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# ON THE PARACOMPACTNESS OF A CLASS OF MONOTONICALLY NORMAL SPACES

PETER NYIKOS AND JERRY E. VAUGHAN

Abstract. We prove that if a monotonically normal space $X$ is both a weak $P(\lambda)$-space and a $\lambda$-perfect space, then $X$ is paracompact. As a corollary, we obtain a new proof that linearly stratifiable spaces are paracompact. The main tool in our proofs is the Balogh–Rudin theorem that says a monotonically normal space is paracompact if and only if it has no closed subsets homeomorphic to a stationary subset of some regular uncountable cardinal $\kappa$.

## 1. Introduction

Jack C. Ceder [3] originally defined a stratifiable space to be a $T_1$-space with a $\sigma$-cushioned pair-base. It is more common, however, to use as the definition the following characterization of Carlos J. R. Borges [2] (or its complementary form in terms of closed sets). We state the definition for all linearly stratifiable spaces.

**Definition 1.1** ([11], c.f. [2]). A $T_1$-space $(X, \mathcal{T})$ is said to be *stratifiable over* $\lambda$, where $\lambda$ is an infinite cardinal number, provided there exists a function $S : \lambda \times \mathcal{T} \to \mathcal{T}$ satisfying for all $\beta < \lambda$ and $U \in \mathcal{T}$,

**$LS_1$:** $cl_X[S(\beta, U)] \subset U$;
**$LS_2$:** $\bigcup\{S(\beta, U) : \beta < \lambda\} = U$;
**$LS_3$:** if $U, W \in \mathcal{T}$ and $U \subset W$, then $S(\beta, U) \subset S(\beta, W)$;
**$LS_4$:** if $\gamma < \beta < \lambda$, then $S(\gamma, U) \subset S(\beta, U)$.

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The map $S$ is called a *$\lambda$-stratification for $X$*. A space that is stratifiable over some $\lambda$ is called *linearly stratifiable*. A space $(X, \mathcal{T})$ can be stratifiable over many different cardinals $\lambda$ (e.g., see [12]). In particular, if $C \subset \lambda$ is a cofinal subset of $\lambda$, then the restricted map $S \restriction (C \times \mathcal{T})$ satisfies $LS_1$–$LS_4$. Thus, for most purposes, we may assume that $\lambda$ is a regular cardinal, and we now make that assumption.

A space is stratifiable over $\omega$ if and only if it is a stratifiable space in the sense of Ceder and Borges.

A well-known class of $\lambda$-stratifiable spaces is what Felix Hausdorff [7] called $\omega_\mu$-metrizable spaces. The use of the notation $\omega_\mu$ has persisted for them, partly to avoid confusion with the class of $\kappa$-metrizable spaces, where $\kappa$ does not refer to a cardinal number.

Recall that a *paracompact space* is a Hausdorff space in which every open cover has a locally finite open refinement [4].

We define two concepts which are probably known and certainly well known in the countable case.

**Definition 1.2.** A space $X$ is said to be a *$\lambda$-perfect space* provided every open subset of $X$ is the union of at most $\lambda$ closed sets.

In the countable case, a space $X$ is an $\omega$-perfect (or perfect) space if every open set is an $F_\sigma$-set (the union of countably many closed sets). The word "perfect" has also been used to designate a closed, dense-in-itself subset of a (usually metric) space, so that it is useful at times to have the expression $\omega$-"perfect" available.

**Definition 1.3.** A space is called a *weak $P(\lambda)$-space* if every subset of $X$ of cardinality less than $\lambda$ is closed [4].

In the countable case, every space is a weak $P(\omega)$-space, and a space $X$ is a weak $P(\omega_1)$-space (or weak P-space) if every countable set is closed [8].

In the next section, we prove a number of results about stationary subsets of regular cardinals and about monotonically normal spaces, culminating in Theorem 2.5. This theorem is used to get a new proof, simpler than the earlier ones, of an important old result (see Theorem 2.6).

The standard proof that stratifiable spaces are paracompact uses a characterization of paracompactness by E. Michael [9] that says a space is paracompact if and only if every open cover has a cushioned refinement. Results of Ceder and Michael are extended to the general case of "stratifiable over $\lambda$" by J. E. Vaughan [11], resulting in a proof that linearly stratifiable spaces are paracompact when combined with a theorem of M. Jeanne Harris [5] which says that every linearly stratifiable space has a linearly cushioned pair-base such that the ordered pairs in the pair-base form a function. A simpler proof, also using Michael's theorem on cushioned refinements (but not pair-bases), is given in [12].

Our proof that linearly stratifiable spaces are paracompact does not use pair-bases or cushioned collections.

For basic topological notions, we refer to [4]. Our limited use of set theory is essentially self contained. These results were presented by the second author at the 32nd Summer Conference on Topology and its Applications 2017.

## 2. Stationary Sets and Linearly Stratifiable Spaces

Let $\kappa$ be a regular uncountable cardinal. We use the usual order topology on ordinals.

Recall that the set of all closed and unbounded subsets (club sets) of $\kappa$ generates a filter base $\mathcal{C}$ (called the club filter). A set $E \subset \kappa$ is said to be a *stationary set* provided $C \cap E \neq \emptyset$ for all $C \in \mathcal{C}$. Three trivial facts about stationary sets are

(1) stationary sets are unbounded in $\kappa$,
(2) the intersection of a stationary set with a club set is a stationary set, and
(3) any set that contains a stationary set is a stationary set.

Recall that a space $X$ is called *scattered* provided that for every non-empty $Y \subset X$, $Y$ has at least one isolated point in its subspace topology. It follows immediately from this definition that in a scattered space, the set of isolated points is dense. Clearly, every well-ordered set with the order topology is scattered (given a non-empty subset $Y$, the minimum element of $Y$ is clearly isolated in the subspace topology on $Y$). So every subspace of an ordinal is a scattered space; in particular, every stationary set is scattered.

We are grateful to the referee for providing the following two lemmas, and their proofs, to simplify the proof of our main theorem.

**Lemma 2.1.** *Let $E$ be a stationary subset of the regular uncountable cardinal $\kappa$. Then every set that is closed in the relative topology of $E$ and is a subset of the set $E^0$ of isolated points is bounded in $\kappa$.*

*Proof.* Suppose $H$ is unbounded in $E^0$. The set $C$ of limit points of $H$ in $\kappa$ is club, so $C \cap E \neq \emptyset$. It follows that $H$ is not closed in $E$. □

**Lemma 2.2.** *The set $E^0$ of isolated points in a stationary set $E$ in a regular uncountable cardinal $\kappa$ is not the union of fewer than $\kappa$-many subsets that are closed in $E$.*

*Proof.* If $\alpha < \kappa$, then the least element of $E$ strictly larger than $\alpha$ is an isolated point of $E$. Hence, $E^0$ is unbounded. Now the result is immediate from Lemma 2.1 and the regularity of $\kappa$. $\square$

The following theorem may be of independent interest. The referee suggested the first conclusion; the second is our original conclusion.

**Theorem 2.3.** *Let $E$ be a stationary subset of the regular uncountable cardinal $\kappa$. Then $E$ is not $\lambda$-perfect for any $\lambda < \kappa$, nor weak $P(\lambda)$ for any $\lambda \geq \kappa$. In particular, $E$ is not both $\lambda$-perfect and weak $P(\lambda)$ for any $\lambda$.*

*Proof.* That $E$ is not $\lambda$-perfect for any $\lambda < \kappa$ is immediate from Lemma 2.2. By either lemma above, $E$ is not discrete. Let $\delta \in E$ be a limit point of $E$. Then $[0, \delta]$ is not closed discrete and has cardinality $< \kappa$. Hence, $E$ is not weak $P(\lambda)$ for any $\lambda \geq \kappa$. $\square$

Next, we recall a powerful theorem by Z. Balogh and M. E. Rudin [1].

**Theorem 2.4.** *A monotonically normal space is paracompact if and only if it has no closed subsets homeomorphic to a stationary subset of some regular uncountable cardinal.*

Thanks to Theorem 2.4, we do not need the definition of monotone normality for our main theorem.

**Theorem 2.5.** *If a monotonically normal space $X$ is both a weak $P(\lambda)$-space and a $\lambda$-perfect space, then $X$ is paracompact.*

*Proof.* This is immediate from Theorem 2.3, the Balogh–Rudin theorem, and the observation that $\lambda$-perfect and weak $P(\lambda)$ are hereditary properties. $\square$

**Theorem 2.6.** *Every linearly stratifiable space is paracompact.*

*Proof.* Let $(X, \mathcal{T})$ be stratifiable over $\lambda$ with stratification map $S : \lambda \times \mathcal{T} \to \mathcal{T}$. Since linearly stratifiable spaces are monotonically normal [11], it suffices by Theorem 2.5 to show that $X$ is both a weak $P(\lambda)$-space and a $\lambda$-perfect space. That $X$ is $\lambda$-perfect follows immediately from $LS_1$ and $LS_2$.

To see that $X$ is a weak $P(\lambda)$-space, let $Y \subset X$ with $|Y| < \lambda$, and let $x \in X$. Since $X$ is $T_1$, we can show that $Y$ is closed by showing that $Y \setminus \{x\}$ is closed, so we may assume that $x \notin Y$. Let $U = X \setminus \{x\}$, an open set. By $LS_2$, for each $y \in Y$, there exists $\alpha_y < \lambda$ such that $y \in S(\alpha_y, U)$. Since $\lambda$ is regular, $\alpha = \sup\{\alpha_y : y \in Y\} < \lambda$; hence, by $LS_4$ and $LS_1$,

$$Y \subset \bigcup\{S(\alpha_y, U) : y \in Y\} \subset S(\alpha, U) \subset \overline{S(\alpha, U)} \subset U = X \setminus \{x\}.$$

Thus, $x$ is not a limit point of $Y$, and this shows that $Y$ is a closed set. $\square$

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