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CELLULAR COMPACTNESS IN FUNCTION SPACES

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Abstract. Given a Tychonoff space X, we provide some necessary and sufficient conditions (in terms of the topology of the space X) for $C_p(X,[0,1])$ to be cellular-compact. We show that countable compactness of $C_p(X,[0,1])$ implies its cellular compactness, but cellular compactness of $C_p(X,[0,1])$ does not imply existence of a dense countably compact subspace in the space $C_p(X,[0,1])$. We also establish that $C_p(X)$ is σ-cellular-compact if and only if X is finite. Besides, pseudocompleteness of $C_p(X)$ implies that $C_p(X,[0,1])$ is cellular-compact.

1. Introduction

Angelo Bella and Santi Spadaro introduce in [1] the class of cellular-Lindelöf spaces. Recall that a space X is *cellular-Lindelöf* if for any disjoint family $\mathcal{U}$ of non-empty open subsets of X, there exists a Lindelöf subspace $L \subset X$ such that $L \cap U \neq \emptyset$ for any $U \in \mathcal{U}$. Wei-Feng Xuan and Yan-Kui Song construct in [12] an example of a weakly Lindelöf space that is not cellular-Lindelöf. V. V. Tkachuk proves in [10] that cellular-Lindelöf spaces need not be weakly Lindelöf, while Bella and Spadaro establish in [2] that every cellular-Lindelöf monotonically normal space is Lindelöf and prove, under $2^{<\mathfrak{c}} = \mathfrak{c}$, that every normal cellular-Lindelöf first countable space has cardinality not greater than $\mathfrak{c}$.

In [11], Tkachuk and R. G. Wilson use the idea of Bella and Spadaro to define the class of cellular-compact spaces: They call a space X *cellular-compact* if for any disjoint family $\mathcal{U}$ of non-empty open subsets of X,

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there exists a compact set $K \subset X$ such that $K \cap U \neq \emptyset$ for any $U \in \mathcal{U}$. It is immediate that every cellular-compact space is cellular-Lindelöf and pseudocompact. It is shown in [11] that cellular-compact spaces have nice categorical properties: They are preserved by continuous images, finite unions, and extensions. It is also established in [11] that every first countable regular cellular-compact space is countably compact and has cardinality at most $\mathfrak{c}$.

In this paper, we study cellular compactness in function spaces. Although $C_p(X)$ is never pseudocompact, the space $C_p(X, [0,1])$ is even compact when X is discrete. Pseudocompactness of $C_p(X, [0,1])$ is equivalent to all countable subsets of X being closed and C^*-embedded (see [6, Problem 398]), while the space $C_p(X, [0,1])$ is countably compact if and only if X is a P-space; i.e., all G_δ-subsets of X are open (see [6, Problem 397]). We show, among other things, that countable compactness of $C_p(X, [0,1])$ implies that it is cellular-compact and there are spaces X such that $C_p(X, [0,1])$ is pseudocompact but not cellular-compact. We also give an example of a space X such that $C_p(X, [0,1])$ is cellular-compact, but there is no dense countably compact subspace in $C_p(X, [0,1])$.

2. Notation and Terminology

All spaces are assumed to be Tychonoff. Given a space X, the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. As usual, $\mathbb{R}$ is the set of reals, $\mathbb{I} = [0,1] \subset \mathbb{R}$, and $\mathbb{D} = \{0,1\} \subset \mathbb{I}$. For any space X, the cardinal $l(X) = \min\{\kappa :$ every open cover of X has a subcover of cardinality $\leq \kappa\}$ is called the *Lindelöf number* of X. The space X is *Lindelöf* if $l(X) \leq \omega$. The cardinal $wl(X) = \min\{\kappa :$ for every open cover $\mathcal{U}$ of the space X, there exists $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $\overline{\bigcup\mathcal{V}} = X\}$ is the *weak Lindelöf number* of X. A space X is *weakly Lindelöf* if $wl(X) \leq \omega$.

Given a space X, a set $A \subset X$ is *C^*-embedded* in X if every continuous function $f : A \to \mathbb{I}$ has a continuous extension over X; i.e., there exists a continuous function $\hat{f} : X \to \mathbb{I}$ such that $\hat{f}|A = f$. The set A is *C-embedded* in X if every continuous function $f : A \to \mathbb{R}$ has a continuous extension $\hat{f} : X \to \mathbb{R}$. The set A is *bounded* in X if $f(A)$ is bounded in $\mathbb{R}$ for any continuous function $f : X \to \mathbb{R}$.

A space X is *ω-bounded* if $\overline{A}$ is compact for every countable set $A \subset X$. We say that X is a *P-space* if every G_δ-subset of X is open in X. The expression υX denotes the Hewitt realcompactification of the space X. The space X is *Fréchet–Urysohn* if for any $A \subset X$ and $x \in \overline{A}$, there exists a sequence $\{a_n : n \in \omega\} \subset A$ such that $a_n \to x$. It is said that X has countable tightness; i.e., $t(X) \leq \omega$ if, for any $A \subset X$ and any point $x \in \overline{A}$, there is a countable set $B \subset A$ such that $x \in \overline{B}$. A family $\mathcal{B} \subset \tau^*(X)$ is a *π-base* in X if for every $U \in \tau^*(X)$, there exists $B \in \mathcal{B}$ such that $B \subset U$. The cardinal $c(X) = \sup\{|\mathcal{U}| : \mathcal{U}$ is a disjoint family of non-empty open subsets of $X\}$ is called the *Souslin number of* X. Given spaces X and Y, a map $f : X \to Y$ is a *condensation* if f is a continuous bijection. In this case, we say that X *condenses* onto Y. The cardinal $iw(X) = \min\{w(Y) : X$ condenses onto $Y\}$ is the *i-weight of* X.

Given spaces X and Y, we denote by $C(X,Y)$ the set of all continuous functions from X to Y; we write $C(X)$ instead of $C(X,\mathbb{R})$ and the set of all real-valued continuous bounded functions on X is denoted by $C^*(X)$. The space $C_p(X,Y)$ is the set $C(X,Y)$ endowed with the pointwise convergence topology; the standard base in $C_p(X,Y)$ consists of the sets

$$[x_1,\ldots,x_n;O_1,\ldots,O_n] = \{f \in C_p(X,Y) : f(x_i) \in O_i \text{ for all } i \le n\};$$

here $x_1,\ldots,x_n \in X$ and $O_1,\ldots,O_n \in \tau(Y)$. Analogously, $C_p(X)$ is the set $C(X)$ with the topology inherited from $\mathbb{R}^X$. If X is a space and $Y \subset X$, then $\pi_Y : C_p(X) \to C_p(Y)$ is the restriction map. If X and Y are spaces and $\varphi : X \to Y$ is a continuous onto map, then $\varphi^* : C_p(Y) \to C_p(X)$ is the dual map of φ defined by $\varphi(f) = f \circ \varphi$ for any $f \in C_p(Y)$.

The rest of our notation is standard and follows [3]; the reader can find more information on cardinal invariants in the survey of R. Hodel [4]. All necessary facts and notions of C_p-theory can be found in [6]–[8].

3. Countable Compactness of $C_p(X,\mathbb{I})$ Implies Its Cellular-Compactness

The purpose of this section is to show that countable compactness of $C_p(X,\mathbb{I})$ implies its cellular compactness but not vice versa. It is convenient to start with formulations of some basic properties of cellular-compact spaces.

Proposition 3.1 ([11, propositions 3.3 and 3.4])**.** *Suppose that X is a cellular-compact space. Then*

(a) *X is pseudocompact;*
(b) *any continuous image of X is cellular-compact;*
(c) *any regular closed subset of X is cellular-compact.*

Proposition 3.2 ([11, Proposition 3.5])**.** *Given any space X,*

(a) *if X is a finite union of cellular-compact spaces, then X is cellular-compact;*
(b) *if X contains a dense cellular-compact space, then X is cellular-compact.*

Proposition 3.3. *If X is a space such that $C_p(X,\mathbb{I})$ is cellular-compact, then $C_p(Y,\mathbb{I})$ is cellular-compact for any $Y \subset X$.*

Proof. The set $Z = \pi_Y(C_p(X,\mathbb{I}))$ is cellular-compact by Proposition 3.1(b). Since Z dense in $C_p(Y,\mathbb{I})$, we can apply Proposition 3.2(b) to conclude that $C_p(Y,\mathbb{I})$ is cellular-compact. $\square$

Proposition 3.4. *If $C_p(X,\mathbb{I})$ has a dense ω-bounded subspace, then it is cellular-compact.*

Proof. Let Y be a dense ω-bounded subspace of $C_p(X,\mathbb{I})$. Then $c(Y) \leq \omega$ and hence, for every disjoint family $\mathcal{U} \subset \tau^*(Y)$, there exists a countable set $A \subset Y$ such that $A \cap U \neq \emptyset$ for every $U \in \mathcal{U}$. Therefore, the compact set $K = \mathrm{cl}_Y(A)$ witnesses that Y is cellular-compact. Finally, apply Proposition 3.2(b) to see that $C_p(X,\mathbb{I})$ is also cellular-compact. $\square$

Not every countably compact space is cellular-compact: The ordinal ω_1 with the order topology is a counterexample. However, in function spaces, cellular compactness shows a better behavior.

Corollary 3.5. *If $C_p(X,\mathbb{I})$ is countably compact, then it is cellular-compact.*

Proof. If $C_p(X,\mathbb{I})$ is countably compact, then it follows from [6, Problem 397] that X is a P-space. Applying [6, S.310, Fact 2], we conclude that $C_p(X,\mathbb{I})$ is ω-bounded so it is cellular-compact by Proposition 3.4. $\square$

Corollary 3.6. *If there exists a condensation of X onto a P-space, then $C_p(X,\mathbb{I})$ is cellular-compact. In particular, $C_p(X,\mathbb{I})$ is cellular-compact for any P-space X.*

Proof. If $\varphi : X \to Y$ is a condensation of X onto a P-space Y, then the dual map $\varphi^* : C_p(Y,\mathbb{I}) \to C_p(X,\mathbb{I})$, defined by $\varphi^*(f) = f \circ \varphi$ for any $f \in C_p(Y,\mathbb{I})$, is a dense embedding of $C_p(Y,\mathbb{I})$ into $C_p(X,\mathbb{I})$. Finally, observe that $C_p(Y,\mathbb{I})$ is ω-bounded by [6, S.310, Fact 2], so the space $C_p(X,\mathbb{I})$ is cellular-compact by Proposition 3.4. $\square$

We saw from Corollary 3.5 that countable compactness of $C_p(X,\mathbb{I})$ implies its cellular compactness. It turns out that a cellular-compact separable $C_p(X,\mathbb{I})$ need not be countably compact.

Example 3.7. It is proved in [6, S.480, Fact 2] that there exists a dense-in-itself space X that condenses onto $\mathbb{R}$ and onto a P-space. Therefore, $d(C_p(X,\mathbb{I})) = iw(X) = \omega$; i.e., $C_p(X,\mathbb{I})$ is separable while every point of X is a G_δ-set which is not open. This shows that X is not a P-space and hence, $C_p(X,\mathbb{I})$ is not countably compact by [6, Problem 397]. However, $C_p(X,\mathbb{I})$ is cellular-compact by Corollary 3.6. We will later show that there exists a space with stronger properties (see Theorem 5.4).

4. Cellular Compactness vs. Pseudocompactness

In this section, we give a characterization for the space $C_p(X,\mathbb{I})$ to be cellular-compact; it easily implies that cellular compactness in spaces $C_p(X,\mathbb{I})$ is preserved by arbitrary products. We also show that there exists a space X such that $C_p(X,\mathbb{I})$ is pseudocompact but not cellular-compact. However, if X is normal, then cellular compactness of $C_p(X,\mathbb{I})$ is equivalent to its pseudocompactness.

Theorem 4.1. *For any space X, the following conditions are equivalent:*

(a) $C_p(X,\mathbb{I})$ *is cellular-compact;*
(b) *for every countable non-empty set $A \subset X$, there exists a compact subspace $K \subset C_p(X,\mathbb{I})$ such that $\pi_A(K) = \mathbb{I}^A$;*
(c) *if $\mathcal{U} \subset \tau^*(C_p(X,\mathbb{I}))$ is a countable family, then there exists a compact set $K \subset C_p(X,\mathbb{I})$ such that $K \cap U \neq \emptyset$ for any $U \in \mathcal{U}$.*

Proof. Assume first that $C_p(X,\mathbb{I})$ is cellular-compact and X is countable. Then it follows from pseudocompactness of $C_p(X,\mathbb{I})$ that it is compact and hence, X is discrete so, for the compact set $K = C_p(X,\mathbb{I})$, we have the equality $\pi_A(K) = \mathbb{I}^A$ for any $A \subset X$.

If $C_p(X,\mathbb{I})$ is cellular-compact and X is uncountable, then fix any countable non-empty set $A \subset X$ and pick a base $\{B_n : n \in \omega\} \subset \tau^*(\mathbb{I}^A)$ for the space $\mathbb{I}^A$. By [9, Proposition 3.9], for each $n \in \omega$, there exists a set $U_n \subset \pi_A^{-1}(B_n)$ such that $U_n \in \tau^*(C_p(X,\mathbb{I}))$ and the family $\mathcal{U} = \{U_n : n \in \omega\}$ is disjoint. If $K \subset C_p(X,\mathbb{I})$ is a compact set such that $K \cap U_n \neq \emptyset$, then $\pi_A(K) \cap B_n \neq \emptyset$ for every number $n \in \omega$ and therefore, the compact set $\pi_A(K)$ is dense in $\mathbb{I}^A$; i.e., $\pi_A(K) = \mathbb{I}^A$, which proves that (a)$\Longrightarrow$(b).

Now assume that the condition (b) is satisfied. Then, trivially, we have the equality $\pi_A(C_p(X,\mathbb{I})) = \mathbb{I}^A$ for any countable set $A \subset X$. This shows that all countable subsets of X are closed and C^*-embedded in X. Let $\mathcal{U}$ be a countable family of non-empty open subsets of $C_p(X,\mathbb{I})$. There is no loss of generality to assume that all elements of $\mathcal{U}$ are standard open sets; i.e., $\mathcal{U} = \{U_n : n \in \omega\}$ and $U_n = [x_1^n, \dots, x_{k_n}^n; O_1^n, \dots, O_{k_n}^n]$ where $x_i^n \in X$ and $O_i^n \in \tau^*(\mathbb{I})$ for all $i \leq k_n$ and $n \in \omega$.

The set $A = \{x_i^n : i \leq k_n \text{ and } n \in \omega\}$ is countable and $\pi_A^{-1}(\pi_A(U_n)) = U_n$ for each $n \in \omega$. There exists a compact set $K \subset C_p(X,\mathbb{I})$ such that $\pi_A(K) = \mathbb{I}^A$ and hence, $\pi_A(K) \cap \pi_A(U_n) \neq \emptyset$ for each $n \in \omega$. This implies that $K \cap U_n \neq \emptyset$ for every $n \in \omega$, so K witnesses that (c) holds; i.e., we settled (b)$\Longrightarrow$(c).

Finally, assume that condition (c) is satisfied and $\mathcal{U} \subset \tau^*(C_p(X,\mathbb{I}))$ is a disjoint family. It follows from $c(C_p(X,\mathbb{I})) = \omega$ that $|\mathcal{U}| \leq \omega$ and therefore, (c) is applicable to find a compact set $K \subset C_p(X, \mathbb{I})$ such that $K \cap U \neq \emptyset$ for every $U \in \mathcal{U}$. Thus, $C_p(X, \mathbb{I})$ is cellular-compact. $\square$

Neither pseudocompactness nor countable compactness is preserved by finite products in general spaces. However, they are both invariant for arbitrary products in spaces $C_p(X, \mathbb{I})$. It turns out that an analogous result is true for cellular compactness.

Theorem 4.2. *Suppose that X_t is a space such that $C_p(X_t, \mathbb{I})$ is cellular-compact for every $t \in T$. Then $\prod\{C_p(X_t, \mathbb{I}) : t \in T\}$ is cellular-compact.*

Proof. Let $X = \bigoplus\{X_t : t \in T\}$; we will identify every X_t with the respective clopen subspace of X. Since the space $\prod\{C_p(X_t, \mathbb{I}) : t \in T\}$ is homeomorphic to $C_p(X, \mathbb{I})$ (see [6, Problem 114]), it suffices to show that $C_p(X, \mathbb{I})$ is cellular-compact.

Take a countable set $A \subset X$ and let $A_t = A \cap X_t$ for every $t \in T$; we will need the set $S = \{t \in T : A_t \neq \emptyset\}$. By Theorem 4.1, there exists a compact space $K_t \subset C_p(X_t, \mathbb{I})$ such that $\pi_{A_t}(K_t) = \mathbb{I}^{A_t}$ for each $t \in S$. The set $K = \{f \in C_p(X, \mathbb{I}) : f|X_t \in K_t$ for every $t \in S$ and $f(X_t) = \{0\}$ for all $t \in T \backslash S\}$ is compact being homeomorphic to $\prod\{K_t : t \in S\}$, and it is straightforward that $\pi_A(K) = \mathbb{I}^A$, so we can apply Theorem 4.1 again to conclude that $C_p(X, \mathbb{I})$ is cellular-compact. $\square$

Since cellular compactness of $C_p(X, \mathbb{I})$ implies its pseudocompactness by Proposition 3.1(a), it is natural to ask whether pseudocompactness of the space $C_p(X, \mathbb{I})$ implies its cellular compactness. To answer this question, we will need the following result.

Theorem 4.3. *If $C_p(X, \mathbb{I})$ is cellular-compact, then every countable subset of X is closed in υX.*

Proof. Assume the contrary and let $Y = \upsilon X$. There exists a countably infinite set $D \subset X$ such that $\mathrm{cl}_Y(D) \backslash D \neq \emptyset$. It follows from pseudocompactness of $C_p(X, \mathbb{I})$ (see [6, Problem 398]) that all countable subsets of X are closed and C^*-embedded and, in particular, D is C^*-embedded in $E = \mathrm{cl}_Y(D)$.

By Theorem 4.1, there is a compact set $K \subset C_p(X, \mathbb{I})$ such that $\pi_D(K) = \mathbb{I}^D$; consider the restriction map $\mu : C_p(Y, \mathbb{I}) \to C_p(X, \mathbb{I})$. The set $L = \mu^{-1}(K)$ is countably compact by [8, Problem 228] and hence, so is the set $L' = \pi_E(L)$, where $\pi_E : C_p(Y, \mathbb{I}) \to C_p(E, \mathbb{I})$ is also the restriction map.

Since E is separable, we have $iw(C_p(E)) = d(E) = \omega$ and hence, $iw(L') = \omega$, which shows that L' is compact and metrizable. Let $\lambda : C_p(E, \mathbb{I}) \to C_p(D, \mathbb{I})$ be the restriction map. Since D is C^*-embedded in E, the set $\lambda(C_p(E, \mathbb{I}))$ coincides with $\mathbb{I}^D$. It is straightforward that

$\lambda(L') = (\lambda \circ \pi_E)(L) = \pi_D(K) = \mathbb{I}^D$, so it follows from bijectivity of λ that $C_p(E, \mathbb{I}) = L'$ and hence, $C_p(E, \mathbb{I})$ is compact. However, compactness of the space $C_p(E, \mathbb{I})$ implies discreteness of E according to [6, Problem 396]; this contradiction shows that any countable subset of X is closed in υX. $\square$

Corollary 4.4. *If $C_p(X, \mathbb{I})$ is cellular-compact, then any bounded subset of X is finite. In particular, any pseudocompact subset of X is finite.*

Proof. Assume the contrary and let $Y = \upsilon X$. Then there exists a countably infinite bounded set $D \subset X$. The space $C_p(X, \mathbb{I})$ being pseudocompact, all countable subsets of X are closed in X by [6, Problem 398], so D is a closed discrete subset of X. We apply [6, Problem 415] to convince ourselves that the set $K = \mathrm{cl}_Y(D)$ is compact. However, D is closed in Y by Theorem 4.3, so D is an infinite closed discrete subset of K, which is a contradiction. $\square$

Corollary 4.5. *There exists a space X such that $C_p(X, \mathbb{I})$ is pseudocompact but not cellular-compact.*

Proof. D. B. Shakhmatov establishes in [5] that there exists a dense pseudocompact subspace X of the Tychonoff cube $\mathbb{I}^{\mathfrak{c}}$ in which all countable subsets are closed and C^*-embedded. This implies that $C_p(X, \mathbb{I})$ is pseudocompact by [6, Problem 398]. Since X is pseudocompact and infinite, the space $C_p(X, \mathbb{I})$ cannot be cellular-compact by Corollary 4.4. $\square$

Given a space X and a set $A \subset X$, say that A has a *discrete open expansion* if there exists a discrete family $\mathcal{U} = \{U_a : a \in A\} \subset \tau(X)$ such that $a \in U_a$ for each $a \in A$. The following fact is well known, but we give here its simple proof because we could not find the respective reference.

Proposition 4.6. *If X is a space and A is a countable discrete subspace of X, then A has a discrete open expansion if and only if it is C-embedded in X.*

Proof. There is no loss of generality to consider that the set A is infinite. Observe first that the necessity is an immediate consequence of [7, T.132, Fact 5]. Now, if A is C-embedded in X and $\{a_n : n \in \omega\}$ is its faithful enumeration, then there exists a continuous function $f : X \to \mathbb{R}$ such that $f(a_n) = n$ for each $n \in \omega$. It is immediate that the family $\{f^{-1}((n - \frac{1}{3}, n + \frac{1}{3})) : n \in \omega\}$ is a discrete expansion of A. $\square$

Our next step is to describe a nice class of spaces X for which $C_p(X, \mathbb{I})$ is cellular-compact.

Proposition 4.7. *Suppose that X is a space in which every countable subset is closed and has a discrete open expansion. Then $C_p(X,\mathbb{I})$ is cellular-compact.*

Proof. Let $A = \{a_n : n \in \omega\}$ be a faithfully indexed subset of the space X and fix a discrete family $\{U_n : n \in \omega\} \subset \tau(X)$ such that $a_n \in U_n$ for every $n \in \omega$; choose a function $h_n : X \to \mathbb{I}$ such that $h_n(a_n) = 1$ and $h_n(X \backslash U_n) = \{0\}$. For any $f \in \mathbb{R}^A$, let $e(f)(x) = \sum\{f(a_n)\cdot h_n(x) : n \in \omega\}$ for any $x \in X$. It is proved in [7, T.132, Fact 5] that $e(f) \in C_p(X)$ for any $f \in \mathbb{R}^A$ and $e : \mathbb{R}^A \to C_p(X)$ is a linear continuous map such that $\pi_A(e(f)) = f$ for any $f \in \mathbb{R}^A$. It is easy to see that $K = e(\mathbb{I}^A) \subset C_p(X,\mathbb{I})$ and hence, $\pi_A(K) = \mathbb{I}^A$, so the compact set K witnesses that condition (b) of Theorem 4.1 is satisfied and therefore, $C_p(X,\mathbb{I})$ is cellular-compact. □

Corollary 4.8. *If X is a normal space, then the following conditions are equivalent:*

(a) *$C_p(X,\mathbb{I})$ is pseudocompact;*
(b) *every countable subset of X is closed;*
(c) *$C_p(X,\mathbb{I})$ is cellular-compact.*

Proof. It follows from [6, Problem 398] that (a)$\Longrightarrow$(b).

If (b) holds and $A \subset X$ is countable, then A is closed, discrete, and C-embedded in X by normality of X. By Proposition 4.6, the set A has an open discrete expansion and hence, $C_p(X, amsI)$ is cellular-compact by Proposition 4.7. This settles (b)$\Longrightarrow$(c).

Finally, (c)$\Longrightarrow$(a) by Proposition 3.1(a). □

Corollary 4.9. *If X is a space and $C_p(X,\mathbb{I})$ has countable tightness, then $C_p(X,\mathbb{I})$ is cellular-compact if and only if it is pseudocompact.*

Proof. It follows from $t(C_p(X,\mathbb{I})) \leq \omega$ that X is Lindelöf and hence, normal (see [6, Problem 149]); apply Corollary 4.8 to complete the proof. □

Recall that a space Z is called *pseudocomplete* if there exists a sequence $\{\mathcal{B}_n : n \in \omega\}$ of π-bases in Z such that $\bigcap_{n\in\omega} U_n \neq \emptyset$ whenever $U_n \in \mathcal{B}_n$ and $\overline{U}_{n+1} \subset U_n$ for every $n \in \omega$. Pseudocompleteness is stronger than the Baire property and weaker than Čech-completeness. If $C_p(X)$ is Čech-complete, then X is discrete and countable (see [6, Problem 265]). However, there exist non-discrete uncountable spaces X such that $C_p(X)$ is pseudocomplete.

Corollary 4.10. *If $C_p(X)$ is pseudocomplete, then $C_p(X,\mathbb{I})$ is cellular-compact.*

Proof. Pseudocompleteness of $C_p(X)$ is equivalent to every countable subset of X being closed and C-embedded in X by [6, Problem 485]. Therefore, if $C_p(X)$ is pseudocomplete, then every countable subset of X is closed and has a discrete open expansion by Proposition 4.6. Proposition 4.7 does the rest. $\square$

5. Countable Additivity of Cellular Compactness

It is known that σ-countable compactness of $C_p(X,\mathbb{I})$ is equivalent to its countable compactness [6, Problem 397] and that σ-pseudocompactness of $C_p(X,\mathbb{I})$ is the same as its pseudocompactness [6, Problem 398]. We will show that cellular compactness demonstrates the same behavior in $C_p(X,\mathbb{I})$ and conclude the paper with an example of a space X such that $C_p(X,\mathbb{I})$ is cellular-compact but has no dense countably compact subspace.

Following an established tradition, we say that a space Z is *σ-cellular-compact* if $Z=\bigcup_{n\in\omega} Z_n$ where every Z_n is cellular-compact.

Theorem 5.1. *For any X, the space $C_p(X,\mathbb{I})$ is σ-cellular-compact if and only if it is cellular-compact.*

Proof. We must only prove necessity, so assume that we have a representation $C_p(X,\mathbb{I})=\bigcup\{Z_n : n\in\omega\}$ where every Z_n is cellular-compact. Since $\overline{Z}_n$ is also cellular-compact by Proposition 3.2(b), we can assume, without loss of generality, that every Z_n is closed in $C_p(X,\mathbb{I})$.

For any $f,g\in C^*(X)$, let $d(f,g)=\sup\{|f(x)-g(x)| : x\in X\}$; then the function d is a complete metric on $C^*(X)$ by [6, Problem 248]. We will denote the respective complete metric space by $C^*_u(X)$. Observe that the ball $B(f,\varepsilon)=\{g\in C^*(X) : d(f,g)<\varepsilon\}$ is open in the space $C^*_u(X)$ and the set $I(f,\varepsilon)=\{g\in C^*(X) : |g(x)-f(x)|\le\varepsilon\}$ is closed in $C^*_u(X)$ for any $f\in C^*(X)$ and $\varepsilon>0$.

Let $w(x)=\frac{1}{2}$ for any $x\in X$; then $w\in C^*(X)$ and $B(w,\frac{1}{2})\subset C(X,\mathbb{I})$. Every Z_n is closed in $C_p(X,\mathbb{I})$ and hence, in $C^*_u(X)$, so it follows from the inclusion $B(w,\frac{1}{2})\subset\bigcup\{Z_n : n\in\omega\}$ and the Baire property of $B(w,\frac{1}{2})$ as a subspace of $C^*_u(X)$ that there exist $\varepsilon>0$ and $f\in B(w,\frac{1}{2})$ such that $B(f,2\varepsilon)\subset Z_n$ for some $n\in\omega$. It is immediate that $I(f,\varepsilon)\subset B(f,2\varepsilon)$ and hence, $I(f,\varepsilon)\subset Z_n$.

Now, if we consider that $I(f,\varepsilon)$ is a subspace of $C_p(X)$, then it is a retract of $C_p(X)$ by [6, S.398, Fact 3]. Therefore, $I(f,\varepsilon)$ is a retract of Z_n, so it is cellular-compact by Proposition 3.1(b). The space $I(f,\varepsilon)$ being homeomorphic to $C_p(X,\mathbb{I})$ by [6, S.398, Fact 3], we convince ourselves that $C_p(X,\mathbb{I})$ is cellular-compact. $\square$

Recall that σ-countable compactness of $C_p(X)$ is equivalent to the space X being finite (see [6, Problem 186]), while there exist infinite spaces X such that $C_p(X)$ is σ-pseudocompact (see [6, Problem 400]). Our next result shows that σ-cellular-compactness of $C_p(X)$ is also equivalent to X being finite.

Theorem 5.2. *The following conditions are equivalent for any space X:*

(a) $C_p(X)$ *is σ-compact;*
(b) $C_p(X)$ *is σ-countably compact;*
(c) $C_p(X)$ *is σ-cellular-compact;*
(d) X *is finite.*

Proof. It follows from [6, Problem 186] that properties (a), (b), and (d) are equivalent. Since (a) trivially implies (c), it suffices to show that (c)$\Longrightarrow$(d), so assume that $C_p(X)$ is σ-cellular-compact.

The space $C_p(X, \mathbb{I})$ is a retract of $C_p(X)$ by [6, Problem 092], so it follows from Proposition 3.1(b) that $C_p(X, \mathbb{I})$ must be σ-cellular-compact. Applying Theorem 5.1, we conclude that $C_p(X, \mathbb{I})$ is cellular-compact. Since every cellular-compact space is pseudocompact, the space $C_p(X)$ is σ-pseudocompact and hence, X is pseudocompact by [6, Problem 399]. Finally, apply Corollary 4.4 to see that X is finite. □

Proposition 5.3. *Suppose that X is a space in which all countable subsets are closed. If there exists a Lindelöf P-space Y which is not homeomorphic to X but condenses onto X, then $C_p(X, \mathbb{I})$ is a cellular-compact space that has no dense countably compact subspace. In particular, X cannot be condensed onto a P-space.*

Proof. The space X is Lindelöf being a continuous image of a Lindelöf space. Therefore, X is normal and hence, we can apply Corollary 4.8 to conclude that $C_p(X, \mathbb{I})$ is cellular-compact.

Let $r : Y \to X$ be a condensation. Since the map r is not a homeomorphism, $Z = r^*(C_p(X, \mathbb{I}))$ is a proper dense subset of $C_p(Y, \mathbb{I})$; observe that the space Z is homeomorphic to $C_p(X, \mathbb{I})$ by [6, Problem 163]. The space $C_p(Y, \mathbb{I})$ has the Fréchet–Urysohn property by [7, Problem 135]. If $C_p(X, \mathbb{I})$ has a dense countably compact subspace Q, then $r^*(Q)$ is a proper dense countably compact subspace of $C_p(Y, \mathbb{I})$. However, in Fréchet–Urysohn spaces, all countably compact subspaces are closed—this is well known and easy to see. As a consequence, $Z = C_p(Y, \mathbb{I})$, which implies that r is a homeomorphism, a contradiction.

Finally, observe that if there exists a condensation $\varphi : X \to E$ of X onto a P-space E, then $\varphi^*(C_p(E, \mathbb{I}))$ is a dense countably compact subspace of $C_p(X, \mathbb{I})$, which is again a contradiction. □

The following theorem is the main result of this section; it should be compared with Corollary 3.6.

Theorem 5.4. *There exists a Lindelöf space X with the following properties:*

(a) *all countable subsets of X are closed;*
(b) *X is not a P-space;*
(c) *there exists a Lindelöf P-space that condenses onto X.*

Therefore, $C_p(X, \mathbb{I})$ is a cellular-compact space that has no countably compact dense subspace and, in particular, X cannot be condensed onto a P-space.

Proof. Let $S = \{x \in \mathbb{D}^{\omega_1} : |x^{-1}(1)| < \omega\}$ be the σ-product in the Cantor cube $\mathbb{D}^{\omega_1}$. If $S_n = \{x \in S : |x^{-1}(1)| \leq n\}$, then S_n is a scattered compact space for every $n \in \omega$. Let $a(\alpha) = 1$ for any $\alpha < \omega_1$; then $a \in \mathbb{D}^{\omega_1}$ and

(1) $a \notin \overline{B}$ for any countable set $B \subset S$ (the bar denotes the closure in $\mathbb{D}^{\omega_1}$).

Let Y be the set S with the topology generated by all G_δ-subsets of S; i.e., Y is the ω-modification of S. If Y_n is the set S_n considered as a subspace of Y, then Y_n is Lindelöf for every $n \in \omega$ (see [7, Problem 128]). Therefore, $Y = \bigcup_{n\in\omega} Y_n$ is a Lindelöf P-space. Since the topology of Y is strictly stronger than the topology of S, the identity map $i : Y \to S$ is a condensation. Let $j : \beta Y \to \mathbb{D}^{\omega_1}$ be the continuous extension of the map i; it is clear that $j(\beta Y) = \mathbb{D}^{\omega_1}$ and hence, there exists a point $z \in \beta Y \backslash Y$ such that $j(z) = a$.

If $z \in \overline{A}$ for some countable $A \subset Y$, then $a = j(z) \in \overline{i(A)}$, which is a contradiction with (1). Besides, if $z \in \overline{Y}_n$, then $a = j(z) \in \overline{j(Y_n)} = \overline{S}_n = S_n$, which is impossible because $a \notin S_n$. Therefore, $z \notin \overline{Y}_n$ and hence, Y_n is closed in $X = Y \cup \{z\}$ for every $n \in \omega$.

Since $z \notin \overline{A}$ for any countable $A \subset Y$, in the space X, all countable subsets are closed. Besides, the space X is Lindelöf because so is Y. Observe also that $\{z\} = \bigcap_{n\in\omega}(X\backslash Y_n)$ which shows that $\{z\}$ is a G_δ-subset of X with empty interior; i.e., X is not a P-space. Furthermore, $Z = Y \oplus \{z\}$ is easily seen to be a Lindelöf P-space that condenses onto the space X and hence, we can apply Proposition 5.3 to see that $C_p(X, \mathbb{I})$ is a cellular-compact space that has no dense countably compact subspace; therefore, X cannot be condensed onto a P-space. □

6. Open Questions

The author hopes to have convinced the reader that cellular-compact spaces constitute a very interesting subclass of pseudocompact spaces.

We saw that cellular-compactness behaves much better in function spaces apart from bringing new information about countable compactness and pseudocompleteness. This work, however, has only scratched the surface of the realm of cellular-compact function spaces as can be seen from the following list of open questions.

Question 6.1. Suppose that X is a space and $C_p(X, \mathbb{I})$ has a dense countably compact subspace. Must $C_p(X, \mathbb{I})$ be cellular-compact?

Question 6.2. Suppose that X is a space and $C_p(X, \mathbb{I})$ has a dense countably compact subspace. Does there exist a condensation of X onto a P-space?

Question 6.3. Suppose that X is a space and $C_p(X, \mathbb{I})$ has a dense ω-bounded subspace. Does there exist a condensation of X onto a P-space?

Question 6.4. Suppose that $C_p(X, \mathbb{I})$ has a dense countably compact subspace. Must every countable subset of X have a discrete open expansion?

Question 6.5. Suppose that $C_p(X, \mathbb{I})$ has a dense ω-bounded subspace. Must every countable subset of X have a discrete open expansion?

Question 6.6. Suppose that X is a space such that $C_p(X, \mathbb{I})$ is cellular-compact. Must every countable subset of X have a discrete open expansion?

Question 6.7. Suppose that X is a space such that $C_p(X, \mathbb{I})$ is cellular-compact and separable. Must every countable subset of X have a discrete open expansion?

Question 6.8. Suppose that X is a space such that $C_p(X, \mathbb{I})$ is cellular-compact and separable. Must $C_p(X, \mathbb{I})$ have a dense countably compact subspace?

Question 6.9. Suppose that X is a space such that $C_p(X)$ is pseudocomplete and separable. Must $C_p(X, \mathbb{I})$ have a dense countably compact subspace?

Question 6.10. Suppose that X and Y are spaces such that $C_p(X, \mathbb{I})$ and $C_p(Y, \mathbb{I})$ have a dense countably compact subspace. Is it true that the space $C_p(X, \mathbb{I}) \times C_p(Y, \mathbb{I})$ has a dense countably compact subspace?

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