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STABLE GROUP ACTIONS ON UNIFORM SPACES

PRAMOD DAS AND TARUN DAS

Abstract. We extend the notions of topological stability, shadowing and persistence for homeomorphisms to finitely generated group actions on uniform spaces and prove that an expansive action with either shadowing or persistence is topologically stable. Using the concept of null set of a Borel measure μ, we extend the notions of μ-expansivity, μ-topological stability, μ-shadowing and μ-persistence for homeomorphisms to finitely generated group actions on uniform spaces and prove that a μ-expansive action with either μ-shadowing or μ-persistence is μ-topologically stable.

1. Introduction

Topological stability is a fundamental notion of a dynamical system which guarantees that the qualitative behavior of trajectories remains unaffected by continuous small perturbations. In Lyapunov stability, one considers perturbations of initial conditions in a fixed system. Unlike in Lyapunov stability, in topological stability, one considers perturbations of the system itself. Walters' stability theorem applied to Anosov diffeomorphisms of compact smooth manifolds is one of the finest results in differentiable dynamics.

Theorem 1.1 ([15, Theorem 1]). *Let M be a compact manifold and let $f : M \to M$ be a diffeomorphism. If f is Anosov, then it is topologically stable.*

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The following result can be regarded as an extension of Theorem 1.1 to homeomorphisms on compact metric spaces because of the well-known fact that Anosov diffeomorphisms are expansive and have shadowing.

Theorem 1.2 ([1, Theorem 2.4.5]). *Let X be a compact metric space and let $f : X \to X$ be a homeomorphism. If f is expansive and has shadowing, then it is topologically stable.*

In [5], the authors prove the above result for homeomorphisms with persistence, a weaker notion than shadowing.

Theorem 1.3 ([5, Theorem 2]). *Let X be a compact metric space and let $f : X \to X$ be a homeomorphism. If f is expansive and persistent, then it is topologically stable.*

In [9], the authors generalize Theorem 1.2 above to homeomorphisms on non-compact, non-metrizable spaces which is as follows.

Theorem 1.4 ([9, Theorem 22]). *Let X be a first countable, locally compact, paracompact, Hausdorff uniform space and let $f : X \to X$ be a homeomorphism. If f is expansive and has shadowing, then it is topologically stable.*

In [6] and [2], the authors extend Theorem 1.2 and Theorem 1.3, respectively, to finitely generated group actions on compact metric spaces as follows.

Theorem 1.5 ([6, Theorem 2.8]). *Let X be a compact metric space and let Φ be a continuous action of a finitely generated group on X. If Φ is expansive and has shadowing, then it is topologically stable.*

Theorem 1.6 ([2, Theorem A]). *Let X be a compact metric space and let Φ be a continuous action of a finitely generated group on X. If Φ is expansive and persistent, then it is topologically stable.*

Our first aim is to extend Theorem 1.5 and Theorem 1.6 to finitely generated group actions on non-compact, non-metrizable spaces.

Theorem 1.7. *Let X be a first countable, locally compact, paracompact, Hausdorff uniform space and let Φ be a uniformly continuous action of a finitely generated group on X. If Φ is expansive and has either shadowing or persistence, then it is topologically stable.*

In the recent past, the notion of expansivity of a homeomorphism has been extended to Borel measure [12]. On the other hand, Keonhee Lee and C. A. Morales introduce in [11] the notions of shadowing and topological stability for Borel measures on compact metric spaces. They prove that expansive measures with shadowing are topologically stable.

Theorem 1.8 ([11, Theorem 3.1]). *Let X be a compact metric space and let $f : X \to X$ be a homeomorphism. If a measure μ is expansive and has shadowing with respect to f, then it is topologically stable with respect to f.*

Very recently, the authors of [8] have extended this theorem to finitely generated group actions on compact metric spaces.

Theorem 1.9 ([8, Theorem 3.2]). *Let X be a compact metric space and let Φ be a continuous action of a finitely generated group on X. If Φ is μ-expansive and has μ-shadowing, then it is μ-topologically stable.*

Our second aim is to extend these results to finitely generated group actions on non-compact, non-metrizable spaces.

Theorem 1.10. *Let X be a first countable, locally compact, paracompact, Hausdorff uniform space and let Φ be a uniformly continuous action of a finitely generated group on X. If Φ is μ-expansive and has either μ-shadowing or μ-persistence, then it is μ-topologically stable.*

The paper is organized as follows. In §2, we discuss preliminaries for self-containment of the paper. In §3, we define the notions of topological stability, μ-topological stability, shadowing, μ-shadowing, μ-expansivity, persistence, and μ-persistence for finitely generated group actions on uniform spaces. In §4, we prove Theorem 1.7 and Theorem 1.9 and provide examples.

2. Preliminaries

The following terminologies regarding uniform space (in general, phase space of a dynamical system) is helpful to understand the dynamics.

Let $\Delta(X) = \{(x, x) \in X \times X \mid x \in X\}$ be the diagonal of a non-empty set X. For $T \subset X \times X$, we define $T^{-1} = \{(y, x) \in X \times X \mid (x, y) \in T\}$. A subset T of $X \times X$ is called *symmetric* if $T = T^{-1}$. For two subsets T_1 and T_2 of $X \times X$, the set $T_1 \circ T_2 = \{(x, y) \in X \times X \mid \text{there is } z \in X$ satisfying $(x, z) \in T_1$ and $(z, y) \in T_2\}$ is called the *composition* of T_1 and T_2. In the present paper, we assume that the phase space of a dynamical system is a uniform space $(X, \mathcal{U})$, where $\mathcal{U}$ is a collection of subsets of $X \times X$ satisfying the following properties [10]:

(i) If $T \in \mathcal{U}$, then $T \supset \Delta(X)$.
(ii) If $T_1 \in \mathcal{U}$ and $T_2 \supset T_1$, then $T_2 \in \mathcal{U}$.
(iii) If $T_1, T_2 \in \mathcal{U}$, then $T_1 \cap T_2 \in \mathcal{U}$.
(iv) If $T \in \mathcal{U}$, then $T^{-1} \in \mathcal{U}$.
(v) For every $T_1 \in \mathcal{U}$. there is a symmetric $T_2 \in \mathcal{U}$ such that $T_2 \circ T_2 \subset T_1$.

The members of $\mathcal{U}$ are called *entourages* of X. The set of closed symmetric entourages form a basis for the uniformity $\mathcal{U}$. If $(X,\mathcal{U})$ is a uniform space, then it has a topology in which a subset $Y \subset X$ is open if and only if there is an entourage T of X such that, for each $x \in Y$, the cross section $T[x] = \{y \in X \mid (x,y) \in T\}$ is contained in Y. An entourage M is said to be *proper* if for any compact subset $K \subset X$, $M[K] = \bigcup_{x \in K} M[x]$ is compact. An entourage T is said to be *wide* if there is a compact subset $K \subset X$ such that $T \cup (K \times X) = X \times X$. In other words, T is wide if there is a compact subset $K \subset X$ such that, for any $x \notin K$, we have $T[x] = X$. A measure μ on X is said to be *non-atomic* if there exists no point $x \in X$ such that $\mu(\{x\}) > 0$. A uniform space X is called *non-atomic* if there exists a non-atomic measure on X.

Throughout the paper, G denotes a finitely generated group and X a uniform space with uniformity $\mathcal{U}$. We shall consider a finite symmetric generating set S of the group G.

A map $\Phi : G \times X \to X$ is said to be a *continuous action* of G on X if the following hold:

(i) The map $\Phi_g = \Phi(g,.)$ is a homeomorphism for any $g \in G$.
(ii) $\Phi_e(x) = x$ for all $x \in X$, where e is the identity element of the group G.
(iii) $\Phi_{g_1 g_2}(x) = \Phi_{g_1}(\Phi_{g_2}(x))$ for all $x \in X$ and $g_1, g_2 \in G$.

An action Φ is said to be *uniformly continuous* if for each $g \in G$, Φ_g is a uniform equivalence. It follows that any continuous action on a compact uniform space is uniformly continuous. We denote by $Act(G,X)$, the set of all uniformly continuous actions of a finitely generated group G on the uniform space X.

Let X and Y be two uniform spaces. An action $\Phi \in Act(G,X)$ is said to be *uniformly conjugate* to another action $\Psi \in Act(G,Y)$ if there is a uniform equivalence $h : X \to Y$ such that $h \circ \Phi_g = \Psi_g \circ h$ for all $g \in G$.

Tullio Ceccherini-Silberstein and Michel Coornaert [4] study expansivity of group actions on uniform spaces and their definition follows.

An action $\Phi \in Act(G,X)$ is called *expansive* if there is a closed entourage D such that for any distinct $x, y \in X$, there is $g \in G$ such that $(\Phi_g(x), \Phi_g(y)) \notin D$. Such an entourage D is called an *expansive entourage* for Φ.

3. Definitions and General Properties

We first define the notion of μ-expansivity of group actions on uniform spaces.

Definition 3.1. An action $\Phi \in Act(G,X)$ is called *μ-expansive* for some non-atomic measure μ if there is a closed entourage D such that $\mu(\Gamma_D(x)) = 0$ for all $x \in X$, where $\Gamma_D(x) = \{y \in X \mid (\Phi_g(x), \Phi_g(y)) \in D$ for all $g \in G\}$. Such D is called a *μ-expansive entourage* for Φ.

The most important result which allows us to extend the argument in proving the stability theorems to non-compact spaces is given in the following lemma. This result is partially proved for homeomorphisms in [9].

Lemma 3.2. *Let X be a first countable, locally compact, paracompact, Hausdorff uniform space and let μ be a non-atomic measure on X. If $\Phi \in Act(G,X)$ is expansive (μ-expansive), then there is a proper expansivity (μ-expansivity) entourage for Φ.*

Proof. Let $E \in \mathcal{U}$ be an expansivity (μ-expansivity) entourage for Φ. Since X is locally compact, each $x \in X$ has an open, relatively compact neighborhood U_x. Since X is paracompact, the open cover $\{U_x\}_{x \in X}$ has a closed (and hence compact) locally finite refinement V_α [10, Chapter 5, Theorem 28]. Then $A = E \cap (\bigcup_\alpha (V_\alpha \times V_\alpha))$ is a proper expansivity (μ-expansivity) entourage. □

Next, we introduce the notions of shadowing, persistence, and topological stability for group actions on uniform spaces and their measurable versions.

Definition 3.3. (i) Let $\Phi \in Act(G,X)$ and let $D \in \mathcal{U}$ be given. Then $\{x_g\}_{g \in G}$ is said to be a *D-pseudo orbit* (with respect to the generating set S of G) for Φ if $(x_{sg}, \Phi_s(x_g)) \in D$ for all $s \in S$ and $g \in G$. On the other hand, $\{x_g\}_{g \in G}$ is said to be *D-shadowed* by some point $x \in X$ if $(x_g, \Phi_g(x)) \in D$ for all $g \in G$. A D-pseudo orbit $\{x_g\}_{g \in G}$ is said to be through $B \subset X$ if $x_e \in B$, where e is the identity element of the group G.

(ii) Let $\mathcal{P}(X)$ be the power set of X and let $H : X \to \mathcal{P}(X)$ be a set valued map of X. We define the *domain* of H by $Dom(H) = \{x \in X \mid H(x) \neq \phi\}$. H is said to be *compact valued* if $H(x)$ is compact for each $x \in X$. We write $(Id, H) \in D$ for some $D \in \mathcal{U}$ if $H(x) \subset D[x]$ for each $x \in X$. H is called *upper semi-continuous* if for every $x \in Dom(H)$ and every open neighborhood O of $H(x)$ there exists $D \in \mathcal{U}$ such that $H(y) \subset O$ for all $y \in X$ with $(x,y) \in D$.

Definition 3.4. Let X be a uniform space with uniformity $\mathcal{U}$. Then

(i) $\Phi \in Act(G,X)$ is said to have *shadowing* (with respect to the generating set S of G) if for every $E \in \mathcal{U}$ there exists $D \in \mathcal{U}$ such that every D-pseudo orbit is E-shadowed by some point $x \in X$.

(ii) $\Phi \in Act(G, X)$ is said to be *topologically stable* (with respect to the generating set S of G) if for every $E \in \mathcal{U}$ there exists $D \in \mathcal{U}$ such that if $\Psi \in Act(G, X)$ is another action satisfying $(\Psi_s(x), \Phi_s(x)) \in D$ for all $x \in X$ and $s \in S$, then there exists a continuous map $f : X \to X$ such that $\Phi_g \circ f = f \circ \Psi_g$ for all $g \in G$ and $(x, f(x)) \in E$ for all $x \in X$.

(iii) $\Phi \in Act(G, X)$ is said to have *μ-shadowing* for some Borel measure μ (with respect to the generating set S of G), if for every $E \in \mathcal{U}$ there exists $D \in \mathcal{U}$ and a measurable set $B \subset X$ with $\mu(X \setminus B) = 0$ such that every D-pseudo orbit through B is E-shadowed by some point $x \in X$.

(iv) $\Phi \in Act(G, X)$ is said to be *μ-topologically stable* for some Borel measure μ (with respect to the generating set S of G), if for every $E \in \mathcal{U}$ there exists $D \in \mathcal{U}$ such that if $\Psi \in Act(G, X)$ is another action satisfying $(\Psi_s(x), \Phi_s(x)) \in D$ for all $x \in X$ and $s \in S$, then there is an upper semi-continuous compact valued map H of X with measurable domain such that the following conditions hold: (a) $\mu(X \setminus Dom(H)) = 0$; (b) $\mu \circ H = 0$; (c) $(Id, H) \in E$; (d) $\Phi_g \circ H = H \circ \Psi_g$ for all $g \in G$.

(v) $\Phi \in Act(G, X)$ is said to be *persistent* (with respect to the generating set S of G) if for every $E \in \mathcal{U}$ there is $D \in \mathcal{U}$ such that if $\Psi \in Act(G, X)$ is another action satisfying $(\Psi_s(x), \Phi_s(x)) \in D$ for all $x \in X$ and $s \in S$, then for every $x \in X$ there is $y \in X$ such that $(\Psi_g(x), \Phi_g(y)) \in E$ for all $g \in G$.

(vi) $\Phi \in Act(G, X)$ is said to be *μ-persistent* for some Borel measure μ (with respect to the generating set S of G) if for every $E \in \mathcal{U}$ there exists $D \in \mathcal{U}$ and a measurable set $B \subset X$ with $\mu(X \setminus B) = 0$ such that if $\Psi \in Act(G, X)$ is another action satisfying $(\Psi_s(x), \Phi_s(x)) \in D$ for all $x \in X$ and $s \in S$, then for each $x \in B$, there exists $y \in X$ such that $(\Psi_g(x), \Phi_g(y)) \in E$ for all $g \in G$.

Lemma 3.5. *Let S and T be two distinct generating sets of G. Then for any $\Phi \in Act(G, X)$,*

(i) *Φ has shadowing with respect to S if and only if it has shadowing with respect to T.*

(ii) *Φ is topologically stable with respect to S if and only if it is topologically stable with respect to T.*

(iii) *Φ is persistent with respect to S if and only if it is persistent with respect to T.*

(iv) *Φ has μ-shadowing with respect to S if and only if it has μ-shadowing with respect to T.*

(v) Φ *is* μ*-topologically stable with respect to* S *if and only if it is* μ*-topologically stable with respect to* T*.*
(vi) Φ *is* μ*-persistent with respect to* S *if and only if it is* μ*-persistent with respect to* T*.*

Proof. (i) Suppose Φ has shadowing with respect to S. Let $E \in \mathcal{U}$ be given and $D' \in \mathcal{U}$ be given for E by the shadowing of Φ. It suffices to show that there exists $D \in \mathcal{U}$ such that every D-pseudo orbit with respect to T is a D'-pseudo orbit with respect to S. Put $m := \max_{s\in S} l_T(s)$, where l_T is the word length metric on G induced by T. Choose $D_1 \in \mathcal{U}$ such that $D_1^m \subset D'$. Since Φ_g is a uniform equivalence for each $g \in G$ and S and T are finite, there exists $D \subset D_1$ such that $(\Phi_h(x), \Phi_h(y)) \in D_1$ for all $x, y \in X$ with $(x, y) \in D$ and $h \in G$ with $l_T(h) \leq m$. Any $s \in S$ can be written as $s = t_1t_2t_3...t_{l(s)}$, where $l(s) = l_T(s) \leq m$ and $t_i \in T$, $i = 1, 2, 3, ..., l(s)$. Let $\{y_g\}_{g\in G}$ be a D-pseudo orbit for Φ with respect to T, i.e, $(y_{tg}, \Phi_t(y_g)) \in D$ for all $t \in T$ and $g \in G$.

Claim. $(y_{sg}, \Phi_s(y_g)) = (y_{t_1...t_{l(s)}g}, \Phi_{t_1...t_{l(s)}}(y_g)) \in D'$ for all $s \in S$ and $g \in G$. Observe that

$$(y_{t_1...t_{l(s)}g}, \Phi_{t_1}(y_{t_2...t_{l(s)}g})) \in D \subset D_1,$$
$$(\Phi_{t_1}(y_{t_2...t_{l(s)}g}), \Phi_{t_1}(\Phi_{t_2}(y_{t_3...t_{l(s)}g})) \in D_1, ...,$$
$$(\Phi_{t_1...t_{l(s)-1}}(y_{l(s)g}), \Phi_{t_1...t_{l(s)-1}}(\Phi_{l(s)}(y_g))) \in D_1.$$

Then the claim holds by the definition of the composition of m-number of entourages and the fact that $D_1^m \subset D'$.

(ii) Suppose Φ is S-topologically stable. Let $E \in \mathcal{U}$ be given and $D' \in \mathcal{U}$ be given for E by the topological stability of Φ. It suffices to show that there exists $D \in \mathcal{U}$ such that for any $\Psi \in Act(G, X)$ if $(\Psi_t(x), \Phi_t(x)) \in D$ for all $t \in T$ and $x \in X$, then $(\Psi_s(x), \Phi_s(x)) \in D'$ for all $s \in S$ and $x \in X$. Put $m := \max_{s\in S} l_T(s)$, where l_T is the word length metric on G induced by T. Choose $D_1 \in \mathcal{U}$ such that $D_1^m \subset D'$. Since Φ_g is a uniform equivalence for each $g \in G$ and S and T are finite, there exists $D \in \mathcal{U}$ such that $(\Phi_h(x), \Phi_h(y)) \in D_1$ for all $x, y \in X$ with $(x, y) \in D$ and $h \in G$ with $l_T(h) \leq m$. Any $s \in S$ can be written as $s = t_1t_2t_3...t_{l(s)}$, where $l(s) = l_T(s) \leq m$ and $t_i \in T$, $i = 1, 2, 3, ..., l(s)$.

Claim. $(\Psi_s(x), \Phi_s(x)) = (\Psi_{t_1...t_{l(s)}}(x), \Phi_{t_1...t_{l(s)}}(x)) \in D_1^m \subset D'$ for all $s \in S$ and $x \in X$.

$$\text{Observe that } (\Psi_{t_1...t_{l(s)-1}}\Psi_{t_{l(s)}}(x), \Psi_{t_1...t_{l(s)-1}}\Phi_{t_{l(s)}}(x)) \in D_1,$$
$$(\Psi_{t_1...t_{l(s)-2}}\Psi_{t_{l(s)-1}}\Phi_{t_{l(s)}}(x)), \Psi_{t_1...t_{l(s)-2}}\Phi_{t_{l(s)-1}}\Phi_{t_{l(s)}}(x))) \in D_1...,$$
$$(\Psi_{t_1}\Phi_{t_2...t_{l(s)}}(x), \Phi_{t_1...t_{l(s)}}(x)) \in D_1.$$

Then the claim holds by the definition of the composition of m number of entourages and the fact that $D_1^m \subset D'$.

The proofs for (iii), (iv), (v), and (vi) follow similarly as those for (i) and (ii). □

This lemma shows that shadowing, topological stability, μ-shadowing, μ-topological stability, persistence, and μ-persistence of an action $\Phi \in Act(G, X)$ are independent of generators of G.

Remark 3.6. We call the above introduced notions as topological versions of the corresponding metric notions. (See [6] for expansivity, shadowing and topological stability; [2] for persistence; [8] for μ-expansivity, μ-shadowing, and μ-topological stability.) If we invoke the natural uniformity on a metric space, then the topological notions coincide with the corresponding metric notions. Since uniformity on a compact Hausdorff space is unique, topological notions on a compact metric space always coincide with the corresponding metric notions. Further, the reader can verify that the above introduced notions are invariant under uniform conjugacy.

Observe that shadowing implies μ-shadowing and topological stability implies persistence. We now show that μ-topological stability implies μ-persistence.

Proposition 3.7. *If an action $\Phi \in Act(G, X)$ is μ-topologically stable, then it is μ-persistent.*

Proof. Let $E \in \mathcal{U}$ be symmetric and let $D \in \mathcal{U}$ be given for E by the μ-topological stability of Φ. Let $\Psi \in Act(G, X)$ be another action such that $(\Psi_s(x), \Phi_s(x)) \in D$ for all $x \in X$ and $s \in S$. Let H be the set-valued map as in the definition of μ-topological stability of Φ. If $B = Dom(H)$, then $\mu(X \setminus B) = 0$ and, for every $x \in B$, there exists $y \in X$ such that $y \in H(x)$, which implies $\Phi_g(y) \in \Phi_g(H(x))$ for all $g \in G$. This gives $\Phi_g(y) \in H(\Psi_g(x)) \subset E[\Psi_g(x)]$ for all $g \in G$, which implies $(\Psi_g(x), \Phi_g(y)) \in E$ for all $g \in G$. That means Φ is μ-persistent. □

Theorem 3.8. *Let X be a non-atomic uniform space and let $\Phi \in Act(G, X)$. If Φ is topologically stable, then it is μ-topologically stable for any non-atomic measure μ on X.*

Proof. Suppose $\Phi \in Act(G, X)$ is topologically stable and μ is a non-atomic measure on X. We want to show that Φ is μ-topologically stable. Let E be given symmetric entourage and let $D \in \mathcal{U}$ be given for E by the topological stability of Φ. Let $\Psi \in Act(G, X)$ be another action such that $(\Psi_s(x), \Phi_s(x)) \in D$ for all $x \in X$ and $s \in S$. Then, there exists a continuous map $f : X \to X$ such that $\Phi_g \circ f = f \circ \Psi_g$ for all $g \in G$. Define the compact valued map $H : X \to \mathcal{P}(X)$ given by $H(x) = \{f(x)\}$ for all $x \in X$. Then H is upper semi-continuous because f is continuous.

Observe that $Dom(H) = X$ and, hence, $\mu(X \setminus Dom(H)) = 0$. Since μ is non-atomic, $\mu(H(x)) = \mu(\{f(x)\}) = 0$ for all $x \in X$, proving $\mu \circ H = 0$. Since $(x, f(x)) \in E$ for all $x \in X$, $(x, H(x)) \in E$ for all $x \in X$ and, hence, $(Id, H) \in E$. Further, since $\Phi_g \circ f = f \circ \Psi_g$ for all $g \in G$, we must have $\Phi_g \circ H = H \circ \Psi_g$ for all $g \in G$. This completes the proof. □

Corollary 3.9. *A complete separable metric space admitting topologically stable action which is not μ-topologically stable is at most countable.*

4. Stable Group Actions

In this section, our first aim is to prove the following theorem which extends Theorem 1.6.

Theorem 4.1. *Let X be a first countable, locally compact, paracompact, Hausdorff uniform space. If $\Phi \in Act(G, X)$ is expansive and persistent, then it is topologically stable.*

The following two lemmas will be useful to prove the theorem. Since the first one is [9, Lemma 17], we prove only the second lemma.

Lemma 4.2. *Let X be a first countable, locally compact, paracompact, Hausdorff uniform space. A map $f : X \to X$ is continuous if for any wide $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ such that $(f \times f)(V) \subset U$.*

Lemma 4.3. *Let X be a first countable, locally compact, paracompact, Hausdorff uniform space and let Φ be an action of G on X with proper expansive entourage A. Then, for any wide $U \in \mathcal{U}$, there exists a non-empty finite set $F \subset G$ such that $V_F(A) \subset U$, where $V_F(A) = \{(x, y) \in X \times X \mid (\Phi_g(x), \Phi_g(y)) \in A \text{ for all } g \in F\}$.*

Proof. If possible, suppose there exists a wide $U \in \mathcal{U}$ such that for each non-empty finite set $F \subset G$, there exists $(x_F, y_F) \in V_F(A) \cap ((X \times X) \setminus U)$. Choose a sequence of non-empty finite subsets $F_1 \subset F_2 \subset F_3 \subset ...$ of G such that $G = \bigcup_{n \geq 1} F_n$. Let $L = \{(x_n, y_n) \in V_{F_n}(A) \cap ((X \times X) \setminus U) \mid n \geq 1\}$. Since $U \in \mathcal{U}$ is wide, there exists a compact subset $K \subset X$ such that $U \cup (K \times X) = X \times X$. Thus, $L \subset ((X \times X) \setminus U) \cap A \subset K \times A[K]$, which is a compact subset of $X \times X$. So there exists a sequence $(x_{n_k}, y_{n_k}) \in L$ converging to $(x, y) \notin \Delta(X)$. On the other hand, we have $(\Phi_g(x), \Phi_g(y)) \in A$ for all $g \in G$, which leads to a contradiction of the fact that A is an expansive entourage for Φ. □

Proof of Theorem 4.1. Let $B \in \mathcal{U}$ be given. Lemma 3.2 proves the existence of a proper expansive entourage A for Φ. Choose symmetric entourage E such that $E^3 \subset A \cap B$. Let $D \in \mathcal{U}$ be given for E by the persistence of Φ, and let $\Psi \in Act(G, X)$ be another action satisfying

$(\Psi_s(x), \Phi_s(x)) \in D$ for all $x \in X$ and $s \in S$. Then, for each $x \in X$, there is unique $y \in X$ such that $(\Psi_g(x), \Phi_g(y)) \in E$ for all $g \in G$. Define a map $f : X \to X$ as given by $f(x) = y$. Observe that $(x, f(x)) \in E \subset E^3 \subset B$ for all $x \in X$.

We now show that $f(\Psi_h(x)) = \Phi_h(f(x))$ for all $x \in X$ and $h \in G$. We have $(\Psi_{gh}(x), \Phi_g(f(\Psi_h(x))) = (\Psi_g(\Psi_h(x)), \Phi_g(f(\Psi_h(x))) \in E$ for all $g, h \in G$. Also, $(\Psi_{gh}(x), \Phi_g(\Phi_h(f(x)))) = (\Psi_{gh}(x), \Phi_{gh}(f(x))) \in E$ for all $g, h \in G$. Since E is symmetric, $(\Phi_g(f(\Psi_h(x)), \Phi_g(\Phi_h(f(x))) \in E^2 \subset A$ for all $g, h \in G$. Since A is expansive, we have $f(\Psi_h(x)) = \Phi_h(f(x))$ for all $x \in X$ and $h \in G$.

Finally, we prove that $f : X \to X$ is continuous by using Lemma 4.2. Let U be a wide entourage. Then, by Lemma 4.3, there exists non-empty finite set $F \subset G$ such that $V_F(A) \subset U$. Let $W = \bigcap_{g \in F} (\Psi_g \times \Psi_g)(E)$ and we show that $(f \times f)(W) \subset U$. Let $(x, y) \in W$. Then, for any $g \in F$, $(\Psi_g(x), \Phi_g(f(x))) \in E$, $(\Psi_g(x), \Psi_g(y)) \in E$, and $(\Psi_g(y), \Phi_g(f(y))) \in E$. Since E is symmetric, $(\Phi_g(f(x)), \Phi_g(f(y)) = (f(\Psi_g(x)), f(\Psi_g(y))) \in E^3 \subset A$, which implies $(f \times f)(W) \subset V_F(A) \subset U$. This finishes the proof. $\square$

Corollary 4.4. *Let X be a first countable, locally compact, paracompact, Hausdorff uniform space. If $\Phi \in Act(G, X)$ is expansive and has shadowing, then it is topologically stable.*

Example 4.5. (i) Let $G =< a, b \mid ba = a^2 b >$ be an infinite solvable group, and let the action $\Phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ be defined by $\Phi_a((x_1, x_2, ..., x_n)) = (x_1, x_2, ..., x_n)$ and $\Phi_b((x_1, x_2, ..., x_n)) = (mx_1, mx_2, ..., mx_n)$, where $m > 1$. One can follow the same steps as in [3, Proposition 2.2] to show that Φ has shadowing. It is also easy to check that Φ is expansive. So by Corollary 4.4, Φ is topologically stable.

(ii) Let $G =< a, b : ba = a^n b >$, $\lambda > n > 1$ and

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} \lambda & 0 \\ 0 & n\lambda \end{bmatrix}$$

If $\Phi : G \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $\Phi_a(x) = Ax$ and $\Phi_b(x) = Bx$ for all $x \in \mathbb{R}^2$, then by [14], it has shadowing. Since Φ_b is hyperbolic, Φ is expansive. Therefore, by Corollary 4.4, Φ is topologically stable.

(iii) Let $\mathbb{R}$ be the real line and let $f : \mathbb{R} \to \mathbb{R}$ be the homeomorphism given by $f(x) = 2x$. Then the action $\Phi \in Act(\mathbb{Z}, X)$ given by $\Phi_n(x) = f^n(x)$ is expansive and has shadowing. Therefore, by Corollary 4.4, Φ is topologically stable.

Our next aim is to prove the following measurable version of Walters' stability theorem which extends Theorem 1.9.

Theorem 4.6. *Let X be a first countable, locally compact, paracompact, Hausdorff uniform space and let μ be a non-atomic measure on X. If $\Phi \in Act(G,X)$ is μ-persistent and μ-expansive, then it is μ-topologically stable.*

Proof. Let $E \in \mathcal{U}$ be given and let A be a μ-expansive entourage for Φ. Further, let $F \in \mathcal{U}$ be a closed symmetric entourage such that $F^2 \subset A \cap E$. Then, by Lemma 3.2, there is a proper entourage $E' \in \mathcal{U}$ such that $E'^2 \subset F$. Let $D \in \mathcal{U}$ and $B \subset X$ be given for E' by μ-persistence of Φ. Fix $\Psi \in Act(G,X)$ such that $(\Psi_s(x), \Phi_s(x)) \in D$ for all $x \in X$ and $s \in S$. Define a compact-valued map $H : X \to \mathcal{P}(X)$ by $H(x) = \bigcap_{g \in G} \Phi_{g^{-1}}(E'[\Psi_g(x)])$.

Let us first prove that $Dom(H)$ is measurable. Let $\{x_k\}_{k \in \mathbb{N}}$ be a sequence in $Dom(H)$ such that $x_k \to x$ as $k \to \infty$. Then, for each $k \in \mathbb{N}$, we can choose y_k such that $(\Psi_g(x_k), \Phi_g(y_k)) \in E'$ for all $g \in G$. In particular, $y_k \in E'[x_k]$ for all $k \in \mathbb{N}$. By local compactness and Hausdorff property of X, there exists a compact neighborhood K of x. Since $x_k \to x$ as $k \to \infty$, there is $N \in \mathbb{N}$ such that $x_k \in K$ for all $k \geq N$. Therefore, $y_k \in E'[K]$ for all $k \geq N$, and since $E'[K]$ is compact, we can assume that y_k converges to a point, say y in X. Taking $k \to \infty$, we have that $(\Psi_g(x), \Phi_g(y)) \in E'$ for all $g \in G$, which implies that $y \in H(x)$. Thus, $H(x) \neq \phi$ and, hence, $x \in Dom(H)$. By the first countability of X, we have $Dom(H)$ is closed and so is measurable.

Next, we prove that $\mu(X \setminus Dom(H)) = 0$. Since $(\Psi_s(x), \Phi_s(x)) \in D$ for all $x \in X$ and $s \in S$, for fixed $x \in B$, there exists $y \in X$ such that $(\Psi_g(x), \Phi_g(y)) \in E'$ for all $g \in G$. This shows that $H(x) \neq \phi$ for each $x \in B$ and, hence, $B \subset Dom(H)$. Therefore, $\mu(X \setminus Dom(H)) \leq \mu(X \setminus B) = 0$.

To prove the upper semi-continuity of H, fix $x \in Dom(H)$ and an open neighborhood O of $H(x)$. Since G is finitely generated, there are non-empty subsets $F_1 \subset F_2 \subset F_3 \subset \ldots$ such that $G = \bigcup_{n \geq 1} F_n$. If we define $H(y) = \bigcap_{m \geq 1} H_m(y)$, where $H_m(y) = \bigcap_{g \in F_m} \Phi_{g^{-1}}(E'[\Psi_g(y)])$, then we observe that each $H_m(y)$ is compact and $H_{m+1}(y) \subset H_m(y)$ for all $m \geq 1$. Taking $y = x$, we get that $H_m(x) \subset O$ for some $m \geq 1$. We claim that there exists $D' \in \mathcal{U}$ such that $H_m(y) \subset O$ for all $y \in X$ with $(x,y) \in D'$. Otherwise, there exists a sequence $y_k \to x$ and $z_k \in H_m(y_k) \setminus O$ for all $k \geq 1$. Since $y_k \to x$, for any compact set K containing x there exists natural number N such that $y_k \in K$ for all $k \geq N$. Thus, for all $k \geq N$, we have $z_k \in H_m(K)$. Since $H_m(K)$ is compact, we can assume that z_k converges to a point, say z, and observe that $z \notin O$. But $z_k \in H_m(y_k)$ implies that $(\Psi_g(y_k), \Phi_g(z_k)) \in E'$ for all $k \geq 1$ and $g \in F_m$. Letting $k \to \infty$, we have $(\Psi_g(x), \Phi_g(z)) \in E'$ for all $g \in F_m$. So $z \in H_m(x) \subset O$, a contradiction. Therefore, we conclude that $H(y) \subset H_m(y) \subset O$, whenever $(x, y) \in D'$. Thus, H is upper semi-continuous.

To prove that $\mu \circ H = 0$, fix $x \in X$ and let $y \in H(x)$. If $z \in H(x)$, then $(\Psi_g(x), \Phi_g(z)) \in E' \subset F$ for all $g \in G$. Since $y \in H(x)$, we have $(\Psi_g(x), \Phi_g(y)) \in E' \subset F$ for all $g \in G$. Since F is symmetric, we get $(\Phi_g(z), \Phi_g(y)) \in F^2 \subset A$ for all $g \in G$. Since A is symmetric, $H(x) \subset \Gamma_A(y)$, and since A is μ-expansive entourage for Φ, $\mu(H(x)) \leq \mu(\Gamma_A(y)) = 0$ for all $x \in X$. Also, observe that $H(x) \subset E'[x]$, and since $E' \subset E$, we have that $(Id, H) \in E$.

Finally, we prove that $\Phi_h \circ H = H \circ \Psi_h$ for all $h \in G$. If $x \in Dom(H)$, then $H(x) \neq \phi$ and, hence,

$$\Phi_h(H(x)) = \Phi_h(\bigcap_{g \in G} \Phi_{g^{-1}}(E'[\Psi_g(x)]))$$

$$= \bigcap_{g \in G} \Phi_{hg^{-1}}(E'[\Psi_g(x)])$$

$$= \bigcap_{g \in G} \Phi_{g^{-1}}(E'[\Psi_{gh}(x)])$$

$$= \bigcap_{g \in G} \Phi_{g^{-1}}(E'[\Psi_g(\Psi_h(x))]) = H(\Psi_h(x))$$

for all $h \in G$. On the other hand, if $x \notin Dom(H)$, then $\Psi_h(x) \notin Dom(H)$ and, hence, $\Phi_h(H(x)) = \phi = H(\Psi_h(x))$. This completes the proof. $\square$

Corollary 4.7. *Let X be a first countable, locally compact, paracompact, Hausdorff uniform space, and let μ be a non-atomic measure on X. If $\Phi \in Act(G, X)$ is μ-expansive and has μ-shadowing, then it is μ-topologically stable.*

Example 4.8. Let (M, d_0) be a locally compact metric space and let $g : M \to M$ be an expansive homeomorphism with shadowing. Further, assume that g has infinitely many periodic points, say $\{p_k\}_{k \in \mathbb{N}}$, which we can suppose belong to different orbits. Then the metric space X, similarly defined as in [7, Theorem A], is locally compact. Moreover, the same function $f : X \to X$ defined therein is an N-expansive uniform equivalence with shadowing property. Therefore, the action $\Phi \in Act(\mathbb{Z}, X)$ given by $\Phi_n(x) = f^n(x)$ is μ-expansive with shadowing. So by Corollary 4.7, Φ is μ-topologically stable.

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