http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 56, 2020

Pages 97–104

http://topology.nipissingu.ca/tp/

Asymmetrics, Coherence, and Metrization

by

H. H. Hung

Electronically published on August 9, 2019

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS

Volume 56 (2020)
Pages 97-104

E-Published on August 9, 2019

ASYMMETRICS, COHERENCE, AND METRIZATION

H. H. HUNG

Abstract. There is Niemytzki, the fountain-head of all classical metrization results, and there is Collins-Reed-Roscoe-Rudin. Neither one implies the other. In response to Nagata's call for unification, we provide here a theorem to which both are corollaries.

In [11] Professor Nagata registered his "surprise" over the metrization condition of Collins-Reed-Roscoe-Rudin (CRRR): T_1-spaces with an open point-network are metrizable [2], [4]. (We adopt here Balogh's terminology in preference to the designation "open decreasing (G)," the latter being so tentative). In CRRR, conspicuously absent is any indication of uniformity, a *sine qua non* among the "classical" results. Professor Nagata suspected something fundamental was setting apart CRRR from the "classical" conditions. And yet he asked for unification.

These "classical" conditions are all topological realizations [6], [7] of Niemytzki's coherent symmetrics [14] and the uniformity among them is there to account for the symmetry of the symmetric. On the other hand, CRRR is not and there is no need for it to have any uniformity.

To have unification, we need to banish the symmetry in the symmetric of Niemytzki's and study coherence anew in the context of what is to be called an asymmetric. Of our results, Corollary 3.1.1 accounts for, Theorem 3.1 improves upon, while Corollary 3.1.3 and Theorem 3.2 supplement CRRR. Corollary 3.1.4 is clearly an improvement on Niemytzki. Thus our Theorem 3.1 unites the "classical" results via Niemytzki on the one hand and CRRR on the other as its corollaries, as Professor Nagata wished. Apparently, efforts to derive CRRR from Niemytzki came to nil [11]. Neither does Niemytzki follow from CRRR, in spite of what Balogh [2] says. It is in this context that Theorem 3.1 is important.

2010 *Mathematics Subject Classification.* Primary 54E35.
Key words and phrases. asymmetric, coherence, metric, metrization, symmetric.

1. Preliminaries

We exhibit here an unpublished strengthening of a factorization of the property of having a base of countable order (Theorem 2.1 of [8]). Notation adopted in a later paper [9] is also used.

Definition 1.1. We say a T_1-space X has property $(*)$, if, given $x(1) \in X$ we let $U_1 \equiv X$; for $x(2) \in U_1 \setminus \{x(1)\}$, we can *choose* an open neighbourhood $U_2 \subset U_1 \setminus \{x(1)\}$; for any $x(3) \in U_2 \setminus \{x(2)\}$, we can *choose* an open neighbourhood $U_3 \subset U_2 \setminus \{x(2)\}$ ... and so on *ad infinitum*, such that if $\langle x(n) \rangle$ clusters to some $x \in \bigcap\{U_n : n \in \mathbb{N}\}$, the family $\{U_n : n \in \mathbb{N}\}$ is a base at x.

Clearly, quasi-metrizable spaces (10.1 of [5]) have property $(*)$. For, we can let $U_n, n > 1$, be any open ball at $x(n)$ of radius less than $\frac{1}{n}$ within $U_{n-1} \setminus \{x(n-1)\}$. Likewise, a T_1-space that admits a base of countable order (BCO) [1] has property $(*)$. T_1 o-spaces [9] also have property $(*)$. For, we can let $U_n, n > 1$, be $\mathbf{A}(x(n), U_{n-1} \setminus \{x(n-1\})$.

Definition 1.2. We say a T_1-space X has property $(\dagger)$, if, given $x(1) \in X$ we let $U_1 \equiv X$; for $x(2) \in U_1 \setminus \{x(1)\}$, we can *choose* an open neighbourhood $U_2 \subset U_1 \setminus \{x(1)\}$; for any $x(3) \in U_2 \setminus \{x(2)\}$, we can *choose* an open neighbourhood $U_3 \subset U_2 \setminus \{x(2)\}$ ... and so on *ad infinitum*, such that, if $x \in \bigcap\{U_n : n \in \mathbb{N}\}$, $\langle x(n) \rangle$ clusters to x.

Clearly, semi-stratifiable spaces have property $(\dagger)$, for, we can let U_n, $n > 1$, be $(U_{n-1} \setminus \{x(n-1)\}) \cap g(n, x(n))$ (notation as in Theorem 5.8 of [5]). Likewise $MP4-$ *spaces* of [8] have property $(\dagger)$.

Theorem 1.3. *First countable T_1-spaces with properties $(*)$ and $(\dagger)$ admit bases of countable order.*

Remark. Recall that the task of constructing a BCO is made much easier (see remarks after Definition 6.2 of [5]) with the introduction of the *sieve* [3] where there is an all-important requirement that every element of the *sieve* be the union of its immediate successors.

If we let every element of the sieve be an open neighbourhood $U(x)$ of a specific point x, and every $y \in U(x) \setminus \{x\}$ contribute an open neighbourhood within $U(x) \setminus \{x\}$ to make up the family of immediate successors of $U(x)$, then the requirement that the elements of every branch constitute a local base at the common point can be clearly and easily met, provided we have first countability, $(*)$ and $(\dagger)$.

2. Asymmetrics and Coherence

2.1. The concept of an asymmetric.

Simply put, if the triangle inequality of the *metric* is allowed to lapse, we have instead a *symmetric*. If, further, we allow the axiom of symmetry to lapse too, we have (what is going to be called) an *asymmetric*. To keep things simple, we are going to restrict the range of the distance function to the inverses of the natural numbers plus 0. Formally,

Definition 2.1.1. An *asymmetric* d on a set X is a function $d : X \times X \to \{1, \frac{1}{2}, \frac{1}{3}...., 0\}$ such that for all $x, y \in X, d(x, y) = 0 \Leftrightarrow x = y$ (cf. Nedev's o-metric [12]).

The (unqualified) *ball* of radius $\frac{1}{n}$ about x is $B_d(x, \frac{1}{n}) \equiv \{y \in X : d(x, y) \leq \frac{1}{n}\}$, subscript d often omitted.

These decreasing balls of course define the asymmetric as much as the asymmetric the balls. We write $\mathcal{B}_d$ for the collection $\{B_d(x, \frac{1}{n}) : n \in \mathbb{N}, x \in X\}$. The *smallest* ball about x to contain y, $y \neq x$, written $S(x, y)$, is $B(x, \frac{1}{n})$ for such an n that $y \in B(x, \frac{1}{n})$ and $y \notin B(x, \frac{1}{n+1})$.

Definition 2.1.2. A topology $\mathcal{J}$ (on X) is said to be generated by an asymmetric d if $U \in \mathcal{J} \Leftrightarrow$ for every $x \in U$, there is contained in U some *ball* about x.

Clearly, such a topology is T_1 and given $\langle x(n) \rangle$ and x on X, $d(x, x(n)) \to 0 \Rightarrow x(n) \to x$. If the topology is Hausdorff and limit points are unique, the converse is true. For, otherwise, $d(x, x(n)) \nrightarrow 0$, there is a subsequence outside some ball at x that nonetheless converges to x, even though the points of this subsequence form a closed set. When we speak of *an asymmetric on a topological space*, it is understood that the asymmetric generates the topology.

2.2. A special asymmetric.

The $\mathcal{B}_d$ of an asymmetric d on a topological space X is a *point-network* of Balogh [2] if there is a *shrinking* $\mathbf{A}$ *of open neighbourhoods* [8] such that, for every $x \in X$ and every open neighbourhood $U, S(y, x) \subset U$ if $y \in \mathbf{A}(x, U) \setminus \{x\}$.

Clearly, if $x(n+1) \in U_{n+1} \subset \mathbf{A}(x(n), U_n) \setminus \{x(n)\}$ for all $n \in \mathbb{N}$ and $x \in \bigcap\{U_n : n \in \mathbb{N}\}$, we have $d(x, x(n)) \to 0$ and X has property (†) as defined in §1.

Clearly, if U and V are open neighbourhoods of, respectively, x and y, $\mathbf{A}(x, U) \cap \mathbf{A}(y, V) \neq \emptyset \Rightarrow$ either $y \in U$ or $x \in V$; and, if U_n is an open neighbourhood of $x(n)$, for all $n \in \mathbb{N}, x(i) \in U_j$ and $x(j) \notin U_i$ whenever $i < j \Rightarrow \bigcap\{\mathbf{A}(x(n), U_n) : n \in \mathbb{N}\} = \emptyset$. And X is paracompact [2], [4], [8].

2.3. Various kinds of coherence.

We study the concept of coherence as Niemytzki did when he sought to generalize the seventh theorem of Pitcher and Chittenden [15] where the term coherence was first used. Only now we have more possibilities, having broken loose from the stricture of the requirement of symmetry. Thus we say the asymmetric d on a topological space X has

(1) 1-*coherence* if, given $\langle x(n)\rangle$, $\langle y(n)\rangle$ and x so that $x(n) \to x$, we have: $d(x(n), y(n)) \leq d(x(n), x)$ for all $n \in \mathbb{N} \Rightarrow y(n) \to x$;
(2) 2-*coherence* if, given $\langle x(n)\rangle$, $\langle y(n)\rangle$ and x so that $x(n) \to x$, we have: $d(x(n), x) \to 0$ *and* $d(x(n), y(n)) \leq d(x(n), x)$ for all $n \in \mathbb{N} \Rightarrow y(n) \to x$;
(3) (a) 3a-*coherence* if, given $\langle x(n)\rangle$, $\langle y(n)\rangle$ and x so that $d(x, x(n)) \to 0$, we have: $d(x(n), y(n)) \to 0 \Rightarrow d(x, y(n)) \to 0$;
(b) 3b-*coherence* if, given $\langle x(n)\rangle$, $\langle y(n)\rangle$ and x so that $d(x, x(n)) \to 0$, we have: $d(y(n), x(n)) \to 0 \Rightarrow y(n) \to x$;
(4) 4-*coherence* if, given $x, y \in X$ and $n \in \mathbb{N}$ such that $d(x, y) \leq \frac{1}{n}$, we have such an $m \in \mathbb{N}$ that $d(y, z) \leq \frac{1}{m} \Rightarrow d(x, z) \leq \frac{1}{n}$;
(5) 5-*coherence* if, given $\langle x(n)\rangle$ and x, we have $d(x, x(n)) \to 0 \Rightarrow d(x(n), x) \to 0$.

If the $\mathcal{B}_d$ of an asymmetric d on a topological space is a *point-network*, then d has 1-coherence (Proposition 2.3.2 below). Conversely, if we have first countability (Proposition 2.3.1 below). Thus 1-coherence is quite strong.

On the other hand, 3a-coherence on an asymmetric d is also quite strong. For one thing, it is strong enough to ensure that the balls are all neighbourhoods (Proposition 2.3.3 below) and, if these neighbourhoods are not open, there is *another* asymmetric δ with open balls *and* with 3a-coherence. Indeed, if d is, in addition, symmetric, then 3a-coherence by itself is capable of bringing forth metrizability in the topology (as Niemytzki would assert). What we designate 3a-coherence on our asymmetric is known as Condition (Π) in Nedev and Cŏban [13].

In contrast, the property 2-coherence is a property common to both 1-coherence and 3a-coherence and is relatively weak.

In the presence of 5-coherence, 3a-coherence implies 1-coherence. Even the weaker 2-coherence does, when $x(n) \to x \Rightarrow d(x, x(n)) \to 0$ (Proposition 2.3.5).

Clearly, 4-coherence on an asymmetric makes the balls open and open (topologically) balls endow 4-coherence on it. With this 4-coherence, an asymmetric d with 2-coherence implies ($*$) on X (Proposition 2.3.4).

Proposition 2.3.1. *On a first countable space, if an asymmetric d has 1-*coherence*, then $\mathcal{B}_d$ is a* point-network.

Proof. Suppose otherwise. Suppose $\mathcal{B}_d$ is not a point-network. Suppose there is an x, an open neighbourhood $U(x)$, a decreasing open base $\{V_n(x), n \in \mathbb{N}\}$ at x and $x(n) \in V_n(x)$ for every n so that, for every n, there is $y(n) \in S(x(n), x) \setminus U(x)$. Clearly $\langle y(n) \rangle$ does not converge to x even though 1-coherence says it should. □

Proposition 2.3.2. *On a topological space X, if an asymmetric d is such that $\mathcal{B}_d$ is a* point-network *then it has 1*-coherence.

Proof. Suppose otherwise. Suppose, given $\langle x(n) \rangle$, $\langle y(n) \rangle$ and x so that $x(n) \to x$ and $d(x(n), y(n)) \leq d(x(n), x)$ for all n, i.e., $y(n) \in S(x(n), x)$ for all n, we have $y(n) \nrightarrow x$. There is such an open neighbourhood $U(x)$ of x that $y(n) \notin U(x)$ for some arbitrarily large n. For this $U(x)$, we cannot have the $V(x)$ required in the definition of a point-network. □

Proposition 2.3.3. *If an asymmetric d on a topological space X has 3a-*coherence*, then all members of $\mathcal{B}_d$ are neighbourhoods generating a* first countable *topology.*

Proof. Given $B_d(x, \frac{1}{n})$. Let $Y \subset B_d(x, \frac{1}{n})$ be such that $y \in Y \Rightarrow B_d(y, \frac{1}{m}) \setminus B_d(x, \frac{1}{n}) \neq \emptyset$ for every $m \in \mathbb{N}$. For any $z \in B_d(x, \frac{1}{n}) \setminus Y$, we cannot have $B_d(z, \frac{1}{l}) \setminus [B_d(x, \frac{1}{n}) \setminus Y] \neq \emptyset$ for every $l \in \mathbb{N}$, because of the 3a-coherence of d. Therefore $B_d(x, \frac{1}{n}) \setminus Y$ is open and x is in it i.e., $B_d(x, \frac{1}{n})$ is a neighbourhood of x. □

Remark.

(1) cf. Theorem 2 of Martin [10] and Lemma 3 of Nedev and Cǒban [13].
(2) For any sequence $\langle x(n) \rangle, x(n) \to x \Rightarrow d(x, x(n)) \to 0$.
(3) An asymmetric d with 3a-*coherence* begets an asymmetric δ with 3a- *and* 4-*coherence*: We can let δ be such that $B_\delta(x, \frac{1}{n}) \equiv B_d^\circ(x, \frac{1}{n})$, for all $x \in X, n \in \mathbb{N}$, noting $d(x, y(n)) \to 0 \Rightarrow y(n) \to x$ and $y(n) \to x \Rightarrow \delta(x, y(n)) \to 0$. This δ also has 2-*coherence*.

Proposition 2.3.4. *If there is an asymmetric d on a topological space X with 2- and 4-*coherence*, then X has property $(*)$.*

Proof. We can, given $n > 1$, let U_n be any open neighbourhood of $x(n)$ so long as it is contained in $\bigcap\{S(x(m), x(n)) : m < n\}$, in $B(x(n), \frac{1}{n})$ and in $U_{n-1} \setminus \{x(n-1)\}$. For, for every m, there is an $n > m$ such that $x(n) \in S(x(m), x)$, if $\langle x(n) \rangle$ clusters to some $x \in \bigcap\{U_n : n \in \mathbb{N}\}$ and it is clear that $S(x(m), x(n)) = S(x(m), x)$ and $\{S(x(m), x) : m \in \mathbb{N}\}$ is a base at x. □

Remark. In particular, if d has 3a-*coherence*, then X has property $(*)$ (Remark (3) on 2.3.3).

Proposition 2.3.5. *If, on a topological space X, an asymmetric d has 2- and 5-*coherence*, it also has 1-*coherence*, provided* $x(n) \to x \Rightarrow d(x, x(n)) \to 0$.

Proof. Given $\langle x(n) \rangle$, $\langle y(n) \rangle$ and x so that $x(n) \to x$ and $d(x(n), y(n)) \leq d(x(n), x)$ for all n. We have $d(x, x(n)) \to 0$ and $d(x(n), x) \to 0$ and, because of 2-coherence, $y(n) \to x$, i.e., 1-coherence. □

Remark. In particular, if d has 3a- and 5-*coherence*, it also has 1-*coherence*.

3. Coherence on Asymmetrics and Metrization

It is well known and easily demonstrated that a T_1-space X is paracompact and has property (†) as defined in §1, if on it is a *point-network* of Balogh [2], [4], [8]. In view of Theorem 1.3, therefore, we need only (∗) as defined in §1 and first countability on such an X to conclude its metrizability. Thus where an asymmetric with 1-coherence falls short in its effort towards generating a metrizable topology, Proposition 2.3.4 above immediately suggests itself, resulting in the following.

Theorem 3.1. *A topological space X is metrizable if (and only if) on it is an* asymmetric *d with 1-*coherence *and an* asymmetric *δ with 2- and 4-*coherence *(each individually generating the topology).*

Corollary 3.1.1. *A topological space X is metrizable if (and only if) on it is an* asymmetric *d with 1- and 4-*coherence*. (Essentially Collins-Reed-Roscoe-Rudin.)*

Corollary 3.1.2. *A topological space X is metrizable if (and only if) on it is an* asymmetric *d with 2-, 4- and 5-*coherence.

Proof. Result follows from Proposition 2.3.5. □

Corollary 3.1.3. *A topological space X is metrizable if (and only if) on it is an* asymmetric *d with 1-*coherence *and an* asymmetric *δ with 3a-*coherence.

Proof. Asymmetric δ with 3a-coherence begets an asymmetric δ' with 3a- *and* 4-coherence (Remark 3 on 2.3.3) and *a fortiori* 2-coherence. Therefore Theorem 1 applies. □

Corollary 3.1.4. *A topological space X is metrizable if (and only if) on it is an* asymmetric *d with 3a- and 5-*coherence.

Proof. Clearly d also has 1-coherence (Remark on Proposition 2.3.5). There is a δ with 3a- *and* 4-coherence (Remark 3 on 2.3.3) and *a fortiori* 2-coherence. Therefore Theorem 1 applies. □

Remark. Clearly, Niemytzki's that a coherent symmetric generates a metrizable topology is a particular case.

Theorem 3.2. *A first countable space X is metrizable if (and only if) on it is an* asymmetric *d with 1- and 3b*-coherence*.*

Proof. $\mathcal{B}_d$ is a *point-network* on X (Proposition 2.3.1). There is therefore a shrinking $\mathbf{A}$ of open neighbourhoods such that, if $x_{n+1} \in U_{n+1} \subset \mathbf{A}(x_n, U_n) \setminus \{x_n\}$ for all $n \in \mathbb{N}$ and if there is an $x \in \bigcap\{U_n : n \in \mathbb{N}\}$, $d(x, x_n) \to 0$ (2.2). The family $\{\mathbf{A}(x_n, U_n) : n \in \mathbb{N}\}$ is a *base* at x. For, otherwise, there is such an open neighbourhood U of x that there is $y_n \in \mathbf{A}(x_n, U_n) \setminus U$ for every $n \in \mathbb{N}$. Clearly, $d(y_n, x_n) \leq \frac{1}{n}$ (because $y_n \in \mathbf{A}(x_m, U_m)$ for all $m \leq n$) and $d(y_n, x_n) \to 0$ and therefore in view of 3b-coherence $y_n \to x$ and we have a contradiction. With such a base we can claim $(*)$. $\square$

Clearly, Cor. 3.1.4 is (an improvement of) Niemytzki and Cor. 3.1.1 is essentially CRRR. Both are corollaries of Theorem 3.1. Thus we have the long-sought unification.

References

[1] A. V. Arhangel'skiĭ, *Some metrization theorems* (Russian), Uspehi Mat. Nauk **18** (1963), no. 5(113), 139–145.

[2] Zoltán T. Balogh, *Topological spaces with point-networks*, Proc. Amer. Math. Soc. **94** (1985), no. 3, 497–501.

[3] J. Chaber, M. M. Čoban, and K. Nagami, *On monotonic generalizations of Moore spaces, Čech complete spaces and p-spaces*, Fund. Math. **84** (1974), no. 2, 107–119.

[4] P. J. Collins, G. M. Reed, A. W. Roscoe, and M. E. Rudin, *A lattice of conditions on topological spaces*, Proc. Amer. Math. Soc. **94** (1985), no. 3, 487–496.

[5] Gary Gruenhage, *Generalized metric spaces* in Handbook of Set-Theoretic Topology. Ed. Kenneth Kunen and Jerry E. Vaughan. Amsterdam: North-Holland, 1984. 423–501.

[6] H. H. Hung, *A contribution to the theory of metrization*, Canadian J. Math. **29** (1977), no. 6, 1145–1151.

[7] H. H. Hung, *Topological symmetry and the theory of metrization* in Papers on General Topology and Applications (Gorham, ME, 1995). Ed. Susan Andima et al. Annals of the New York Academy of Sciences, 806. New York: New York Acad. Sci., 1996. 207–213.

[8] H. H. Hung, *Shrinkings of open neighbourhoods and a topological description of metrizability*, Topology Proc. **22** (1997), Summer, 139–153.

[9] H. H. Hung, *The question of Morton Brown and the developability of $w\Delta$-spaces*, Topology Proc. **25** (2000), Summer, 257–270 (2002).

[10] Harold W. Martin, *Metrization of symmetric spaces and regular maps*, Proc. Amer. Math. Soc. **35** (1972), 269–274.

[11] Jun-iti Nagata, *A survey of metrization theory II*, Questions Answers Gen. Topology **10** (1992), no. 1, 15–30.

[12] S. Nedev, *Generalized-metrizable spaces* (Russian), C. R. Acad. Bulgare Sci. **20**, (1967), 513–516.

[13] S. Ĭ. Nedev and M. M. Čoban, *On the theory of o-metrizable spaces. III* (Russian), Vestnik Moskov. Univ. Ser. I Mat. Meh. **27** (1972), no. 3, 10–15.

[14] V. W. Niemytzki, *On the "third axiom of metric space,"* Trans. Amer. Math. Soc. **29** (1927), no. 3, 507–513.

[15] A. D. Pitcher and E. W. Chittenden, *On the foundations of the calcul fonctionnel of Fréchet*, Trans. Amer. Math. Soc. **19** (1918), no. 1, 66–78.

Concordia University; Montreal, Quebec, Canada H3G 1M8
Email address: juliathung@gmail.com