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UNDECIDABILITY OF THE CARDINALITY OF C^*-EMBEDDED DISCRETE SUBSETS IN PRODUCTS OF NATURAL NUMBERS

YASUSHI HIRATA AND YUKINOBU YAJIMA

Abstract. Let $\mathbb{N}$ denote $\{1, 2, \cdots\}$ with the discrete topology. In this paper, we prove that it is undecidable in ZFC that every C^*-embedded (or C-embedded) discrete subset in $\mathbb{N}^{\mathfrak{c}}$ is countable, where $\mathfrak{c} = 2^\omega$. We also show that there is a C^*-embedded discrete subset in $\mathbb{N}^\kappa$ with cardinality κ only if κ has countable cofinality.

1. Introduction

For a set A, we denote by $|A|$ the cardinality of A. All cardinals dealt with here (κ and τ, etc.) are assumed to be infinite. In particular, ω, ω_1, and $\mathfrak{c}$ denote the first infinite cardinal, the first uncountable cardinal, and the cardinal of continuum, respectively.

We denote by CH the continuum hypothesis (i.e., $\mathfrak{c} = \omega_1$). Let $\mathbb{N}$ denote $\{1, 2, \cdots\}$ with the discrete topology. For a space X, we denote by X^κ the product of κ many copies of X. In particular, we deal with $\mathbb{N}^\kappa$. Let $w(X)$ denote the weight of X. For a Tychonoff space X, we denote by βX the Čech-Stone compactification of X.

For a space X, $A \subset X$ is *C^*-embedded* (*C-embedded*) in X if every continuous function from A to the unit interval $\mathbb{I} = [0, 1]$ (the real-line $\mathbb{R}$) can be continuously extended over X.

In [10], Elżbieta Pol and Roman Pol prove the following result.

(1) There is a closed discrete countable subset in $\mathbb{N}^{\mathfrak{c}}$ which is C^*-embedded but not C-embedded in $\mathbb{N}^{\mathfrak{c}}$.

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On the other hand, we prove the following in our previous paper [5].

(2) Under $\mathsf{MA}+\neg\mathsf{CH}$, every C^*-embedded subset in $\mathbb{N}^{\omega_1}$ is C-embedded in $\mathbb{N}^{\omega_1}$, where MA denotes Martin's axiom.

By (1) and (2), it is undecidable in ZFC that there is a C^*-embedded but not C-embedded (closed or closed discrete) subset in $\mathbb{N}^{\omega_1}$.

Let D_{ω_1} denote a discrete space of cardinality ω_1. It should be noticed by a known result (see Theorem 2.4) that βD_{ω_1} can be embedded in $\mathbb{N}^{2^{\omega_1}}$, which means the following.

(3) Under $2^{\omega_1} = \mathfrak{c}$, there is an uncountable C^*-embedded discrete subset in $\mathbb{N}^{\mathfrak{c}}$.

Since $2^{\omega_1} = \mathfrak{c}$ is inconsistent with CH, it is natural to ask the following question.

Problem 1.1. Under CH, is every C^*-embedded discrete subset in $\mathbb{N}^{\mathfrak{c}}$ countable?

Now, we recall a special case of D. P. Baturov's result in [2].

(4) For a cardinal τ, $2^\tau \leq \mathfrak{c}$ holds if and only if there is a (hereditarily) normal dense subspace of $\mathbb{N}^{\mathfrak{c}}$ which contains a closed discrete subset with cardinality τ.

In this paper, being stimulated by (4), we prove a more general result than that as follows.

(5) For a cardinal κ with $\kappa^\omega = \kappa$ and a cardinal τ, $2^\tau \leq \kappa$ holds if and only if there is a C^*-embedded discrete subset in $\mathbb{N}^\kappa$ with cardinality τ.

As an immediate consequence of (5), we can obtain an affirmative answer to Problem 1.1.

Throughout this paper, all spaces are assumed to be T_1.

2. Preliminaries

Let X be a space with a non-empty $A \subset X$. For a cover $\mathcal{U}$ of X, let us denote $\mathcal{U} \restriction A = \{U \cap A : U \in \mathcal{U}\}$. For a continuous function f on X, $f \restriction A$ denotes the restriction of f on A.

The following is well known as the characterizations of C^*-embedding and C-embedding (see [1]).

Lemma 2.1. *Let X be a space with a non-empty $A \subset X$. Then A is C^*-embedded (C-embedded) in X if and only if for every binary (countable) cozero cover $\mathcal{U}$ of A, there is a finite (countable) cozero cover $\mathcal{V}$ of X such that $\mathcal{V} \restriction A$ refines $\mathcal{U}$.*

Let $X = \prod_{\lambda\in\Lambda} X_\lambda$ be a product of spaces. For each $\Gamma \subset \Lambda$, let $X_\Gamma = \prod_{\lambda\in\Gamma} X_\lambda$ and π_Γ denotes the projection from X onto X_Γ.

We will make use of the following classical result.

Lemma 2.2 (Gleason, see [6]). *Let $X = \prod_{\lambda\in\Lambda} X_\lambda$ be a product of separable spaces. Let Y be a Hausdorff space, each point of which is a G_δ-set. Let f be a continuous map from X to Y. Then there are an $R \subset \Lambda$ and a continuous map g from X_R to Y such that $|R| \leq \omega$ and $f = g \circ \pi_R$.*

For two spaces X and E, we denote by $C(X, E)$ the set of all continuous maps from X to E. Let $C(X, E) = \{f_\xi : \xi \in \kappa\}$. We define $\Delta_{\xi\in\kappa} f_\xi : X \to E^\kappa$ by $(\Delta_{\xi\in\kappa} f_\xi)(x) = \langle f_\xi(x)\rangle_{\xi\in\kappa}$ for each $x \in X$. The map $\Delta_{\xi\in\kappa} f_\xi$ is called the *diagonal map* of $C(X, E)$.

The following is quite well known as the diagonal theorem.

Theorem 2.3 (see [4, Theorem 2.3.20] and [3, Theorem 2.2]). *Let X be a Tychonoff space with $|C(X, \mathbb{I})| = \kappa$, where $\mathbb{I} = [0, 1]$. Let φ be the diagonal map of $C(X, \mathbb{I})$. Then φ is a homeomorphic embedding of X to $\mathbb{I}^\kappa$.*

Let X and Y be spaces with $X \subset Y$. We say that X is *E-embedded* in Y if every continuous function from X to E is continuously extendable over Y. As is well known, a Tychonoff space X is $\mathbb{I}$-embedded in βX as a dense subspace.

A space X is *E-compact* if X is homeomorphic to a closed subspace in a product of copies of E. In particular, $\mathbb{R}$-compactness means realcompactness.

A Hausdorff space is *zero-dimensional* if it has a base consisting of clopen sets.

The following is also as well known as the above.

Theorem 2.4 (see [3, Theorem 2.2 and Proposition 2.4]). *Let X be a zero-dimensional space with $|C(X, E)| = \kappa$. Let φ be the diagonal map of $C(X, E)$. Then we have*

(i) *φ is a homeomorphic embedding of X into E^κ;*
(ii) *$\varphi(X)$ is E-embedding of X in E^κ;*
(iii) *if X is E-compact, $\varphi(X)$ is closed in E^κ.*

Remark 2.5. For a space X, we have $|C(X, \mathbb{N})| \leq |C(X, \mathbb{I})| \leq |C(X, \mathbb{R})| \leq 2^{d(X)}$, where $d(X)$ denotes the density of X.

Furthermore, we add two facts related to realcompactness, which we will use in the next section.

Fact 2.6 (see [3, Theorem 3.4] and [4, Exercise 3.11.D(a)]). *A discrete space D is realcompact if and only if $|D|$ is Ulam non-measurable.*

A Tychonoff space X is *strongly zero-dimensional* if each binary (finite, countable or locally finite) cozero cover of X has a pairwise disjoint clopen refinement, which is often denoted by $\dim X = 0$. Every strongly zero-dimensional space is zero-dimensional.

Fact 2.7 (see [3, Theorem 3.10] and [8])**.** *Every strongly zero-dimensional and realcompact space is $\mathbb{N}$-compact.*

Moreover, we make use of the first part in the following results. The proofs are quite standard.

Proposition 2.8. *Let X be a Tychonoff space with $A \subset X$.*

(1) *If A is strongly zero-dimensional and $\mathbb{N}$-embedded in X, then it is C-embedded in X.*
(2) *If A is C-embedded in X which is strongly zero-dimensional, then A is strongly zero-dimensional and $\mathbb{N}$-embedded in X.*

Proof. (1) Let $\{U_i : i \in \mathbb{N}\}$ be a countable cozero cover of A. By $\dim A = 0$, there is a pairwise disjoint clopen cover $\{V_i : i \in \mathbb{N}\}$ of A such that $V_i \subset U_i$ for each $i \in \mathbb{N}$. Take the function $f : A \to \mathbb{N}$ defined by $f(V_i) = \{i\}$ for each $i \in \mathbb{N}$. Since A is $\mathbb{N}$-embedded in X, there is a continuous function $g : X \to \mathbb{N}$ such that $g \restriction A = f$. Let $W_i = g^{-1}(i)$ for each $i \in \mathbb{N}$. Then $\{W_i : i \in \mathbb{N}\}$ is a pairwise disjoint clopen cover of X such that $W_i \cap A = V_i \subset U_i$ for each $i \in \mathbb{N}$.

(2) Recall the fact that $\dim A \le \dim X$ holds for any C^*-embedded subset A in X. Hence, we have $\dim A = 0$. Take a continuous function $f : A \to \mathbb{N}$. Then $\{f^{-1}(i) : i \in \mathbb{N}\}$ is a pairwise disjoint clopen cover of A. Since A is C-embedded in X, there is a countable cozero cover $\{V_i : i \in \mathbb{N}\}$ of X such that $V_i \cap A \subset f^{-1}(i)$ for each $i \in \mathbb{N}$. By $\dim X = 0$, there is a pairwise disjoint clopen cover $\{W_i : i \in \mathbb{N}\}$ of X such that $W_i \subset V_i$ for each $i \in \mathbb{N}$. Take the continuous function $g : X \to \mathbb{N}$ defined by $g(W_i) = \{i\}$ for each $i \in \mathbb{N}$. Then we have $g \restriction A = f$. $\square$

Let $\{0, 1\}$ denote a discrete space of two points.

Proposition 2.9. *Let X be a Tychonoff space with $A \subset X$.*

(1) *If A is strongly zero-dimensional and $\{0, 1\}$-embedded in X, then it is C^*-embedded in X.*
(2) *If A is C^*-embedded in X which is strongly zero-dimensional, then A is strongly zero-dimensional and $\{0, 1\}$-embedded in X.*

The proof is similar to that of Proposition 2.8. In fact, we may need only to replace $\mathbb{N}$ with $\{0, 1\}$ to obtain the proof of Proposition 2.9.

3. Small C^*-Embedded Discrete Subsets in $\mathbb{N}^\kappa$

Hereafter, we mainly deal with C^*-embedded discrete subsets in $\mathbb{N}^\kappa$ with cardinality τ. Since the cardinality of a discrete subset in a space

X does not exceed the weight $w(X)$ of X, we should note by $w(\mathbb{N}^\kappa) = \kappa$ that $\tau \leq \kappa$ always holds.

Lemma 3.1. *For a (Ulam non-measurable) cardinal τ, there is a C-embedded (closed) discrete subset in $\mathbb{N}^{2^\tau}$ with cardinality τ.*

Proof. Let D be a discrete space with $|D| = \tau$. By Theorem 2.4, since $|C(D, \mathbb{N})| = \omega^\tau = 2^\tau$, we may consider $D \subset \mathbb{N}^{2^\tau}$ and D is $\mathbb{N}$-embedded in $\mathbb{N}^{2^\tau}$. Since $\dim D = 0$, it follows from Proposition 2.8(1) that D is C-embedded in $\mathbb{N}^{2^\tau}$. For the parenthetic part, let τ be Ulam non-measurable. It follows from Fact 2.6 that the discrete subspace D in $\mathbb{N}^{2^\tau}$ is realcompact. By Fact 2.7, D is $\mathbb{N}$-compact. Hence, it follows from Theorem 2.4(iii) that D $(= \beta_{\mathbb{N}} D)$ is closed in $\mathbb{N}^{2^\tau}$. □

Lemma 3.2. *If there is a C^*-embedded discrete subset in $\mathbb{N}^\kappa$ with cardinality τ, then $2^\tau \leq \kappa^\omega$ holds.*

Proof. Let D be a C^*-embedded discrete subset in $\mathbb{N}^\kappa$ with $|D| = \tau \geq \omega$. Take an $A \subset D$. We take the continuous function $f_A : D \to \{0, 1\}$ defined by $f_A(A) = \{1\}$ and $f_A(D \setminus A) = \{0\}$. Obviously, $f_A \neq f_{A'}$ whenever $A, A' \subset D$ with $A \neq A'$. Since D is C^*-embedded in $\mathbb{N}^\kappa$, there is a continuous function $g_A : \mathbb{N}^\kappa \to \mathbb{I}$ such that $g_A \restriction D = f_A$. It follows from Lemma 2.2 that there are a countable subset R_A of κ and a continuous function $h_A : \mathbb{N}^{R_A} \to \mathbb{I}$ such that $g_A = h_A \circ \pi_{R_A}$, where π_{R_A} denotes the projection from $\mathbb{N}^\kappa$ onto $\mathbb{N}^{R_A}$. Note that

$$|\{g_A : A \subset D\}| = |\{f_A : A \subset D\}| = 2^{|D|} = 2^\tau.$$

Since $R_A \in [\kappa]^\omega$ for each $A \subset D$, we have $|\{R_A : A \subset D\}| \leq \kappa^\omega$. Since $\mathbb{N}^{R_A}$ is separable (metric) and h_A is continuous, it follows that

$$|\{g_A : A \subset D\}| = |\{h_A \circ \pi_{R_A} : A \subset D\}| \leq \mathfrak{c}^\omega \cdot \kappa^\omega = \kappa^\omega.$$

Hence, we obtain $2^\tau \leq \kappa^\omega$. □

Since $\mathbb{N}^\xi$ is C-embedded in $\mathbb{N}^\kappa$ for any $\xi \leq \kappa$, Lemma 3.1 and Lemma 3.2 immediately yield the following theorem.

Theorem 3.3. *Let κ be a cardinal with $\kappa^\omega = \kappa$. Then the following are equivalent for a (Ulam non-measurable) cardinal τ.*

(a) *$2^\tau \leq \kappa$ holds.*
(b) *There is a C^*-embedded discrete subset in $\mathbb{N}^\kappa$ with cardinality τ.*
(c) *There is a C-embedded (closed) discrete subset in $\mathbb{N}^\kappa$ with cardinality τ.*

Remark 3.4. The condition $\kappa^\omega = \kappa$ is necessary for only (b) $\Rightarrow$ (a) as in Lemma 3.2.

Remark 3.5. Let D_τ be a discrete space of cardinality τ. Then βD_τ can be embedded in $\mathbb{N}^\kappa$ under $2^\tau = \kappa$ (by Theorem 2.4). This means (a) $\Rightarrow$ (b) as stated in the introduction.

Letting $\kappa = \mathfrak{c}$ and $\tau = \omega_1$, since $\mathfrak{c}^\omega = \mathfrak{c}$ and ω_1 is Ulam non-measurable, the following is an immediate consequence of Theorem 3.3.

Corollary 3.6. *The following are equivalent.*

(a) $2^{\omega_1} = \mathfrak{c}$ *holds.*
(b) *There is an uncountable C^*-embedded discrete subset in $\mathbb{N}^{\mathfrak{c}}$.*
(c) *There is an uncountable C-embedded closed discrete subset in $\mathbb{N}^{\mathfrak{c}}$.*

It is well known that $2^{\omega_1} = \mathfrak{c}$ is consistent. On the other hand, since $2^{\omega_1} = 2^{\mathfrak{c}} > \mathfrak{c}$ holds under CH, Corollary 3.6 immediately yields an affirmative answer to Problem 1.1 in the introduction.

Corollary 3.7. *Under* CH*, every C^*-embedded discrete subset in $\mathbb{N}^{\mathfrak{c}}$ is countable.*

From Corollary 3.6 and Corollary 3.7, we obtain the following.

Corollary 3.8. *It is undecidable in* ZFC *that every C^*-embedded (or C-embedded) discrete subset in $\mathbb{N}^{\mathfrak{c}}$ is countable.*

In the proof of Lemma 3.1, consider $\{0,1\}$ instead of $\mathbb{N}$ and use Proposition 2.9(1) instead of Proposition 2.8(1). Then we have the following lemma.

Lemma 3.9. *For a cardinal τ, there is a C^*-embedded discrete subset in $\{0,1\}^{2^\tau}$ with cardinality τ.*

Then we can obtain the following result without any restriction for κ and τ.

Proposition 3.10. *For two cardinals τ and κ, $2^\tau \leq \kappa$ holds if and only if there is a C^*-embedded discrete subset in $\{0,1\}^\kappa$ with cardinality τ.*

Proof. The "only if" part follows from Lemma 3.9. For the "if" part, let D be a C^*-embedded discrete subset in $\{0,1\}^\kappa$ with $|D| = \tau \geq \omega$. Since D is C^*-embedded in $\mathrm{Cl}_{\{0,1\}^\kappa} D$, we have $\mathrm{Cl}_{\{0,1\}^\kappa} D = \beta D$. It follows from [4, Theorem 3.6.11] that $2^\tau = w(\beta D) \leq w(\{0,1\}^\kappa) = \kappa$ holds. $\square$

Remark 3.11. A C-embedded subset in a (pseudo)compact space is pseudocompact. Since every infinite discrete space is not pseudocompact, we cannot replace "C^*-embedded" with "C-embedded" in Proposition 3.10 for any cardinal $\tau \geq \omega$.

4. C-Embedded but Not P-Embedded Subset in $\mathbb{N}^\kappa$

For a space X, $A \subset X$ is P-*embedded* in X if every continuous function from A to a Banach space B can be continuously extended over X. The following is also well known (for example, see [1, Theorem 14.7]).

Lemma 4.1. *Let X be a space with a non-empty $A \subset X$. Then A is P-embedded in X if and only if for every locally finite cozero cover $\mathcal{U}$ of A, there is a locally finite cozero cover $\mathcal{V}$ of X such that $\mathcal{V} \restriction A$ refines $\mathcal{U}$.*

In [5], we prove that every C-embedded subset in $\mathbb{N}^{\omega_1}$ is P-embedded in $\mathbb{N}^{\omega_1}$. So it is natural to ask the following.

Problem 4.2. Is every C-embedded subset in $\mathbb{N}^\kappa$ P-embedded in $\mathbb{N}^\kappa$ for any cardinal $\kappa > \omega_1$?

The following seems to be known. So we state only an outline of its proof.

Lemma 4.3 (folklore)**.** *Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be a product of separable spaces. Then there is no uncountable point-countable collection of non-empty open sets in X.*

Proof. Let $\{U_\gamma : \gamma \in \Gamma\}$ be an uncountable collection of non-empty open sets in X. For each $\gamma \in \Gamma$, we may assume that $U_\gamma = \pi_{r_\gamma}^{-1}(V_\gamma)$ for some $r_\gamma \in [\Lambda]^{<\omega}$ and some open set V_γ in X_{r_γ}. By the Δ-system lemma, there are an uncountable subset Γ' of Γ and a finite subset r of Λ such that $r_\gamma \cap r_{\gamma'} = r$ for any $\gamma, \gamma' \in \Gamma'$ with $\gamma \neq \gamma'$. Since X_r is separable, $\{\pi_r(V_\gamma) : \gamma \in \Gamma'\}$ is not point-countable in X_r. By the choice of $r_\gamma, \gamma \in \Gamma'$, we can show that $\{U_\gamma : \gamma \in \Gamma'\}$ is not point-countable in X. $\square$

Proposition 4.4. *Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be a product of separable spaces. Then every uncountable discrete subset in X is not P-embedded in X.*

Proof. Let D be an uncountable discrete subset in X. Assume that D is P-embedded in X. Since $\mathcal{U} = \{\{d\} : d \in D\}$ is a discrete clopen cover of D, there is a locally finite cozero cover $\mathcal{V}$ of X such that $\mathcal{V} \restriction D$ refines $\mathcal{U}$. For each $d \in D$, take a $V_d \in \mathcal{V}$ with $d \in V_d$. Since $V_d \cap D = \{d\}$ for each $d \in D$, $\{V_d : d \in D\}$ is an uncountable locally finite collection of non-empty cozero sets in X. This contradicts Lemma 4.3. $\square$

Considering $\tau = \omega_1$, Lemma 3.1 and Proposition 4.4 immediately yield a negative answer to Problem 4.2 as follows.

Corollary 4.5. *For each $\kappa \geq 2^{\omega_1}$, there is an uncountable C-embedded closed discrete subset in $\mathbb{N}^\kappa$ which is not P-embedded in $\mathbb{N}^\kappa$.*

5. Large C^*-Embedded Discrete Subsets in $\mathbb{N}^\kappa$

It follows from Theorem 3.3 that there is no C^*-embedded discrete subset in $\mathbb{N}^\kappa$ with cardinality κ when $\kappa^\omega = \kappa$. So it is natural to ask whether this is true without assuming $\kappa^\omega = \kappa$.

Problem 5.1. Let κ be an uncountable cardinal. Is there no C^*-embedded discrete subset in $\mathbb{N}^\kappa$ with cardinality κ?

Remark 5.2. In the case of $\tau = \kappa = \omega$, there is obviously an infinite C-embedded closed discrete subset in $\mathbb{N}^\omega$. In the case of $\tau = \kappa = \omega_1$, it follows from [5, Lemma 4.1] that there is no uncountable C^*-embedded discrete subset in $\mathbb{N}^{\omega_1}$.

Remark 5.3. Without "C^*-embedded" in Problem 5.1, Jan Mycielski [9] proves the existence of a closed discrete subset in $\mathbb{N}^\kappa$ with cardinality κ if κ is less than the first weakly inaccessible cardinal θ_1. However, it is consistent that 2^ω is weakly inaccessible or that $2^\omega > \theta_1$ holds (see [7, p. 34]).

Let X be a space with $A \subset X$ and $\mathcal{U}$ a cover of X. Let us denote $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. In particular, for each $x \in X$, let $\mathrm{St}(x,\mathcal{U}) = \mathrm{St}(\{x\},\mathcal{U})$.

Lemma 5.4. *Let κ be a cardinal. Let X be a space which has a base $\{K_\xi : \xi \in \kappa\}$ such that each binary cozero cover of X is refined by a cover $\{K_\xi : \xi \in R\}$ of X for some $R \subset \mu$ with $\mu < \kappa$. If F is a C^*-embedded subset in X, then each locally finite cozero cover of F has a subcover of cardinality less than κ.*

Proof. Let $\mathcal{V}$ be a locally finite cozero cover of F. We will find a subcover $\mathcal{V}'$ of $\mathcal{V}$ with $|\mathcal{V}'| < \kappa$. By induction, we define an increasing sequences $\{A_\xi : \xi \in \kappa+1\}$ and $\{B_\xi : \xi \in \kappa+1\}$ of subsets of F. Put $A_0 = B_0 = \emptyset$. If $\xi \in \kappa+1$ is a limit ordinal, then put $A_\xi = \bigcup_{\eta\in\xi} A_\eta$ and $B_\xi = \bigcup_{\eta\in\xi} B_\eta$. Let $\xi \in \kappa$ and assume that subsets A_ξ and B_ξ of F are defined. If $K_\xi \cap F \subset \mathrm{St}(A_\xi \cup B_\xi, \mathcal{V})$, then put $A_{\xi+1} = A_\xi$ and $B_{\xi+1} = B_\xi$. Otherwise, take an $a_\xi \in (K_\xi \cap F) \setminus \mathrm{St}(A_\xi \cup B_\xi, \mathcal{V})$ and put $A_{\xi+1} = A_\xi \cup \{a_\xi\}$. If $K_\xi \cap F \subset \mathrm{St}(A_{\xi+1} \cup B_\xi, \mathcal{V})$, then put $B_{\xi+1} = B_\xi$. Otherwise, take a $b_\xi \in (K_\xi \cap F) \setminus \mathrm{St}(A_{\xi+1} \cup B_\xi, \mathcal{V})$ and put $B_{\xi+1} = B_\xi \cup \{b_\xi\}$.

After finishing induction, let $V_A = \mathrm{St}(A_\kappa, \mathcal{V})$ and $V_B = \mathrm{St}(F \setminus V_A, \mathcal{V})$. Since V_A and V_B are the unions of locally finite collections of cozero sets in F, they are also cozero sets in F. Since $\{V_A, V_B\}$ is a binary cozero cover of F, it follows from Lemma 2.1 that there is a binary cozero cover $\{W_A, W_B\}$ of X such that $W_A \cap F \subset V_A$ and $W_B \cap F \subset V_B$. Take an $R \subset \mu$ with $\mu < \kappa$ such that $\{K_\xi : \xi \in R\}$ is a cover of X and refines

$\{W_A, W_B\}$. Let $\mathcal{V}' = \{V \in \mathcal{V} : V \cap (A_\mu \cup B_\mu) \neq \emptyset\}$. By the definition, we see that $|A_\mu|, |B_\mu| \leq |\mu| < \kappa$. So $|\mathcal{V}'| < \kappa$ holds since $\mathcal{V}$ is point finite.

It suffices to show that $\mathcal{V}'$ covers F. Pick an $x \in F$. Take a $\xi \in R$ with $x \in K_\xi$. Then we have $K_\xi \subset W_A$ or $K_\xi \subset W_B$. It follows from $x \in K_\xi \cap F$ and $\mathrm{St}(A_\xi \cup B_\xi, \mathcal{V}) \subset \mathrm{St}(A_{\xi+1} \cup B_\xi, \mathcal{V}) \subset \mathrm{St}(A_\mu \cup B_\mu, \mathcal{V}) = \bigcup \mathcal{V}'$ that $x \in \bigcup \mathcal{V}'$ holds in the case $K_\xi \cap F \subset \mathrm{St}(A_\xi \cup B_\xi, \mathcal{V})$ or $K_\xi \cap F \subset \mathrm{St}(A_{\xi+1} \cup B_\xi, \mathcal{V})$.

In the other case, we have $a_\xi \in (K_\xi \cap F) \setminus \mathrm{St}(A_\xi \cup B_\xi, \mathcal{V})$ and $b_\xi \in (K_\xi \cap F) \setminus \mathrm{St}(A_{\xi+1} \cup B_\xi, \mathcal{V})$. We shall derive a contradiction by showing that $b_\xi \notin W_A$ and $a_\xi \notin W_B$, i.e., $K_\xi \not\subset W_A$ and $K_\xi \not\subset W_B$. It suffices to show that $b_\xi \notin V_A$ and $a_\xi \notin V_B$ since $W_A \cap F \subset V_A$ and $W_B \cap F \subset V_B$.

For each $b \in F \setminus V_A$, it follows from $\mathrm{St}(a_\xi, \mathcal{V}) \subset V_A$ that $b \notin \mathrm{St}(a_\xi, \mathcal{V})$, so $a_\xi \notin \mathrm{St}(b, \mathcal{V})$ holds. Hence, $a_\xi \notin V_B$. Next, pick an $a \in A_\kappa$. Take a $\rho \in \kappa$ such that $a = a_\rho \in K_\rho \cap F \setminus \mathrm{St}(A_\rho \cup B_\rho, \mathcal{V})$ and $A_{\rho+1} = A_\rho \cup \{a_\rho\}$. If $\rho \leq \xi$, then $b_\xi \notin \mathrm{St}(A_{\xi+1} \cup B_\xi, \mathcal{V}) \supset \mathrm{St}(a, \mathcal{V})$ holds since $a = a_\rho \in A_{\rho+1} \subset A_{\xi+1}$. If $\rho > \xi$, then $a = a_\rho \notin \mathrm{St}(A_\rho \cup B_\rho, \mathcal{V}) \supset \mathrm{St}(b_\xi, \mathcal{V})$ since $b_\xi \in B_{\xi+1} \subset B_\rho$, so $b_\xi \notin \mathrm{St}(a, \mathcal{V})$ holds. Hence, we obtain $b_\xi \notin V_A$. $\square$

For a cardinal κ, we denote by $\mathrm{cf}(\kappa)$ the cofinality of κ.

Lemma 5.5. *Let κ be a cardinal with $\mathrm{cf}(\kappa) > \omega$. If F is a C^*-embedded subset in $\mathbb{N}^\kappa$, then each locally finite cozero cover of F has a subcover with cardinality less than κ.*

Proof. For each subset S of κ, let T_S be the set of all pairs $\langle r, t\rangle$ such that r is a finite subset of S and $t \in \mathbb{N}^r$. For each $\langle r, t\rangle \in T_S$, put $K_S(r,t) = \{x \in \mathbb{N}^S : x \restriction r = t\}$. Enumerating the members, let $T_\kappa = \{\langle r_\xi, t_\xi\rangle : \xi \in \kappa\}$ and put $K_\xi = K_\kappa(r_\xi, t_\xi)$ for each $\xi \in \kappa$. Then $\{K_\xi : \xi \in \kappa\}$ is a base of a space $X = \mathbb{N}^\kappa$ satisfying the assumption of Lemma 5.4. Actually, we see from Lemma 2.2 that each binary cozero cover of X is of the form $\mathcal{V} = \{V_i = \{x \in X : x \restriction S \in U_i\} : i \in 2\}$ for some countable subset S of κ and for some binary cozero cover $\{U_0, U_1\}$ of $\mathbb{N}^S$. Since T_S is a countable subset of T_κ, we can take a countable subset R of κ such that

$$\{\langle r, t\rangle \in T_S : K_S(r,t) \subset U_i \text{ for some } i \in 2\} = \{\langle r_\xi, t_\xi\rangle : \xi \in R\}.$$

Since $\mathrm{cf}(\kappa) > \omega$, we obtain a $\mu < \kappa$ with $R \subset \mu$. It is routine to check that $\{K_\xi : \xi \in R\}$ covers X and refines $\mathcal{V}$. $\square$

Theorem 5.6. *Let κ be a cardinal with $\mathrm{cf}(\kappa) > \omega$. Then there is no C^*-embedded discrete subset in $\mathbb{N}^\kappa$ with cardinality κ.*

Proof. Assume that there is a C^*-embedded discrete subset D in $\mathbb{N}^\kappa$ with $|D| = \kappa$. Then $\{\{d\} : d \in D\}$ is a locally finite cozero cover of D and its subcover is itself. By Lemma 5.5, we have $|D| < \kappa$. This is a contradiction. $\square$

As a special case, we have the following corollary.

Corollary 5.7. *Every C^*-embedded discrete subset in* $\mathbb{N}^{\omega_1}$ *is countable.*

This also yields Corollary 3.7 immediately.

Finally, we would like to close with an unsolved problem. The condition $\kappa^\omega = \kappa$ in Theorem 3.3 seems to be so strong that we raise the following question.

Problem 5.8. Let κ be a cardinal with $\kappa^\omega > \kappa \geq \omega$. Assume that there is a C^*-embedded (or C-embedded closed) discrete subset in $\mathbb{N}^\kappa$ with cardinality $\tau \geq \omega$. Is then $2^\tau \leq \kappa$ true?

Remark 5.9. If it would be proved that there is an uncountable C^*-embedded discrete subset in $\mathbb{N}^{\omega_2}$, under $2^\omega = 2^{\omega_1} = \omega_3$, Problem 5.1 would be negative.

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