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by

Leonard Mdzinarishvili

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Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
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ON THE CONTINUITY PROPERTY OF THE EXACT HOMOLOGY THEORIES

LEONARD MDZINARISHVILI

ABSTRACT. It is well known [Samuel Eilenberg and Norman Steenrod, *Foundations of Algebraic Topology.* Princeton, New Jersey: Princeton University Press, 1952] that on the category $\mathcal{K}_c$ of compact pairs and continuous maps there is a continuity property of the partially exact homology [*Foundations of Algebraic Topology*, Definition 2.3.X]. Namely, if the compact pair (X, A) is an inverse limit of the compact pairs (X_α, A_α), then the partially exact homology H_* of (X, A) is an inverse limit of homology groups of (X_α, A_α); i.e., there is an isomorphism

$$H_*(X, A) \xrightarrow{\sim} \varprojlim H_*(X_\alpha, A_\alpha).$$

It has been shown that among all the partially exact theories on the category $\mathcal{K}_C$, the Čech theory is essentially the only one satisfying this continuity axiom [*Foundations of Algebraic Topology*, Theorem 3.1.X].

We define a continuity property of the exact homology theories on the category $\mathcal{K}_C$ and prove that the homology theory on the category $\mathcal{K}_C$, satisfying all the Eilenberg–Steenrod axioms and the continuity property of the exact homology theories, exists.

Let $\mathcal{K}_C$ be the category of compact pairs (X, A) and continuous maps; let H_* be an exact homology theory. Let $\{(X_\alpha, A_\alpha)\}$ be an inverse system of compact pairs (X_α, A_α) and $(X, A) = \varprojlim (X_\alpha, A_\alpha)$. The inverse system $\{(X_\alpha, A_\alpha)\}$ generates an inverse system $\{H_*(X_\alpha, A_\alpha)\}$ and the projection $\pi_\alpha : (X, A) \to (X_\alpha, A_\alpha)$ induces the homomorphism $\pi_{\alpha,*} : H_*(X, A) \to$

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$H_*(X_\alpha, A_\alpha)$. The homomorphisms $\{\pi_{\alpha,*}\}$ induce the homomorphism

$$\pi_* : H_*(X, A) \to \varprojlim H_*(X_\alpha, A_\alpha).$$

Definition 1. The exact homology theory H_* is said to be a continuity on the category $\mathcal{K}_C$, if there is an infinite exact sequence

$$(1) \qquad \cdots \to \varprojlim{}^{(2i+1)} H_{n+i+1}\overline{X}_\alpha \to \cdots \to \varprojlim{}^{(1)} H_{n+1}\overline{X}_\alpha \to H_n(\overline{X})$$
$$\xrightarrow{p_n} \varprojlim H_n\overline{X}_\alpha \to \varprojlim{}^{(2)} H_{n+1}\overline{X}_\alpha \to \cdots \to \varprojlim{}^{(2i)} H_{n+i}\overline{X}_\alpha \to \cdots,$$

where $\overline{X}_\alpha = (X_\alpha, A_\alpha)$, $\overline{X} = (X, A)$.

On the category $\mathcal{K}_C$, there are different definitions of exact homology theories (see [4], [7], [9], [11]) which, on the category $\mathcal{K}_{CM}$ of compact metric spaces, are isomorphic to the Steenrod homology. We will show that these exact homologies are continuity; i.e., there is an infinite exact sequence (1).

For each $Y \in \mathcal{K}_C$, let there be an integral free cochain complex $C^*(Y)$ and for each continuous map $f : Y_1 \to Y_2$, let there be the homomorphism $f^* : C^*(Y_2) \to C^*(Y_1)$. The examples of integral free cochain complexes are the cochain complexes defined in [7], [8], and [11].

Using [5, Theorem III.4.1], we have the universal-coefficient theorem

$$(2) \qquad 0 \to \operatorname{Ext}(H^{q+1}(Y), G) \to H_q(Y, G) \to \operatorname{Hom}(H^q(Y), G) \to 0,$$

where $H_q(Y, G) = H_q(\operatorname{Hom}(C^*(Y), G))$ and $H^q(Y) = H^q(C^*(Y))$.

If $C^*(Y)$ is the free cochain complex defined by the system $S = \{S_\alpha\}$ of all finite decompositions S_α [8], then there is an isomorphism [8, Theorem 2]

$$(3) \qquad \overset{K}{H}_*(Y, G) \approx \overset{ch}{H}_*(Y, G),$$

where $\overset{K}{H}_*$ is the Kolmogorov homology and $\overset{ch}{H}_*$ is the Chogoshvili homology,

$$(4) \qquad \overset{ch}{H}_*(Y, G) = H_*(\operatorname{Hom}(C^*(Y), G)).$$

By [8, Theorem 3], there is an isomorphism

$$(5) \qquad H^*(Y) \approx \check{H}^*(Y).$$

Using the exact sequence (2) and the isomorphisms (3), (4), and (5), we have the exact sequence

$$(6) \qquad 0 \to \operatorname{Ext}(\check{H}^{q+1}(Y), G) \to \overset{K}{H}_q(Y, G) \to \operatorname{Hom}(\check{H}^q(Y), G) \to 0.$$

If $C^*(Y)$ is the free cochain complex of William S. Massey [7], then by [7, Part I, Theorem 4.18], there is the exact sequence

$$0 \to \mathrm{Ext}(\check{H}^{q+1}(Y), G) \to \overset{M}{H}_q(Y, G) \to \mathrm{Hom}(\check{H}^q(Y), G) \to 0, \tag{7}$$

where $\overset{M}{H}$ is the Massey homology.

A continuous map $f : Y_1 \to Y_2$ induces a commutative diagram with exact rows

(8)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H^{q+1}(Y_1), G) \arrow[r] \arrow[d, "f'_q"] & H_q(Y_1, G) \arrow[r] \arrow[d, "f_q"] & \mathrm{Hom}(H^q(Y_1), G) \arrow[r] \arrow[d, "f''_q"] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H^{q+1}(Y_2), G) \arrow[r] & H_q(Y_2, G) \arrow[r] & \mathrm{Hom}(H^q(Y_2), G) \arrow[r] & 0 .
\end{tikzcd}$$

Let $\{X_\alpha\}$ be an inverse system of compact spaces X_α. By diagram (8), the inverse system $\{X_\alpha\}$ generates a short exact sequence of inverse systems

$$\begin{aligned} 0 \to \{\mathrm{Ext}(H^{q+1}(X_\alpha), G)\} &\to \{H_q(X_\alpha, G)\} \\ &\to \{\mathrm{Hom}(H^q(X_\alpha), G)\} \to 0. \end{aligned} \tag{9}$$

Let $X = \varprojlim X_\alpha$. For each $\alpha \in \Omega$, a continuous map $p_\alpha : X \to X_\alpha$ induces the homomorphism $p_{\alpha,*} : H_*(X, G) \to H_*(X_\alpha, G)$. The system $\{p_{\alpha,*}\}$ generates the homomorphism

$$p_* : H_*(X, G) \to \varprojlim H_*(X_\alpha, G). \tag{10}$$

Using the exact sequences (8), (9), and the property of the functor $\varprojlim$, we have a commutative diagram with the exact rows

$$\begin{array}{ccccccc}
0 \to & \mathrm{Ext}(H^{q+1}(X), G) & \to & H_q(X, G) & \to & \mathrm{Hom}(H^q(X), G) & \to 0 \\
& \downarrow p'_q & & \downarrow p_q & & \downarrow p''_q & \\
0 \to & \varprojlim \mathrm{Ext}(H^{q+1}(X_\alpha), G) & \to & \varprojlim H_q(X_\alpha, G) & \to & \varprojlim \mathrm{Hom}(H^q(X_\alpha), G) & \to
\end{array} \tag{11}$$

$$\to \varprojlim{}^{(1)} \mathrm{Ext}(H^{q+1}(X_\alpha), G) \to \varprojlim{}^{(1)} H_q(X_\alpha, G) \to$$
$$\to \varprojlim{}^{(1)} \mathrm{Hom}(H^q(X_\alpha), G) \to \cdots .$$

In the category $\mathcal{K}_C$, the Čech cohomology satisfies the continuity axiom; i.e., there is an isomorphism

$$\varinjlim \check{H}^*(X_\alpha) \xrightarrow{\sim} \check{H}^*(X). \tag{12}$$

Using isomorphisms (5) and (12), we have an isomorphism

$$\varinjlim H^*(X_\alpha) \xrightarrow{\sim} H^*(X). \tag{13}$$

Using the property of the functors $\operatorname{Hom}(-, G)$, the direct limit $\varinjlim$, and isomorphism (13), we have an isomorphism

$$\operatorname{Hom}(H^*(X), G) \approx \operatorname{Hom}(\varinjlim H^*(X_\alpha), G) \approx \varprojlim \operatorname{Hom}(H^*(X_\alpha), G). \tag{14}$$

From diagram (11) and isomorphism (14), it follows that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \operatorname{Ext}(H^{q+1}(X), G) & \to & H_q(X, G) & \to & \operatorname{Hom}(H^q(X), G) & \to 0 \\ & \Big\downarrow p'_q & & \Big\downarrow p_q & & \approx \Big\downarrow p''_q & \\ 0 \to & \varprojlim \operatorname{Ext}(H^{q+1}(X_\alpha), G) & \to & \varprojlim H_q(X_\alpha, G) & \to & \varprojlim \operatorname{Hom}(H^q(X_\alpha), G) & \to 0 \end{array} \tag{15}$$

and

$$\operatorname{Ker} p'_q \approx \operatorname{Ker} p_q, \quad \operatorname{Coker} p'_q \approx \operatorname{Coker} p_q. \tag{16}$$

Therefore, there is an exact sequence

$$0 \to \operatorname{Ker} p'_q \to H_q(X, G) \to \varprojlim H_q(X_\alpha, G) \to \operatorname{Coker} p'_q \to 0. \tag{17}$$

Using the exact sequence, from [2, Proposition 1.2],

$$\begin{aligned} 0 \to \varprojlim{}^{(1)} \operatorname{Hom}(A_\alpha, G) \to \operatorname{Ext}(\varinjlim A_\alpha, G) \\ \to \varprojlim \operatorname{Ext}(A_\alpha, G) \to \varprojlim{}^{(2)} \operatorname{Hom}(A_\alpha, G) \to 0, \end{aligned} \tag{18}$$

isomorphism (13), and also from the exact sequences (17) and (18), we have the exact sequence

$$\begin{aligned} 0 \to \varprojlim{}^{(1)} \operatorname{Hom}(H^{q+1}(X_\alpha, G)) \to H_q(X, G) \\ \to \varprojlim H_q(X_\alpha, G) \to \varprojlim{}^{(2)} \operatorname{Hom}(H^{q+1}(X_\alpha), G) \to 0. \end{aligned} \tag{19}$$

For each $Y \in \mathcal{K}_C$, there is a free cochain complex

$$C^*(Y) = C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \cdots \to C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \to \cdots .$$

Denote $Z^n = \operatorname{Ker} \delta^n$, $B^n = \operatorname{Im} \delta^{n-1}$, and $H^n = Z^n / B^n$.

The functor $\operatorname{Hom}(-, G)$, where G is an abelian group, induces the chain complex

$$\begin{aligned} C_*(Y, G) &= \operatorname{Hom}(C^*(Y), G) \\ &= C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} \cdots \leftarrow C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} \cdots . \end{aligned}$$

Denote $Z_n = \operatorname{Ker} \partial_n$, $B_n = \operatorname{Im} \partial_{n+1}$, and $H_n(C_*) = Z_n / B_n$.

By [10, Lemma 1], if $C^*(Y)$ is a free cochain complex, then there is an exact sequence

$$0 \to \operatorname{Hom}(B^{n+1}, G) \to Z_n \to \operatorname{Hom}(H^n, G) \to 0. \tag{20}$$

By [10, Lemma 2], there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(B^{n+1},G) \arrow[r] \arrow[d] & Z_n \arrow[r] \arrow[d] & \mathrm{Hom}(H^n,G) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H^{n+1},G) \arrow[r] \arrow[d] & H_n \arrow[r] \arrow[d] & \mathrm{Hom}(H^n,G) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

By [10, Theorem 1], if there is a direct system $\{C^*(X_\alpha)\}$ of the free cochain complexes $C^*(X_\alpha)$, then, for $n \in \mathbb{Z}$ and $i \geq 2$, there exists a split exact sequence

$$0 \to \underleftarrow{\lim}{}^{(i)}\, \mathrm{Ext}(H^{n+1}(X_\alpha),G) \to \underleftarrow{\lim}{}^{(i)} H_n(X_\alpha,G) \to \underleftarrow{\lim}{}^{(i)}\, \mathrm{Hom}(H^n(X_\alpha),G) \to 0. \tag{21}$$

Definition 2. A direct system $\underline{C}^* = \{C^*_\alpha\}$ of the cochain complexes C^*_α is said to be associated with a cochain complex C^* if there is a homomorphism $\underline{C}^* \to C^*$ such that, for each $n \in \mathbb{Z}$, the induced homomorphism

$$\underrightarrow{\lim}\, H^n_\alpha \to H^n(C^*)$$

is an isomorphism, where $H^n_\alpha = H^n(C^*_\alpha)$.

Theorem 1. *If a direct system $\underline{C}^* = \{C^*_\alpha\}$ of the free cochain complexes C^*_α is associated with a free cochain complex C^*, then there is an infinite exact sequence*

$$\cdots \to \underleftarrow{\lim}{}^{(2i+1)} H^\alpha_{q+i+1} \to \cdots \to \underleftarrow{\lim}{}^{(1)} H^\alpha_{q+1} \to H_q(C_*) \to \underleftarrow{\lim}\, H^\alpha_q \to \underleftarrow{\lim}{}^{(2)} H^\alpha_{q+1} \to \cdots \to \underleftarrow{\lim}{}^{(2i)} H^\alpha_{q+i} \to \cdots, \tag{22}$$

where $H^\alpha_q = H_q(C^\alpha_*) = H_q(\mathrm{Hom}(C^*_\alpha,G))$ and $H_q(C_*) = H_q(\mathrm{Hom}(C^*,G))$.

Proof. By Definition 2, for all $q \in \mathbb{Z}$, there is an isomorphism

$$\underrightarrow{\lim}\, H^q_\alpha \xrightarrow{\sim} H^q(C^*). \tag{23}$$

Using the formula of universal coefficients, the exact sequence (19), and also isomorphism (23), we have an exact sequence

$$0 \to \underleftarrow{\lim}{}^{(1)}\, \mathrm{Hom}(H^{q+1}_\alpha,G) \to H_q(C_*) \to \underleftarrow{\lim}\, H^\alpha_q \to \underleftarrow{\lim}{}^{(2)}\, \mathrm{Hom}(H^{q+1}_\alpha,G) \to 0\,, \tag{24}$$

where $H_q(C_*) = H_q(\mathrm{Hom}(C^*,G))$ and $H^\alpha_q = H_q(\mathrm{Hom}(C^*_\alpha,G))$.

The method developed in [10] (see the proof of Theorem 2) and the split exact sequence (21), the exact sequence (24), and the isomorphism

$$\varprojlim{}^{(i)} \operatorname{Ext}(H_q^\alpha, G) \approx \varprojlim{}^{(i+2)} \operatorname{Hom}(H_q^\alpha, G), \quad i \geq 1$$

give the statement of the theorem. □

Theorem 2. *Let $\{X_\alpha\}$ be an inverse system of compact spaces X_α and $X = \varprojlim X_\alpha$. If H_* is an exact homology theory generated by an integral free cochain complex C^*, for each compact space and a direct system $\{C^*(X_\alpha)\}$ of integral free cochain complexes $C^*(X_\alpha)$ associated with an integral free cochain complex $C^*(X)$, then there is an infinite exact sequence*

$$\begin{aligned}(25)\qquad \cdots \to \varprojlim{}^{(2i+1)} H_{q+i+1}X_\alpha \to \cdots \to \varprojlim{}^{(1)} H_{q+1}X_\alpha \\ \to H_q(X) \to \varprojlim H_q X_\alpha \to \varprojlim{}^{(2)} H_{q+1}X_\alpha \\ \to \cdots \to \varprojlim{}^{(2i)} H_{q+i}X_\alpha \to \cdots,\end{aligned}$$

where $H_qX = H_q(X,G) = H_q(\operatorname{Hom}(C^*(X),G))$ and $H_qX_\alpha = H_q(X_\alpha,G) = H_q(\operatorname{Hom}(C^*(X_\alpha),G))$ and G is an abelian group.

Using Theorem 1, we get the statement of Theorem 2.

Corollary 1. *The exact homology theories of A. N. Kolmogoroff* [4]*, John W. Milnor* [11]*, and Massey* [7]*, defined on the category $\mathcal{K}_C$ of compact spaces and continuous maps are the homology theories generated by the integral free cochain complexes. Therefore, there is the continuity property of these exact homology theories.*

Corollary 2 ([11, Theorem 4])**.** *Let H be a homology theory satisfying nine axioms, and let*

$$X_1 \leftarrow X_2 \leftarrow X_2 \leftarrow \cdots$$

be an inverse sequence of complex metric spaces with an inverse limit X. Then there is an exact sequence

$$(26)\qquad 0 \to \varprojlim{}^{(1)} H_{q+1}X_i \to H_qX \to \varprojlim H_qX_i \to 0$$

for each integer q. The corresponding assertion holds if each space is replaced by a pair.

Proof. For every inverse sequence $\xi = \{\xi_i\}$ of abelian groups ξ_i, there is $\varprojlim{}^{(i)} \xi = 0$, $i \geq 2$ [6, Theorem 11.52]. Therefore, using Theorem 2, we have the exact sequence (26). □

It is known ([1, Theorem 10.1]) that for each compact pair (X, A) there exists an inverse system $\{(K_\alpha, L_\alpha)\}$ of finite polyhedral pairs (K_α, L_α) such that

$$(X, A) = \varprojlim (K_\alpha, L_\alpha).$$

Definition 3 ([9])**.** An exact homology theory h satisfies the property of partial continuity on the category $\mathcal{K}_C$ of compact pairs, if for all $q \in \mathbb{Z}$, there is an exact sequence

$$0 \to \varprojlim{}^{(1)} h_{q+1}(K_\alpha, L_\alpha) \to h_q(X, A) \to \varprojlim h_q(K_\alpha, L_\alpha) \to 0. \tag{27}$$

Corollary 3. *If H_* is an exact homology theory generated by an integral free cochain complex C^* for each compact space, then for all $q \in \mathbb{Z}$, there is a short exact sequence*

$$\begin{aligned} 0 \to \varprojlim{}^{(1)} H_{q+1}(K_\alpha, L_\alpha) &\to H_q(X, A) \\ &\to \varprojlim H_q(K_\alpha, L_\alpha) \to 0\,, \end{aligned} \tag{28}$$

where $(X, A) = \varprojlim (K_\alpha, L_\alpha)$ and (K_α, L_α) is a finite polyhedral pair.

Proof. Using the conditions of the corollary and an exact sequence (19), we have an exact sequence

$$\begin{aligned} 0 \to \varprojlim{}^{(1)} \operatorname{Hom}(H^{q+1}(K_\alpha), G) &\to H_q(X, G) \\ \to \varprojlim H_q(K_\alpha, G) \to \varprojlim{}^{(2)} &\operatorname{Hom}(H^{q+1}(K_\alpha), G) \to 0\,. \end{aligned} \tag{29}$$

There is an exact sequence

$$\begin{aligned} \cdots \to \varprojlim{}^{(1)} \operatorname{Ext}(H^{q+1}(K_\alpha), G) &\to \varprojlim{}^{(1)} H_q(K_\alpha, G) \\ \to \varprojlim{}^{(1)} \operatorname{Hom}(H^q(K_\alpha), G) &\to \varprojlim{}^{(2)} \operatorname{Ext}(H^{q+1}(K_\alpha), G) \to \cdots\,. \end{aligned} \tag{30}$$

Since the cohomology groups $H^*(K_\alpha)$ are finite generated abelian groups, by [2, Corollary 1.5], there is

$$\varprojlim{}^{(i)} \operatorname{Ext}(H^*(K_\alpha), G) = 0 \quad \text{for } \ i \geq 1. \tag{31}$$

Therefore, from an exact sequence (30) and equality (31), there is an isomorphism

$$\varprojlim{}^{(1)} H_{q+1}(K_\alpha, G) \approx \varprojlim{}^{(1)} \operatorname{Hom}(H^{q+1}(K_\alpha), G). \tag{32}$$

By [2, Corollary 1.5], there is

$$\varprojlim{}^{(i)} \operatorname{Hom}(H^*(K_\alpha), G) = 0 \quad \text{for } \ i \geq 2. \tag{33}$$

Using the exact sequence (28), isomorphism (32), and equality (33), we obtain the statement of the corollary. □

Corollary 4. *Let $\{X_\alpha\}$ be an inverse system of compact spaces X_α and $X = \varprojlim X_\alpha$. If H_* is an exact homology theory from Theorem 2 and G is an injective abelian coefficients group, then there is an isomorphism*

$$H_*(X,G) \xrightarrow{\sim} \varprojlim H_*(X_\alpha, G).$$

Proof. By the exact sequences (11) and (18), the split exact sequence (21), and [5, Theorem III.10.2], we have

$$\varprojlim{}^{(i)} H_q(X_\alpha, G) = 0, \quad i \geq 1.$$

Therefore, using this equality and the infinite exact sequence (25), we have the statement of the corollary. □

Let $\mathcal{K}^\Omega$ be the category of inverse systems $\{A_\alpha\}_{\alpha\in\Omega}$ of abelian groups A_α and let $\varprojlim : \mathcal{K}^\Omega \to \mathcal{K}$ be the inverse limit. On the category $\mathcal{K}^\Omega$, consider a chain complex

$$\underline{C}_* = \{\cdots \leftarrow \underline{C}_{n-1} \overset{\partial_n}{\leftarrow} \underline{C}_n \overset{\partial_{n+1}}{\leftarrow} \underline{C}_{n+1} \leftarrow \cdots\},$$

where $\underline{C}_n = \{C_n^\alpha, \pi_{\beta\alpha}\}_{\alpha\in\Omega} \in \mathcal{K}^\Omega$, $n \in \mathbb{Z}$, and $\Omega = \{\alpha\}$ is a set of indices of the inverse system $\underline{C}_n$.

The spectral homology groups $H_*(\underline{C}_*, s)$ of the chain complex $\underline{C}_*$ are the inverse limit $\varprojlim H_n(C_*^\alpha)$, where $H_n(C_*^\alpha)$ is a homology of chain complexes $C_*^\alpha = \{\cdots \leftarrow C_{n-1}^\alpha \leftarrow C_n^\alpha \leftarrow \cdots\}$. Projective homology groups $H_n(\underline{C}_*, p)$ of the chain complex $\underline{C}_*$ are obtained if we first take the limit of the inverse system $\{\underline{C}_*^\alpha\}$ of the chain complexes $\overline{C}_*^\alpha$; i.e., there is a chain complex

$$\varprojlim \underline{C}_* = \{\cdots \leftarrow \varprojlim \underline{C}_{n-1} \overset{\partial_n}{\leftarrow} \varprojlim \underline{C}_n \leftarrow \cdots\}$$

and then we apply the homological functor, i.e., $H_n(\varprojlim \underline{C}_*)$. The chain complex $\underline{C}_*$ generates for each $n \in \mathbb{Z}$ two exact sequences of the inverse systems

$$(34_n) \qquad 0 \to \underline{Z}_{n+1} \to \underline{C}_{n+1} \xrightarrow{\partial_{n+1}} \underline{B}_n \to 0,$$

$$(35_n) \qquad 0 \to \underline{B}_n \to \underline{Z}_n \to \underline{H}_n \to 0,$$

where $\underline{Z}_n = \operatorname{Ker} \underline{\partial}_n$, $\underline{B}_n = \operatorname{Im} \underline{\partial}_{n+1}$, and $\underline{H}_n = \{H_n(C_*^\alpha), \pi_{\beta\alpha}\}$.

Since the functor $\varprojlim$ is a left exact functor, the exact sequences (34_n) and (35_n) induce, respectively, the exact sequences

$$(36_n) \qquad 0 \to \varprojlim \underline{Z}_{n+1} \to \varprojlim \underline{C}_{n+1} \to \varprojlim \underline{B}_n \to \varprojlim{}^{(1)} \underline{Z}_{n+1} \to \varprojlim{}^{(1)} \underline{C}_{n+1} \to \cdots,$$

$$(37_n) \qquad 0 \to \varprojlim \underline{B}_n \to \varprojlim \underline{Z}_n \to \varprojlim \underline{H}_n \to \varprojlim{}^{(1)} \underline{B}_n$$
$$\to \varprojlim{}^{(1)} \underline{Z}_n \to \varprojlim{}^{(1)} \underline{H}_n \to \varprojlim{}^{(2)} \underline{B}_n \to \cdots .$$

Lemma 1. *If* $\varprojlim{}^{(1)}\underline{C}_n = 0$ *and* $\varprojlim{}^{(i)}\underline{B}_n = 0$*,* $i = 1, 2$*, then for all* $n \in \mathbb{Z}$*, there is an exact sequence*

$$(38) \qquad 0 \to \varprojlim{}^{(1)} \underline{H}_{n+1} \to H_n(\underline{C}_*, p) \to H_n(\underline{C}_*, s) \to 0.$$

Proof. Since the projective homology group $H_n(\underline{C}_*, p)$ defined by $H_n(\underline{C}_*, p) = \varprojlim \underline{Z}_n / \operatorname{Im} \partial_{n+1}$, there is an exact sequence

$$(39) \qquad 0 \to \varprojlim \underline{B}_n / \operatorname{Im} \partial_{n+1} \to H_n(\underline{C}_*, p) \to \varprojlim \underline{Z}_n / \varprojlim \underline{B}_n \to 0.$$

Using the conditions of the lemma and the exact sequences (36_n) and (37_n), we have

$$(40) \qquad \varprojlim \underline{B}_n / \operatorname{Im} \partial_{n+1} \approx \varprojlim{}^{(1)} \underline{Z}_{n+1} \approx \varprojlim{}^{(1)} \underline{H}_{n+1}.$$

Since $\varprojlim{}^{(1)} \underline{B}_n = 0$, using the exact sequence (37_n), we have

$$(41) \qquad \varprojlim \underline{Z}_n / \varprojlim \underline{B}_n \approx \varprojlim \underline{H}_n = H_n(\underline{C}_*, s).$$

Therefore, using the exact sequence (39) and isomorphisms (40) and (41), we get the statement of Lemma 1. □

Corollary 5. *If* $\underline{K} = \{K_\alpha\}$ *is an inverse system of finite polyhedral spaces, then there is an exact sequence*

$$(42) \qquad 0 \to \varprojlim{}^{(1)} H_{n+1}(K_\alpha, G) \to H_n(\underline{K}, G, p) \to H_n(\underline{K}, G, s) \to 0,$$

where $H_n(\underline{K}, G, p) = H_n(\underline{C}_*, G, p) = H_n(\varprojlim C_*^\alpha)$ *is a projective homology group,* $C_*^\alpha = C_*(K_\alpha)$*, and* $H_n(\underline{K}, G, s) = \varprojlim H_n(K_\alpha, G)$*.*

Proof. Since a direct system $\underline{C}^* = \{C^*(K_\alpha)\}$ of the integral cochains complexes $C^*(K_\alpha)$ is a direct system of free cochain complexes, and all of the conditions of [3, Theorem 3] are fulfilled, we find that $\varinjlim \underline{C}^*$ is a free cochain complex. Using [2, Proposition 1.2], for all $n \in \mathbb{Z}$, there is

$$(43) \qquad \varprojlim{}^{(i)} C_n(K_\alpha, G) = \varprojlim{}^{(i)} \operatorname{Hom}(C^n(K_\alpha), G) = 0, \quad i = 1, 2.$$

Since $C^n(K_\alpha)/B^n(K_\alpha)$ is a finitely generated abelian group, $n \in \mathbb{Z}$, by [2, Corollary 1.5], for $i \geq 2$, there is the equality

$$(44) \qquad \varprojlim{}^{(i)} Z_n(K_\alpha, G) = \varprojlim{}^{(i)} \operatorname{Hom}(C^n(K_\alpha)/B^n(K_\alpha), G) = 0.$$

Using the exact sequence (36_n) and equalities (43) and (44), there is $\varprojlim{}^{(i)} B_n(K_\alpha) = 0$, $i = 1, 2$.

Since the conditions of Lemma 1 are fulfilled, we obtain the statement of Corollary 5. □

Corollary 6. *If H_* is an exact homology theory generated by an integral free cochain complex C^* for each compact space, then, for all $n \in \mathbb{Z}$, there is an isomorphism*

$$H_*(X, G) \approx H_*(\underline{K}, G, p), \tag{45}$$

where $X = \varprojlim K_\alpha$, $\underline{K} = \{K_\alpha\}$, K_α is a finite polyhedron.

Proof. Using the exact sequences (28) and (42) and a homomorphism $H_n(X, G) \to H_n(X, G, p)$ induced by the chain homomorphism $C_*(X, G) \to \varprojlim C_*(K_\alpha, G)$, we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim^{(1)} H_{n+1}(K_\alpha, G) \arrow[r] \arrow[d, equal] & H_n(X, G) \arrow[r] \arrow[d] & \varprojlim H_n(K_\alpha, G) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \varprojlim^{(1)} H_{n+1}(K_\alpha, G) \arrow[r] & H_n(\underline{K}, G, p) \arrow[r] & \varprojlim H_n(K_\alpha, G) \arrow[r] & 0 .
\end{tikzcd}$$

Therefore, using the lemma of five homomorphisms, we get the statement of the corollary. □

References

[1] Samuel Eilenberg and Norman Steenrod, *Foundations of Algebraic Topology.* Princeton, New Jersey: Princeton University Press, 1952.

[2] Martin Huber and Willi Meier, *Cohomology theories and infinite CW-complexes*, Comment. Math. Helv. **53** (1978), no. 2, 239–257.

[3] Ludger Kaup and Michael S. Keane, *Induktive Limiten endlich erzeugter freier Moduln*, Manuscripta Math. **1** (1969), 9–21.

[4] A. N. Kolmogoroff, *Les groupes de Betti des espaces localement bicompacts*, C. R. Acad. Sci. Paris Sér. Math. **202** (1936), 1144–1147.

[5] Saunders Mac Lane, *Homology.* Die Grundlehren der mathematischen Wissenschaften, Bd. 114. New York: Academic Press, Inc., Publishers; Berlin-Göttingen-Heidelberg: Springer-Verlag, 1963

[6] Sibe Mardešić, *Strong Shape and Homology.* Springer Monographs in Mathematics. Berlin: Springer-Verlag, 2000.

[7] William S. Massey, *Homology and Cohomology Theory: An Approach Based on Alexander–Spanier Cochains.* Monographs and Textbooks in Pure and Applied Mathematics, Vol. 46. New York-Basel: Marcel Dekker, Inc., 1978.

[8] L. D. Mdzinarišvili, *The relation between the homology theories of Kolmogorov and Steenrod* (Russian), Dokl. Akad. Nauk SSSR **203** (1972), 528–531.

[9] L. D. Mdzinarishvili, *Application of the shape theory in the characterization of exact homology theories and the strong shape homotopic theory* in Shape Theory and Geometric Topology (Dubrovnik, 1981). Ed. S. Mardešić and J. Segal. Lecture Notes in Mathematics, 870. Berlin-New York: Springer, 1981. 253–262.

[10] L. Mdzinarishvili, *The uniqueness theorem for cohomologies on the category of polyhedral pairs*, Trans. A. Razmadze Math. Inst. **172** (2018), no. 2, 265–275.

[11] John W. Milnor, *On the Steenrod homology theory* in Novikov Conjectures, Index Theorems and Rigidity, Volume 1 (Oberwolfach 1993). Ed. Steven C. Ferry, Andrew Ranicki, and Jonathan Rosenberg. London Mathematical Society Lecture Notes 226. Cambridge: Cambridge University Press, 1995. 79–96.

[12] N. E. Steenrod, *Regular cycles of compact metric spaces*, Ann. of Math. (2) **41** (1940), 833–851.

Department of Mathematics; Faculty of Informatics and Control Systems; Georgian Technical University; 77, Kostava St.; Tbilisi, Georgia

Email address: `l.mdzinarishvili@gtu.ge`