon$.

In order to do this, take δ_0 and w_0 as above. Suppose that $K(S,t)\cap p_0x$ has k_0 components. We can also ask that $\delta_0 < \frac{\varepsilon}{4(k_0+2)}$.

Let $K(S,t) = C_1 \cup \dots \cup C_m \cup D$, where $C_1, \dots, C_m$ are the components of $K(S,t)$ intersecting p_0w_0 and $D = K(S,t) \setminus (C_1 \cup \dots \cup C_m)$. Then $m \leq k_0$. By (5), D is compact. Let $\delta_1 > 0$ be such that $\delta_1 < \delta_0$ and $K(p_0w_0, 2\delta_1) \cap D = \emptyset$.

Let $N_1 \in \mathbb{N}$ be such that for each $n \geq N_1$, $H(S_n, S) < \delta_1$, $|t_n - t| < \delta_1$, and $d(x_n, x) < \delta_1$. Since $d(x, w_0) \leq \delta_0$, we have $d(x_n, w_0) \leq 2\delta_0$.

Let $n \geq N_1$.

We are going to show that $K(S_n, t_n) \cap p_0w_0 \subset N(K(S,t) \cap p_0w_0, 2\delta_1)$.

Given $v \in K(S_n, t_n) \cap p_0w_0$, there exists $q \in S_n$ such that $d(v,q) \leq t_n$. Let $p \in S$ be such that $d(p,q) < \delta_1$. Then $d(v,p) \leq d(v,q) + d(q,p) < t + 2\delta_1$.

In the case that $d(v,p) \leq t$, we have $v \in K(S,t) \cap p_0w_0$ and we are done. Then we suppose that $d(v,p) > t$; define w_1 as the only point in vp satisfying that $d(p, w_1) = t$. Notice that $w_1 \in K(S,t)$. Since d is convex, it follows that $d(v, w_1) < 2\delta_1$. Then $w_1 \in K(S,t) \cap K(p_0w_0, 2\delta_1) \subset C_1 \cup \dots \cup C_m$. Thus, we may assume that $w_1 \in C_1$.

If $w_1 \in p_0w_0$, then $v \in N(K(S,t) \cap p_0w_0, 2\delta_1)$. Thus, suppose that $w_1 \notin p_0w_0$. Let $w_2 = \sigma(w_1, p_0w_0)$. Notice that $w_2 \in vw_1$.

Let $z \in C_1 \cap p_0w_0$. By (6), $w_2 \in w_1z$ and $w_2 \in vw_1$. Since C_1 is connected, $w_1z \subset C_1$, so $w_2 \in C_1$ and $d(v, w_2) \leq d(v, w_1) < 2\delta_1$. Hence, $v \in N(K(S,t) \cap p_0w_0, 2\delta_1)$. This completes the proof that $K(S_n, t_n) \cap p_0w_0 \subset N(K(S,t) \cap p_0w_0, 2\delta_1)$.

Since $x_n \in K(x, \delta_1)$, by the choice of w_0, we have $w_0 \in p_0x_n$. Then

$$\begin{gathered} K(S_n, t_n) \cap p_0x_n \subset \\ (K(S_n, t_n) \cap p_0w_0) \cup w_0x_n \subset \\ (N(C_1 \cap p_0w_0, 2\delta_1) \cup \dots \cup N(C_m \cap p_0w_0, 2\delta_1)) \cup w_0x_n. \end{gathered}$$

Thus,

$$\begin{gathered} \chi_{(S_n,t_n)}(x_n) \leq \\ d(w_0, x_n) + 4m\delta_1 + \sum_{i=1}^{m} \text{diameter}(C_i \cap p_0w_0) \leq \\ 2\delta_0 + 4m\delta_1 + \chi_{(S,t)}(x). \end{gathered}$$

Therefore, $\chi_{(S_n,t_n)}(x_n) \leq 2\delta_0 + 4k_0\delta_1 + \chi_{(S,t)}(x) < \chi_{(S,t)}(x) + \varepsilon$. Therefore, for each $n \geq \max\{N, N_1\}$, $\left|\chi_{(S_n,t_n)}(x_n) - \chi_{(S,t)}(x)\right| < \varepsilon$.

This completes the proof of (14).

We prove (15). Since $L(S,t) \subset K(S,t)$, we have $G(S,t) \subset \{\beta(q) : q \in K(S,t)\}$. Given $q \in K(S,t)$, let $D_j(S,t)$ be the component of $K(S,t)$ such that $q \in D_j(S,t)$ and let $p = \sigma(p_0, D_j(S,t))$. Then $p \in L(S,t)$ and $p \in p_0q$; so $p_0q = p_0p \cup pq$. Since $p, q \in D_j(S,t)$, we have $pq \subset D_j(S,t)$. Notice that $D_j(S,t) \cap p_0q = pq$. Thus, diameter$(D_j(S,t) \cap p_0q) = d(p,q)$.

If $i \in \{1, \ldots, m(S,t)\} \setminus \{j\}$, then $D_i(S,t) \cap pq = \emptyset$, so $D_i(S,t) \cap p_0q = D_i(S,t) \cap p_0p$. Hence,

$$\begin{aligned}\chi_{(S,t)}(q) &= \textstyle\sum_{i=1}^{m(S,t)} \text{diameter}(D_i(S,t) \cap p_0q) \\ &= d(p,q) + \textstyle\sum_{i \neq j} \text{diameter}(D_i(S,t) \cap p_0p) \\ &= d(p,q) + \chi_{(S,t)}(p).\end{aligned}$$

Since $d(p_0,q) = d(p_0,p) + d(p,q)$, we have $d(p_0,q) - \chi_{(S,t)}(q) = d(p_0,p) - \chi_{(S,t)}(p)$. Hence, $\beta(q) = \beta(p) \in G(S,t)$.

This finishes the proof of (15).

We now prove (16). Let $\{(S_n,t_n)\}_{n=1}^{\infty}$ be a sequence in $S_c(X) \times [0,1]$ converging to an element $(S,t) \in S_c(X) \times [0,1]$. We consider two cases.

Case 1. $t > 0$.

In this case we suppose that $t_n > 0$ for all $n \in \mathbb{N}$. By (15), $G(S,t) = \{\beta(p) : p \in K(S,t)\}$.

Given $p \in K(S,t)$, by continuity of K, there exists a sequence $\{p_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} p_n = p$ and for each $n \in \mathbb{N}$, $p_n \in K(S_n,t_n)$. Since X is a dendrite, $\lim_{n\to\infty} p_0p_n = p_0p$.

By (14), $\lim_{n\to\infty} \chi_{(S_n,t_n)}(p_n) = \chi_{(S,t)}(p)$. Since $\lim_{n\to\infty} d(p_0,p_n) = d(p_0,p)$, we have $\lim_{n\to\infty} d(p_0,\beta(p_n)) = d(p_0,\beta(p))$. This implies that $\lim_{n\to\infty} \beta(p_n) = \beta(p)$. This proves that $G(S,t) \subset \liminf_{n\to\infty} G(S_n,t_n)$ (for the definition and properties of lim inf and lim sup, see [12, §2, Ch. 4]).

With similar arguments it follows that $\limsup_{n\to\infty} G(S_n,t_n) \subset G(S,t)$. Therefore, $\lim_{n\to\infty} G(S_n,t_n) = G(S,t)$.

Case 2. $t = 0$.

Since $G(S_n,0) = S_n$ and $G(S,0) = S$, we suppose that $0 < t_n$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. Let $x_0 = \lim S$. Take $\delta_0 > 0$ such that $\delta_0 < \frac{\varepsilon}{3}$ and the set $S \setminus K(x_0,\delta_0)$ is nonempty (and finite). Suppose that $S \setminus K(x_0,\delta_0) = \{x_1, \ldots, x_k\}$.

Fix $\delta > 0$ such that $\delta < \frac{\delta_0}{4(k+1)}$ and the sets $K(x_1,2\delta), \ldots, K(x_k,2\delta)$ and $K(K(x_0,\delta_0),2\delta)$ are pairwise disjoint.

Let $N \in \mathbb{N}$ be such that for each $n \geq N$, $t_n < \delta$, $H(S_n,S) < \delta$, and $H(K(S_n,t_n),K(S,0)) = H(K(S_n,t_n),S) < \delta$.

Take $n \geq N$. Given $p \in K(S_n,t_n)$, there exists $u \in S_n$ such that $d(p,u) \leq t_n < \delta$. Let $w \in S$ be such that $d(u,w) < \delta$. Thus, $d(p,w) < 2\delta$. Then $p \in K(S,2\delta) \subset K(x_1,2\delta) \cup \cdots \cup K(x_k,2\delta) \cup K(K(x_0,\delta_0),2\delta)$. This proves that

$$\begin{gathered}K(S_n,t_n) \subset \\ K(S,2\delta) \subset \\ K(x_1,2\delta) \cup \cdots \cup K(x_k,2\delta) \cup K(K(x_0,\delta_0),2\delta).\end{gathered}$$

Given $p \in K(S_n, t_n)$, $K(S_n, t_n) \cap p_0p \subset (K(x_1, 2\delta) \cap p_0p) \cup \cdots \cup (K(x_k, 2\delta) \cap p_0p) \cup (K(K(x_0, \delta_0), 2\delta) \cap p_0p)$. This implies that $\chi_{(S_n,t_n)}(p) \leq 4k\delta + 2\delta_0 + 4\delta < 3\delta_0 < \varepsilon$.

Then $d(p_0, \beta(p)) = d(p_0, p) - \chi_{(S_n,t_n)}(p) > d(p_0, p) - \varepsilon = d(p_0, \beta(p)) + d(\beta(p), p) - \varepsilon$. Thus, $d(\beta(p), p) < \varepsilon$.

By (15), $G(S_n, t_n) \subset K(K(S_n, t_n), \varepsilon) \subset K(S, \delta+\varepsilon)$. Thus, $G(S_n, t_n) \subset K(S, 2\varepsilon)$.

On the other hand, given $q \in S$, there exists $p \in S_n \subset K(S_n, t_n)$ such that $d(q, p) < \delta$. Since $d(p, \beta(p)) < \varepsilon$, by (15), we conclude that $S \subset K(G(S_n, t_n), 2\varepsilon)$.

We have shown that for each $n \geq N$, $H(S, G(S_n, t_n)) < 3\varepsilon$. Therefore, $\lim_{n\to\infty} G(S_n, t_n) = S = G(S, 0)$. This completes the proof of (16).

Define $G_1 : S_c(X) \times [0, 1] \to S_c(X)$ by

$$G_1(S, t) = G(S, t) \cup S.$$

2.6. **Properties of G_1.**

Let $S \in S_c(X)$ and $t \in [0, 1]$. Then

(18) G_1 is continuous,
(19) $G_1(S, t) \in S_c(X)$,
(20) $G_1(S, 0) = S$,
(21) $G_1(S, 1) = S \cup \{p_0\}$.

Fix a point $q_0 \in X \setminus \{p_0\}$.

Let $e_0 = d(p_0, q_0)$. We identify the arc p_0q_0 with the interval $[0, e_0]$, with all the structure that the interval has as a subspace of $\mathbb{R}$, identifying the point p_0 with 0 and q_0 with e_0.

For each $n \in \mathbb{N}$, let $p_n = \frac{1}{n}q_0$, then $p_n \in p_0q_0$. Let $S_0 = \{p_0, p_1, p_2, \ldots\} \in S_c(X)$; note that $\lim S_0 = p_0$.

Let $G_2 : p_0q_0 \times [0, 1] \to X$ be given by

$$G_2(p, t) = tp.$$

Define $G_3 : S_c(X) \times [0, 1] \to S_c(X)$ by

$$G_3(S, t) = G(S, t) \cup G_2(S_0 \times \{t\}).$$

2.7. **Properties of G_3.**

Let $S \in S_c(X)$ and $t \in [0, 1]$. Then

(22) G_3 is continuous,
(23) $G_3(S, t) \in S_c(X)$,
(24) $G_3(S, 0) = S \cup \{p_0\}$,
(25) $G_3(S, 1) = S_0$.

Finally define $F : S_c(X) \times [0, 1] \to S_c(X)$ by

$$F(S,t)=\begin{cases} G_1(S,2t), & \text{if } t\in[0,\frac{1}{2}],\\ G_3(S,2t-1), & \text{if } t\in[\frac{1}{2},1].\end{cases}$$

2.8. **Properties of F.**

Let $S\in S_c(X)$. Then

(26) F is continuous,
(27) $F(S,0)=S$,
(28) $F(S,1)=S_0$.

Therefore, $S_c(X)$ is contractible.

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References

[1] R. H. Bing, *Partitioning a set*, Bull. Amer. Math. Soc. **55** (1949), 1101–1110.

[2] Javier Camargo, David Maya, and Patricia Pellicer-Covarrubias, *Path connectedness, local path connectedness and contractibility of $S_c(X)$*. Available at arXiv:1802.00725v1 [math.GN].

[3] S. García-Ferreira and Y. F. Ortiz-Castillo, *The hyperspace of convergent sequences*, Topology Appl. **196** (2015), part B, 795–804.

[4] S. García-Ferreira and R. Rojas-Hernández, *Connectedness like properties on the hyperspace of convergent sequences*, Topology Appl. **230** (2017), 639–647.

[5] S. García-Ferreira, R. Rojas-Hernández, and Y. F. Ortiz-Castillo, *Categorical properties on the hyperspace of nontrivial convergent sequences*, Topology Proc. **52** (2018), 265–279.

[6] Alejandro Illanes, *Open induced mappings, an example*, Topology Proc. **53** (2019), 37–45.

[7] Alejandro Illanes, *Contractibility of the hyperspace of sequences, harmonic fan.* Preprint.

[8] David Maya, Patricia Pellicer-Covarrubias, and Roberto Pichardo-Mendoza, *Induced mappings on the hyperspace of convergent sequences*, Topology Appl. **229** (2017), 85–105.

[9] David Maya, Patricia Pellicer-Covarrubias, and Roberto Pichardo-Mendoza, *Cardinal functions of the hyperspace of convergent sequences*, Math. Slovaca **68** (2018), no. 2, 431–450.

[10] David Maya, Patricia Pellicer-Covarrubias, and Roberto Pichardo-Mendoza, *General properties of the hyperspace of convergent sequences*, Topology Proc. **51** (2018), 143–168.

[11] Edwin E. Moise, *Grille decomposition and convexification theorems for compact metric locally connected continua*, Bull. Amer. Math. Soc. **55** (1949), 1111–1121.

[12] Sam B. Nadler, Jr., *Hyperspaces of Sets: A Text with Research Questions.* Monographs and Textbooks in Pure and Applied Mathematics, Vol. 49. New York-Basel: Marcel Dekker, Inc., 1978.

[13] Sam B. Nadler, Jr. *Continuum Theory: An Introduction.* Monographs and Textbooks in Pure and Applied Mathematics, Vol. 158. New York: Marcel Dekker, Inc., 1992.

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