http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 56, 2020

Pages 249–261

http://topology.nipissingu.ca/tp/

Topologies Generated by Families of Sets and Strong Poset Models

by

Dongsheng Zhao, Xiaoyong Xi, and Yixiang Chen

Electronically published on January 28, 2020

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 56 (2020)
Pages 249-261

E-Published on January 28, 2020

TOPOLOGIES GENERATED BY FAMILIES OF SETS AND STRONG POSET MODELS

DONGSHENG ZHAO, XIAOYONG XI, AND YIXIANG CHEN

ABSTRACT. A poset model of a topological space X is a poset P such that X is homeomorphic to the maximal point space of P (the set $\mathrm{Max}(P)$ of all maximal points of P equipped with the relative Scott topology of P). The poset models of topological spaces based on other topologies, such as Lawson topology and lower topology, have also been investigated by other people. These models establish various types of new links between posets and topological spaces. In this paper, we introduce the strong Scott topology on a poset and use it to define the strong poset model: A strong poset model of a space X is a poset P such that $\mathrm{Max}(P)$ (equipped with the relative strong Scott topology) is homeomorphic to X. The main aim is to establish a characterization of T_1 spaces with T-generated topologies (such as the Hausdorff k-spaces) in terms of maximal point spaces of posets. A poset P is called *ME-separated* if for any elements x and y of P, $x \leq y$ if and only if $\uparrow y \cap \mathrm{Max}(P) \subseteq \uparrow x \cap \mathrm{Max}(P)$. We consider the topological spaces that have an ME-separated strong poset model. The main result is that a T_1 space has an ME-separated strong poset model if and only if its topology is T-generated. The class of spaces whose topologies are T-generated include all Scott spaces and all Hausdorff k-spaces.

A poset model of a topological space X is a poset P such that the subspace $\mathrm{Max}(P)$ of all maximal points of P of the Scott space ΣP is homeomorphic to X. It has been proved by several authors that a topological space has a poset model if and only if it is a T_1 space (see [1], [3],

---

2010 *Mathematics Subject Classification.* 06B35, 06B30, 54A05.

Key words and phrases. maximal point space; ME-separated poset; strong poset model; strong Scott topology; T-generated space.

This work is supported by NIE AcRF Project (RI 3/16 ZDS); NSF Project of Jiangsu Province, China (BK20170483); NSF of China (11361028, 61300153, 11671008, 11626207, 11701500); and NSF of China (61370100).

and [14]). In [15], it is further proved that every T_1 space has a directed complete poset (dcpo) model. Finding poset models with extra properties can help us better understand the topologies of spaces modeled by these posets.

In [16], the topological spaces that have a bounded complete dcpo model are investigated. In particular, they study the spaces in which all nonempty closed compact subsets form a dcpo model. One of the special features of the dcpo $CK(X)$ of all nonempty closed compact subsets of a T_1 space X is that elements A in $CK(X)$ are determined by the maximal elements above it: $A \subseteq B$ if and only if $(\uparrow B) \cap \mathrm{Max}(CK(X)) \subseteq (\uparrow A) \cap \mathrm{Max}(CK(X))$. A poset P satisfying this condition will be called *ME-separated*. In this paper, we investigate such posets and their maximal point spaces. We introduce the notion of strong Scott topology on a poset and use it to define the strong poset model of a topological space. The main result is that a T_1 space has an ME-separated strong poset model if and only if its topology is T-generated (to be defined). The class of spaces whose topologies are T-generated include all Scott spaces, metric spaces, and even all the Hausdorff k-spaces. The definition of T-generated topologies may suggest a new convenient method of constructing topologies on a set.

1. The Strong Scott Topology on Directed Complete Posets

Recall that a subset U of a poset P is Scott open if (1) $U = \uparrow U = \{x \in P : y \leq x \text{ for some } y \in U\}$ and (2) for any directed subset $D \subseteq P$, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ whenever $\bigvee D$ exists. The Scott open sets of a poset P form a topology on P, denoted by $\sigma(P)$ and called the *Scott topology* on P. The space $(P, \sigma(P))$ is denoted by ΣP, called the *Scott space* of P. A subset F of P is closed with respect to the Scott topology if it is a lower set and closed under suprema of directed subsets (i.e., for any directed set $D \subseteq F$, $\bigvee D \in F$ whenever $\bigvee D$ exists).

A poset is called a *directed complete poset* (*dcpo*) if every directed subset of the poset has a supremum. For more about the Scott topology and dcpos, see [5] and [6].

For any poset P, let $\mathrm{Max}(P)$ denote the set of all maximal points of P. A poset model of a topological space X is a poset such that the subset $\mathrm{Max}(P)$ with the relative Scott topology is homeomorphic to X.

As our main focus will be on the subspace $\mathrm{Max}(P)$, we shall assume that in all the posets P considered below, every element $x \in P$ is below some maximal element. This is equivalent to $\downarrow \mathrm{Max}(P) = P$. Note that every dcpo satisfies this condition.

In order to extend the ways of constructing topologies from order structures, we introduce a new topology on posets, which is finer than the Scott topology in general. The corresponding poset models of topological spaces will be investigated.

Definition 1.1. Let P be a poset. The strong Scott topology $\sigma_s(P)$ on P consists of $U \subseteq P$ such that (1) $U = \uparrow U$ and (2) for any directed set $D \subseteq P$, $\bigvee D \in \mathrm{Max}(P) \cap U$ implies $D \cap U \neq \emptyset$.

It is easy to verify that $\sigma_s(P)$ is indeed a topology on P and is finer than the Scott topology in general.

Remark 1.2. In [8] and [4], the authors first define the strong Scott topology τ_{sSc} on the open set lattice $\mathcal{O}(Y)$ of a topological space Y as follows: $\mathcal{H} \subseteq \mathcal{O}(Y)$ is in τ_{sSc} if and only if (1) it is an upper set of the complete lattice $(\mathcal{O}(Y), \subseteq)$ and (2) for any directed subfamily $\mathcal{D}$ of $\mathcal{O}(Y)$ satisfying $\bigcup \mathcal{D} = Y$, $\mathcal{H} \cap \mathcal{D} \neq \emptyset$.

Example 1.3. (1) Let $P = [0, 1]$ be the unit interval of real numbers with the ordinary order of numbers. Then a subset U of P is in $\sigma_s(P)$ if and only if it has the form $[a, 1]$ or $(a, 1]$ $(a < 1)$. Thus, for this P, $\sigma(P) \neq \sigma_s(P)$.

(2) Let $\mathbb{I} = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ be the set of all nonempty closed intervals of real numbers. Then $(\mathbb{I}, \supseteq)$ is a dcpo.

The set $\{I \in \mathbb{I} : I \subseteq [0, 1]\} \cup \{I \in \mathbb{I} : I \subseteq (-1, 1/2) \cup (1/2, 3/2)\}$ is in $\sigma_s(\mathbb{I})$. But it is not Scott open.

In general, an upper set $\mathcal{U} \subseteq \mathbb{I}$ is in $\sigma_s(\mathbb{I})$ if and only if for any $[x, x] = \{x\} \in \mathcal{U}$, there is $I \in \mathcal{U}$ such that $x \in int(I)$.

Given two elements a and b in a poset P, a is way below b, denoted by $a \ll b$, if for any directed subset D of P, if $\bigvee D$ exists and $b \leq \bigvee D$, then there exists $d \in D$ such that $a \leq d$. An element x is called *compact* if $x \ll x$. The set of all compact elements of P is denoted by $\mathrm{K}(P)$.

A poset P is called a *continuous poset* if for any element $a \in P$, the set $\twoheaddownarrow a = \{x \in P : x \ll a\}$ is a directed set and

$$a = \bigvee \twoheaddownarrow a.$$

A poset P is called an *algebraic poset* if for any element $a \in P$, the set $\{x \in \mathrm{K}(P) : x \leq a\}$ is a directed set and

$$a = \bigvee \{x \in \mathrm{K}(P) : x \leq a\}.$$

Every algebraic poset is continuous.

Given a poset P, the subspace of $(P, \sigma_s(P))$ of all maximal points will be denoted by $\mathrm{Max}_s(P)$. Since the strong Scott topology is finer than the Scott topology, the space $\mathrm{Max}_s(P)$ is always T_1.

Definition 1.4. A strong poset model of a T_1 space X is a poset P such that $\mathrm{Max}_s(P)$ is homeomorphic to X.

Theorem 1.5. *For any continuous poset P, the restrictions of $\sigma(P)$ and $\sigma_s(P)$ on $Max(P)$ are the same.*

Proof. Let $W \in \sigma_s(P)$ and $B = W \cap \mathrm{Max}(P)$. For any $b \in B$, there is a $b^* \in W$ such that $b^* \ll b$. Since P is continuous, each $\{y \in P : b^* \ll y\}$ is Scott open. Let $W* = \bigcup_{b\in B}\{y \in P : b^* \ll y\}$. Then $B = W^* \cap \mathrm{Max}(P)$. This shows that $\sigma_s(P)|_{\mathrm{Max}(P)}$ is contained in $\sigma(P)|_{\mathrm{Max}(P)}$. As the inverse inclusion always holds, we have $\sigma_s(P)|_{\mathrm{Max}(P)} = \sigma(P)|_{\mathrm{Max}(P)}$; thus, the result is proved. $\square$

Therefore, when considering the maximal point spaces of continuous posets, it does not matter whether one uses the Scott topology or the strong Scott topology.

The following is an immediate problem concerning the strong dcpo models: Which T_1 spaces have a strong poset model?

By [14, Theorem 1], every T_1 space has a bounded complete algebraic poset model. Since every algebraic poset is continuous, we obtain the following result by Theorem 1.5.

Proposition 1.6. *Every T_1 space has a strong continuous poset model.*

Remark 1.7. The models of topological spaces with respect to other topologies different from the Scott topology have already been considered by other authors. For instance,

(i) Jimmie Lawson [9] proves that for every Polish space X, there is an ω-domain whose maximal point space with the relative Lawson topology is homeomorphic to X;
(ii) J. H. Liang and K. Keimel [10] prove that a T_1 space has a continuous poset model with respect to the Lawson topology if and only if it is Tychonoff;
(iii) Tsutomu Kamimura and Adrian Tang [7] prove that a T_1 space is compact and second countable if and only if it is homeomorphic to the maximal point space of a bounded complete ω-algebraic dcpo with the relative lower topology;
(iv) Joe Mashburn [11], [12] studies the models with respect to the weakly way below (wwb) topology.

Here, we consider the models with respect to the strong Scott topology and use it to establish new links between domain theory and T_1 spaces.

2. T-Generated Topologies

In [16], the authors define the CK-generated topology and prove that for a Hausdorff space (X, τ), the set of all nonempty compact subsets form a dcpo model (in a specific sense) if and only if τ is CK-generated. In this paper we shall study the general way of using families of sets to generate a topology in a similar manner. We also study their links to the strong poset models of topological spaces.

Definition 2.1. A T-family Ψ on a set X is a collection of nonempty subsets of X such that for any $x \in X$, the family

$$\{A \in \Psi : x \in A\}$$

is a nonempty filter base.

Let $c_\Psi(x) = \bigcap\{A \in \Psi : x \in A\}$ for each $x \in X$.

Definition 2.2. Given a T-family Ψ on a set X, a subset $U \subseteq X$ is called Ψ*-open* if for any filter base $\mathcal{F} \subseteq \Psi$, $\bigcap \mathcal{F} = c_\Psi(a)$ and $a \in U$ implies $F \subseteq U$ holds for some $F \in \mathcal{F}$.

Let τ_Ψ be the collection of all Ψ-open sets of X.

A subset A of a topological space (X, τ) is called saturated if it is an intersection of open sets [5]. The saturation $sat(A)$ of a set A is the intersection of all open sets containing A (it is the smallest saturated set containing A).

Proposition 2.3. *Let Ψ be a T-family of subsets of a set X. Then we have the following:*

(1) *τ_Ψ is a topology on X.*
(2) *$sat(\{x\}) = c_\Psi(x)$ holds for any $x \in X$, where the saturation is the one with respect to τ_Ψ.*
(3) *For any $U \in \tau_\Psi$, $U = \bigcup\{c_\Psi(x) : x \in U\}$.*
(4) *The space (X, τ_Ψ) is T_0 if and only if $x \neq y$ implies $c_\Psi(x) \neq c_\Psi(y)$.*
(5) *The space (X, τ_Ψ) is T_1 if and only if $c_\Psi(x) = \{x\}$ for all $x \in X$.*

Proof. (1) is easy to verify.

(2) If $x \in U$ and U is Ψ-open, then as $\{A \in \Psi : x \in A\}$ is a filter base and $\bigcap\{A \in \Psi : x \in A\} = c_\Psi(x)$, so there is an $F \in \Psi$ such that $x \in F$ and $F \subseteq U$. Now $c_\Psi(x) \subseteq F \subseteq U$ imply $c_\Psi(x) \subseteq U$. Hence, $c_\Psi(x) \subseteq sat(\{x\})$.

Now assume that $y \notin c_\Psi(x)$. Let $U = \{z \in X : y \notin c_\Psi(z)\}$. Then $x \in U$ and $y \notin U$. Note that for any $F \in \Psi$, $F \subseteq U$ if and only if $y \notin F$. We show that U is Ψ-open. Let $\mathcal{F} \subseteq \Psi$ be a filter base such that $\bigcap \mathcal{F} = c_\Psi(d)$ and $d \in U$. Then $y \notin c_\Psi(d)$ by the definition of U. Hence, $y \notin F$ for some $F \in \mathcal{F}$ with $d \in F$. But then $F \subseteq U$. Hence, U is Ψ-open. All these show that $c_\Psi(x) = sat(\{x\})$.

(3) follows from (2) and the general fact that every open set is the union of the saturations of elements in the set.

(4) follows from (2) and the result that a space is T_0 if and only if for any points x and y in the space, $sat(\{x\}) = sat(\{y\})$ implies $x = y$.

(5) follows from (2) and the result that a space is T_1 if and only if $sat(\{x\}) = sat(\{x\})$ holds for any points x in the space. □

A topology β on a set X is called *T-generated* if $\beta = \tau_\Psi$ for some T-family Ψ on X. In this case we also call the space (X, β) T-generated.

Example 2.4. (1) For any set X, the family $\mathcal{P}_0(X)$ of all nonempty subsets of X is a T-family on X. The topology μ on X generated by $\mathcal{P}_0(X)$ is the co-finite topology ($U \in \mu$ if and only if $U = \emptyset$ or $X - U$ is finite).

(2) Let $\mathbb{I} = \{[a, b] : a \leq b, a, b \in \mathbb{R}\}$ be the family of all closed intervals of reals. Then $\mathbb{I}$ is a T-family on $\mathbb{R}$ and the topology generated by $\mathbb{I}$ is the usual Euclidean topology.

(3) Let ω_1 be the set of all countable ordinals and let τ be the co-countable topology on ω_1 ($U \in \tau$ if and only if $U = \emptyset$ or $\omega_1 - U$ is countable). Let Ψ be the family of sets F such that there are $x, y \in \omega_1$ with $x < y$ and $F = \{x\} \cup \{z \in \omega_1 : z \geq y\}$. Then one can easily check that $U \in \tau$ if and only if U is Ψ-open.

(4) If (X, τ) is a Hausdorff first countable space (more general, a k-space), then the family $CK(X)$ of all non-empty closed compact subsets of X is a T-family and it generates the original topology τ (see [16]). In particular, every topology defined by a metric is T-generated.

Example 2.5. For any poset P, let $\Psi = \{\uparrow x : x \in P\}$. Then for any $x \in P$, $\bigcap\{A \in \Psi : x \in A\} = \uparrow x$. It is then easy to check that $U \subseteq P$ is Ψ-open if and only if it is Scott open. Thus, the Scott topology on posets are T-generated.

Probably a little more interesting type of T-family on a poset is the following one.

Example 2.6. Given a poset P, let

$$\Psi_{uf} = \{\uparrow F : F \subseteq P \text{ is a non-empty finite set}\}.$$

Every Ψ_{uf}-open set is clearly Scott open.

Consider the poset P below:

$$P = \{a_1, a_2, \cdots\} \cup \{b\};$$

the order is given by $a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots$ and $a_1 < b$. Then $U = \{b\}$ is Scott open. For each $i \in \mathbb{N}$, let $F_i = \{a_i, b\}$. Then

$\bigcap\{\uparrow F_i : i \in \mathbb{N}\} = \{b\} \subseteq U$, but $F_i \not\subseteq U$ for all i. Therefore, U is not Ψ_{uf}-open.

Proposition 2.7. *For any dcpo P and $U \subseteq P$, $U \in \sigma(P)$ if and only if U is Ψ_{uf}-open.*

Proof. We only need to show that every Scott open set of a dcpo is Ψ_{uf}-open.

Let P be a dcpo (every directed subset has a supremum) and $U \in \sigma(P)$. Let $\{\uparrow A_i : i \in I\}$ be a filtered base with each A_i a nonempty finite subset and $\bigcap\{\uparrow A_i : i \in I\} =\uparrow x \subseteq U$. If $A_i - U \neq \emptyset$ for all $i \in I$, then $\{\uparrow (A_i - U) : i \in I\}$ is a filtered family. By Rudin's Lemma ([5, Lemma III-3.3]), there is a directed subset $D \subseteq \bigcup\{A_i - U : i \in I\}$ such that $D \cap (A_i - U) \neq \emptyset$ for each $i \in I$. Then, as $D \subseteq P - U$ and $P - U$ is Scott closed, $\bigvee D \in P - U$. But $\bigvee D \in \bigcap\{\uparrow A_i : i \in I\} \subseteq U$. This leads to a contradiction. Hence, there must be i such that $\uparrow A_i \subseteq U$, showing that U is Ψ_{uf}-open. □

Remark 2.8. If (X, τ) is a T-generated space and Y is an open subspace of X, then Y is also T-generated (if $\tau = \tau_\Psi$, then the topology on Y is generated by $\Psi \cap Y = \{A \cap Y : A \in \Psi\}$).

It is still unknown whether a G_δ subset of a T-generated space is also T-generated.

Remark 2.9. Given a collection $\mathcal{U}$ of nonempty subsets of set X with $\bigcup \mathcal{U} = X$, let Ψ be the family of nonempty intersections of a finite number of members of $\mathcal{U}$. Then Ψ is a T-family on X and it generates a topology on X.

Although T-generated spaces are quite general, we do have Hausdorff spaces which are not T-generated.

Example 2.10. Let $X = \mathbb{R}$, the set of all real numbers. The topology τ on X is the coarsest one containing the Euclidean topology and the co-countable topology. Equivalently, $U \in \tau$ if and only if $U = V - C$, where V is an open set in the Euclidean topology and C is a countable set. Assume that τ is generated by a T-family Ψ (thus, $U \in \tau$ if and only if U is Ψ-open). Then, as (X, τ) is T_1, $c_\Psi(x) = \{x\}$ for all $x \in X$ by Proposition 2.3(5).

The set $\{0\}$ is clearly not in τ, hence, not Ψ-open. Therefore, there must be a filter base $\mathcal{F} \subseteq \Psi$ such that $\bigcap \mathcal{F} = \{0\}$ and $\{0\} \notin \mathcal{F}$.

Now for each $n \in \mathbb{N}$, the set $(-\frac{1}{n}, \frac{1}{n})$ is Ψ-open (because it is in τ). So there is an $F_n \in \mathcal{F}$ such that $F_n \subseteq (-\frac{1}{n}, \frac{1}{n})$. Since $\mathcal{F}$ is a filter base, we can choose F_n's so that $F_{n+1} \subseteq F_n$ holds for all n's. Now let $\mathcal{F}^* = \{F_n : n \in \mathbb{N}\}$. Then $\mathcal{F}^*$ is a filter base in $\mathcal{F}$ and $\bigcap \mathcal{F}^* = \{0\}$. For each n, as $F_n \neq \{0\}$, we can choose a $b_n \in F_n - \{0\}$. Let $B = \{b_n : n \in \mathbb{N}\}$. Then $\mathbb{R} - B$ is in τ and contains 0. But there exists no $F_n \in \mathcal{F}^*$ that is contained in $\mathbb{R} - B$. This contradicts the assumption that τ is generated by Ψ.

Remark 2.11. By Proposition 1.6, the space X in the above example has a strong continuous poset model. Thus, a space having a strong poset model need not have a T-generated topology. We can even construct a space which has an algebraic dcpo model and whose topology is not T-generated (e.g., the dcpo $Z(X)$ in [13, §4, (2)] is an algebraic model of the space (R, τ), and τ is not T-generated).

In the following, we prove a new link between T-generated topologies and k-spaces. For any topological space (X, τ), let $CK(X)$ be the collection of all nonempty compact subsets of X. It is easy to check that $CK(X)$ is a T-family. In [16], we introduce the CK-filter generated spaces and prove that every Hausdorff k-space (or compactly generated space) is CK-filter generated and, thus, deduce that every Hausdorff k-space has a bounded complete dcpo model. It is still open whether every Hausdorff CK-filter generated space is a k-space ([16, p. 8]). We now give a positive answer to this problem.

By [16], a space (X, τ) is CK-filter generated if a subset U is open if and only if for any filtered family $\mathcal{F} \subseteq CK(X)$ with $\bigcap \mathcal{F} = \{x\}$ and $x \in U$, there is an $F \in \mathcal{F}$ such that $F \subseteq U$.

Note that for any T_1 space X and $a \in X$, $c_{CK(X)}(a) = \bigcap\{K \in CK(X) : a \in K\} = \{a\}$. Thus, in terms of the notion of T-generated topology, a T_1 space (X, τ) is CK-filter generated if and only if the topology τ is T-generated by the family $CK(X)$.

A space (X, τ) is called a k-space if a subset $U \subseteq X$ is open if and only if for any closed compact subspace $A \subseteq X$, $U \cap A$ is open in A.

Theorem 2.12. *Every CK-filter generated T_1 space is a k-space.*

Proof. Let (X, τ) be a CK-filter generated T_1 space. Note that for any $x \in X$, $c_{CK(X)}(x) = \{x\}$ because (X, τ) is T_1.

Assume that U is a subset of X such that for any closed compact subset A of (X, τ), $U \cap A$ is open in the subspace A. Let $\{K_i : i \in I\} \subset CK(X)$ be a filtered family and

$$\bigcap\{K_i : i \in I\} = \{x\} \subseteq U.$$

Without loss of generality, we can assume that all K_i's are contained in some K_{i_0}(otherwise, we can consider the family $\{K_i \cap K_{i_0} : i \in I\}$). Then each K_i is a compact closed subset of the subspace K_{i_0}.

If for each $i \in I$, $K_i \cap U^c \neq \emptyset$, then, as $U \cap K_i$ is open in K_i, $K_i \cap U^c$ is closed in K_i. Hence, $K_i - U$ is closed in K_{i_0}. Now

$$\bigcap\{K_i - U : i \in I\} = \bigcap\{K_i : i \in\} - U = \emptyset.$$

This contradicts that K_{i_0} is compact. Hence, there exists K_i such that $K_i \subseteq U$. By the assumption, we deduce that U is open in X.

Conversely, if $U \subseteq \tau$ and $\{K_i : i \in I\} \subset CK(X)$ is a filtered family such that

$$\bigcap\{K_i : i \in I\} = \{x\} \subseteq U,$$

then trivially we have $K_i \subseteq U$ for some i. Therefore, (X, τ) is a k-space. □

The following result provides a new characterization for Hausdorff K-spaces.

Corollary 2.13. *A Hausdorff space is a k-space if and only if it is CK-filter generated.*

3. ME-Separated Strong Poset Models

Consider the dcpo $(\mathbb{I}, \supseteq)$ in Example 2.4(2). For any $I = [a, b] \in \mathbb{I}$, $\uparrow I \cap \mathrm{Max}(\mathbb{I}) = \{[x, x] : x \in [a, b]\}$. Thus, for any two $I, J \in \mathbb{I}$, $I \supseteq J$ if and only if $\uparrow I \cap \mathrm{Max}(\mathbb{I}) \subseteq \uparrow J \cap \mathrm{Max}(\mathbb{I})$. For any T_1 space X, the dcpo $(CK(X), \supseteq)$ of all nonempty closed compact sets also has this property.

Definition 3.1. A poset P is called *ME-separated* if for any $x, y \in P$,

$$x \leq y \text{ iff } \uparrow y \cap \mathrm{Max}(P) \subseteq \uparrow x \cap \mathrm{Max}(P).$$

Example 3.2. (1) The poset $P = [0, 1]$ of real numbers between 0 and 1, with the usual order, is not ME-separated.

(2) Let $X = \mathbb{R}^n$ be the Euclidean n-space. For each $x \in X$ and $\epsilon \in \mathbb{R}$ with $\epsilon \geq 0$, let $B(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$. Then the set $BX = \{B(x, \epsilon) : x \in X, \epsilon \geq 0\}$ is a dcpo with respect to the reverse inclusion order (see [5, Example V-6.8] for a more general conclusion). It is easy to see that BX is ME-separated.

Lemma 3.3. *If a T_1 space has an ME-separated strong poset model, then X is T-generated.*

Proof. Assume that P is an ME-separated poset and X is homeomorphic to $\mathrm{Max}_s(P)$. To simplify the statements, we assume $X = \mathrm{Max}_s(P)$.

Let $\Psi = \{\uparrow u \cap X : u \in P\}$. For any $x \in X$,

$$\{A \in \Psi : x \in A\}$$

has a smallest member $\{x\}$, so it is a nonempty filter base. Therefore, Ψ is a T-family on X and $c_\Psi(x) = \{x\}$ holds for every $x \in X$. We now show that the restriction of $\sigma_s(P)$ on X is generated by Ψ.

Let $U \in \sigma_s(P)$ and let $\mathcal{F} \subseteq \Psi$ be a filter base such that $\bigcap \mathcal{F} = \{m\} \subseteq U$. Assume that $\mathcal{F} = \{\uparrow p_i \cap X : i \in I\}$. Then $A = \{p_i : i \in I\}$ is a directed subset of P because P is ME-separated.

Since $m \in \bigcap\{\uparrow p_i : i \in I\}$, m is an upper bound of $A = \{p_i : i \in I\}$. Assume that d is any upper bound of A, then $d \in \bigcap\{\uparrow p_i : i \in I\}$ and thus $\uparrow d \subseteq \bigcap\{\uparrow p_i : i \in I\}$. Then

$$\uparrow d \cap X \subseteq \bigcap\{\uparrow p_i : i \in I\} \cap X = \{m\}.$$

As $\uparrow d \cap X \neq \emptyset$ (every element is below some maximal point), it follows that $\uparrow d \cap X = \{m\}$. Noting that $m \in X$, $\uparrow m = \{m\}$. Therefore, $\uparrow d \cap X = \uparrow m \cap X$, which implies $d = m$ because P is ME-separated. All these show that $\bigvee A = m$ (in fact, m is the unique upper bound of A). Since $U \in \sigma_s(P)$, there exists $p_{i_0} \in U$. Then $\uparrow p_{i_0} \subseteq U$ because U is an upper set; thus, $\uparrow p_{i_0} \cap X \subseteq U \cap X$. This shows that $U \cap X \in \tau_\Psi$, the topology generated by Ψ on X.

Conversely, assume that $V \subseteq X$ is nonempty and is in τ_Ψ. Let $V^* = \{p \in P : \uparrow p \cap X \subseteq V\}$. Then V^* contains V and is an upper set of P, again because P is ME-separated. For any directed set $D \subseteq P$ such that $\bigvee D \in X \cap V^*$, the family $\{\uparrow r \cap X : r \in D\}$ is a filter base in Ψ and $\bigcap\{\uparrow r \cap X : r \in D\} = \{\bigvee D\} \subseteq X$ and $\{\bigvee D\} \subseteq V$; thus, there is $d \in D$ such that $\uparrow d \cap X \subseteq V$, which implies $d \in V^*$. Thus, $V^* \in \sigma_s(P)$. Trivially, $V = V^* \cap X$.

The proof is complete. □

Corollary 3.4. *If a T_1 space has an ME-separated domain model, then it is T-generated.*

Proof. Let P be an ME-separated domain model of space X. Then, by Theorem 1.5, P is an ME-separated strong poset model of X, so X is T-generated. □

Lemma 3.5. *If a T_1 space is T-generated, then it has an ME-separated strong poset model.*

Proof. Assume that (X, μ) is a T_1 space such that there is a T-family Ψ satisfying $\mu = \tau_\Psi$. We show that the space X is homeomorphic to $\mathrm{Max}_s(P)$, where $P = \Psi \cup \{\{x\} : x \in X\}$ with the reverse inclusion order. Clearly, $\mathrm{Max}(P) = \{\{x\} : x \in X\}$. Let $f : X \longrightarrow \mathrm{Max}(P)$ be the trivial bijection. Let $\mathcal{U} \in \sigma_s(P)$ and let $\mathcal{F} \subseteq \Psi$ be a filter base such that $\bigcap \mathcal{F} = \{x\} \subseteq f^{-1}(\mathcal{U} \cap \mathrm{Max}(P))$. Then $\mathcal{F}$ is a directed subset of P such that $\sup \mathcal{F} = \{x\} \in \mathcal{U} \cap \mathrm{Max}(P)$. Hence, there is $F \in \mathcal{F} \cap \mathcal{U}$. Then for any $x \in F$, $f(x) = \{x\} \geq_P F$, implying $f(x) \in \mathcal{U} \cap \mathrm{Max}(P)$. So $F \subseteq f^{-1}(\mathcal{U} \cap \mathrm{Max}(P))$. Therefore, $f^{-1}(\mathcal{U} \cap \mathrm{Max}(P)) \in \tau_\Psi = \mu$, showing the continuity of f.

Now let $U \in \tau_\Psi$.

(i) Clearly, $f(U) = \{\{x\} : x \in U\} = U^* \cap \mathrm{Max}(P)$, where $U^* = \{A \in P : A \subseteq U\}$.

(ii) The set U^* is in $\sigma_s(P)$. As a matter of fact, U^* is clearly an upper set in P. Assume that $\mathcal{F} \subseteq P$ is a directed subset such that $sup_P\mathcal{F} = \{x_0\} \in U^*$. Then $x_0 \in \bigcap \mathcal{F}$. If $y \in \bigcap \mathcal{F}$, then $\{y\}$ is an upper bound of $\mathcal{F}$, so $x_0 = y$. All these show that $\{x_0\} = \bigcap \mathcal{F}$. Since $\{x_0\} \in U^*$, $x_0 \in U$. If one of $F_0 \in \mathcal{F}$ is a singleton, then $F_0 = \{x_0\}$, so $F_0 \in U^*$. Otherwise, as $\mathcal{F} \subseteq \Psi$ is a filter base and $U \in \tau_\Psi$, we have that $F \subseteq U$ holds for some $F \in \mathcal{F}$, which implies $F \in U^*$. Therefore, $U^* \in \sigma_s(P)$. Hence, $f(U)$ is open in $\mathrm{Max}_s(P)$.

In all, we have shown that f is a homeomorphism.

Therefore, $(P, \supseteq)$ is a strong poset model of X. In addition, P is easily seen to be ME-separated. □

The combination of the above two lemmas leads to the following theorem.

Theorem 3.6. *A T_1 space is T-generated if and only if it has an ME-separated strong poset model.*

In [16] it is proved that for any Hausdorff k-space, a subset U is open if and only if for any filter base $\mathcal{F}$ of closed compact sets, $\bigcap \mathcal{F} \subseteq U$ implies $F \subseteq U$ for some $F \in \mathcal{F}$. This means exactly that the topology of every Hausdorff k-space is generated by the family $CK(X)$ of all nonempty closed compact sets of X. Also, as $CK(X)$ is closed under the intersection of filter bases, $CK(X)$ is a dcpo with respect to the reverse inclusion order. Thus, we have the following corollary.

Corollary 3.7. *Every Hausdorff k-space has an ME-separated strong dcpo model.*

We close the paper with some problems for further work on this topic.

The metric space $\mathbb{R}$ of real numbers with the Euclidean topology has an ME-separated domain model, i.e., $(\mathbb{I}, \supseteq)$. For any complete metric space (Y, d), the domain BY of all closed formal balls of Y is a domain model of Y [2] (and, hence, also a strong dcpo model). But, in general, BY need not be ME-separated.

Thus, we have the following question.

Question 3.8. Which complete metrizable spaces have an ME-separated domain model?

We hope to know more about the T-generated spaces. The following is a very natural question on T-generated space.

Question 3.9. Is the product of any two T-generated spaces also T-generated?

Acknowledgment. The referee has read the manuscript very carefully and provided us with a lot of constructive comments and suggestions for improvement. We are very grateful to that.

References

[1] M. Ali-Akbari, B. Honari, and M. Pourmahdian, *Any T_1 space has a continuous poset model*, Topology Appl. **156** (2009), no. 13, 2240–2245.

[2] Abbas Edalat and Reinhold Heckmann, *A computational model for metric spaces*, Theoret. Comput. Sci. **193** (1998), no. 1-2, 53–73.

[3] Marcel Erné, *Algebraic models for T_1-spaces*, Topology Appl. **158** (2011), no. 7, 945–962.

[4] D. N. Georgiou and S. D. Iliadis, *Topologies on posets and function spaces*, Topology Proc. **30** (2006), no. 2, 467–473.

[5] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *Continuous Lattices and Domains.* Encyclopedia of Mathematics and its Applications, 93. Cambridge: Cambridge University Press, 2003.

[6] Jean Goubault-Larrecq, *Non-Hausdorff Topology and Domain Theory: Selected Topics in Point-Set Topology.* New Mathematical Monographs, 22. Cambridge: Cambridge University Press, 2013.

[7] Tsutomu Kamimura and Adrian Tang, *Total objects of domains*, Theoret. Comput. Sci. **34** (1984), no. 3, 275–288.

[8] Panos Th. Lambrinos and Basil Papadopoulos, *The (strong) Isbell topology and (weakly) continuous lattices* in Continuous Lattices and Their Applications (Bremen, 1982). Ed. Rudolf-E. Hoffmann and Karl H. Hofmann. Lecture Notes in Pure and Appl. Math., 101. New York: Dekker, 1985. 191–211.

[9] Jimmie Lawson, *Spaces of maximal points*, Math. Structures Comput. Sci. **7** (1997), no. 5, 543–555.

[10] J. H. Liang and K. Keimel, *Order environments of topological spaces*, Acta Math. Sin. (Engl. Ser.) **20** (2004), no. 5, 943–948.

[11] Joe Mashburn, *A comparison of three topologies on ordered sets*, Topology Proc. **31** (2007), no. 1, 197–217.

[12] Joe Mashburn, *Linearly ordered topological spaces and weak domain representability*, Topology Proc. **35** (2010), 149–164.

[13] Xiaoyong Xi and Dongsheng Zhao, *Maximal point spaces of dcpos satisfying the Lawson condition*, Topology Appl. **230** (2017), 417–424.

[14] Dongsheng Zhao, *Poset models of topological spaces* in Proceedings of the International Conference on Quantitative Logic and Quantification of Software. Hong Kong: Global-Link Publisher, 2009. 229–238.

[15] Dongsheng Zhao and Xiaoyong Xi, *Directed complete poset models of T_1 spaces*, Math. Proc. Cambridge Philos. Soc. **164** (2018), no. 1, 125–134.

[16] (Zhao Dongsheng and Xi Xiaoyong) Dongsheng Zhao and Xiaoyong Xi, *On topological spaces that have a bounded complete dcpo model*, Rocky Mountain J. Math. **48** (2018), no. 1, 141–156.

(Zhao) Mathematics and Mathematics Education; National Institute of Education; Nanyang Technological University; 1 Nanyang Walk; Singapore 637616

Email address: dongsheng.zhao@nie.edu.sg

(Xi) School of Mathematics and Statistics; Jiangsu Normal University; Jiangsu, China

Email address: littlebrook@jsnu.edu.cn

(Chen) School of Computer Science and Software Engineering; East China Normal University; #3663, Zhongshan Road (North); Shanghai, China

Email address: yxchen@sei.ecnu.edu.cn