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by

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TOPOLOGICAL MONOIDS ARE TRANSFINITELY π_1-COMMUTATIVE AT THE IDENTITY ELEMENT

JEREMY BRAZAS AND PATRICK GILLESPIE

Abstract. Infinite products and infinite products of commutators play an important role in the homotopy theory of Peano continua and other locally path-connected spaces. In this paper, we identify an analogue of the Eckmann-Hilton Principle that applies to infinite products in fundamental groups of topological monoids and slightly more general objects called pre-Δ-monoids. In particular, we show that every pre-Δ-monoid M is "transfinitely π_1-commutative" in the sense that permutation of the factors of any infinite loop-concatenation indexed by a countably infinite order and based at the identity $e \in M$ is a homotopy invariant action.

1. Introduction

The Eckmann-Hilton Principle [4] states that if a set M is equipped with two unital binary operations $*$ and $\cdot$ satisfying the distributive law $(a \cdot b) * (c \cdot d) = (a * c) \cdot (b * d)$, then the operations $*$ and $\cdot$ agree and are associative and commutative. Applying this principle to the fundamental group $\pi_1(X, e)$ of any H-space (X, e), it follows that $\pi_1(X, e)$ is abelian. Since every loop space $\Omega(X, e)$ is an H-space, one has as a corollary that all higher homotopy groups are abelian.

In the homotopy theory of Peano continua and other "wild" spaces, infinite product operations on homotopy classes play an important role.

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Even if the fundamental group of a locally path-connected metric space X is abelian, there may be loop products $\prod_{n=1}^{\infty} \alpha_n = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdots$, constructed from an infinite shrinking sequence of loops α_n based at $x \in X$, for which some infinite reordering of the sequence $\alpha_1, \alpha_2, \alpha_3, \dots$ changes the homotopy class of the product. When all such reorderings leave the homotopy class of any given infinite product based at x unchanged, the space X is said to be *infinitely* π_1*-commutative at* x. If all reorderings of the factors of more general "transfinite" loop products indexed over an arbitrary countable linear order leave the homotopy class unchanged, then X is said to be *transfinitely* π_1*-commutative at* x.

This paper is the second in a sequence of papers on infinitary commutativity and abelianization in fundamental groups. In the prequel [1], the authors introduce the two notions of π_1-commutativity described above, show they are equivalent to each other, and also give a characterization in terms of factorization through the Specker group (see Theorem 2.3 below). Section 2 below is dedicated to a recollection of the definitions and characterizations of these properties.

A space X, which is transfinitely π_1-commutative at x, should be thought of as being highly structured at x, having enough homotopies to infinitely commute all linearly ordered products of shrinking sequences of loops based at x. Examples from [1] include:

- spaces with cotorsion-free fundamental groups, e.g. the infinite torus, are transfinitely π_1-commutative at all of their points,
- arbitrary products of semilocally simply connected spaces with abelian fundamental groups, e.g. $(\mathbb{RP}^2)^{\mathbb{N}}$ are transfinitely π_1-commutative at all of their points,
- based loop spaces $\Omega(X, x)$, including all higher loop spaces, are transfinitely π_1-commutative at the constant loop.

In the current paper, we add a significant class of examples to this list by showing that the property of being transfinitely π_1-commutative holds in all topological monoids as well as in slightly more general objects. A *pre-*Δ*-monoid* is a monoid M with topology whose operation $(a, b) \mapsto a*b$ may not be continuous but for which the function product $\alpha * \beta : I \to M$ of two continuous paths $\alpha, \beta : I \to M$ is guaranteed to be continuous (Definition 3.2). In categorical terms, M is a pre-Δ-monoid if and only if its coreflection into the category of Δ-generated spaces [2, 8] is a monoid object in that category. Certainly, every topological monoid is a pre-Δ-monoid. Our main result is the following.

Theorem 1.1. *Every pre-*Δ*-monoid* M *is transfinitely* π_1*-commutative at its identity element.*

We consider the infinitary "square shuffle" argument used to prove Theorem 1.1 an infinite analogue of the Eckmann-Hilton Principle. A crucial part of our argument is maintaining control over the sizes of the homotopies being used at each step. Our proof of Theorem 1.1 is also inspired by the highly technical infinite "n-cube shuffle" argument used in [6, 12], which implies that n-loop spaces are transfinitely π_1-commutative at the constant loop. Our work with general monoids also implies that n-loop spaces are transfinitely π_1-commutative (Remark 3.12) but depends on much simpler arguments.

We are particularly motivated to work with pre-Δ-monoids, since the primary intention of Theorem 1.1 is to lay the foundation for further work on the homotopy and homology groups of topologized monoids such as the James reduced products $J(X)$ [11], infinite symmetric products $SP(X)$ [3], and the group-theoretic analogues, for path-connected Hausdorff X. These topological-algebraic constructions, which are always pre-Δ-monoids, provide geometric models of infinitary π_1-abelianizations that are closely related to the infinitary abelianization defined in [1]. However, even for separable path-connected metric spaces X, the free monoid $J(X)$ and the free commutative monoid $SP(X)$ with their usual weak topologies need not be topological monoids (see Example 3.3). Specifically, in a sequel paper on fundamental groups of James reduced products, the authors will prove that for any based path-connected Hausdorff space (X, e), the inclusion $i : X \to J(X)$ into the James reduced product induces a surjection $i_\# : \pi_1(X, e) \to \pi_1(J(X), e)$ on fundamental groups. The proof is long and technical and has nothing to do with transfinite π_1-commutativity. Despite the fact that the surjectivity of $i_\#$ provides the second ingredient for our main supply of non-trivial computations (Theorem 1.1 is the first ingredient), we believe its inclusion would overwhelm the streamlined arguments of the current paper. Hence, the most striking implications of this paper will be found in the sequel.

2. The Hawaiian earring and infinite commutativity

All notation in this paper will agree with that in [1]. If $\alpha_1, \alpha_2, \ldots, \alpha_n : I \to X$ is a sequence of paths satisfying $\alpha_i(1) = \alpha_{i+1}(0)$, we write $\prod_{n=1}^{m} \alpha_n$ or $\alpha_1 \cdot \alpha_2 \cdots \alpha_n$ for the n-fold *concatenation* defined to be α_i on the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$. We write $\alpha^-(t) = \alpha(1-t)$ for the *reverse* of a path α. If $[a,b], [c,d] \subseteq I$ and $\alpha : [a,b] \to X$, $\beta : [c,d] \to X$ are maps, we write $\alpha \equiv \beta$ if $\alpha = \beta \circ \lambda$ for some increasing homeomorphism $\lambda : [a,b] \to [c,d]$. We write $\Omega(X,x)$ for the space of based loops $(I, \partial I) \to (X,x)$ with the compact-open topology. A sequence of paths $\{\alpha_n\}_{n \in \mathbb{N}}$ is *null at x* if for every neighborhood U of x, we have $Im(\alpha_n) \subseteq U$ for all but finitely many $n \in \mathbb{N}$. If $\alpha_n \in \Omega(X,x)$ is a null-sequence of loops based at x, then we may form the *infinite concatenation* $\prod_{n=1}^{\infty} \alpha_n$, which is defined as α_n on $\left[\frac{n-1}{n}, \frac{n}{n+1}\right]$ and maps 1 to x. If $f : (X, x) \to (Y, y)$ is a based map, then $f_{\#} : \pi_1(X, x) \to \pi_1(Y, y)$ denotes the homomorphism induced on the fundamental group. Throughout this paper, any space in which paths are considered is assumed to be path-connected.

If $C_n = \{(x, y) \in \mathbb{R}^2 \mid (x - 1/n)^2 + y^2 = 1/n^2\}$, then the *Hawaiian earring* is the space $\mathbb{H} = \bigcup_{n\in\mathbb{N}} C_n$ with basepoint $b_0 = (0, 0)$. We fix the following loops in $\mathbb{H}$:

- For each $n \in \mathbb{N}$, let $\ell_n \in \Omega(C_n, b_0)$ be the canonical counterclockwise loop traversing the circle C_n.
- Let $\ell_\infty \in \Omega(\mathbb{H}, b_0)$ denote the loop defined as ℓ_n on the interval $\left[\frac{n-1}{n}, \frac{n}{n+1}\right]$ and $\ell_\infty(1) = b_0$.
- Let $\mathcal{C} \subseteq I$ be the middle third Cantor set. Write $I\backslash\mathcal{C}$ as the union $\bigcup_{n\geqslant 1} \bigcup_{k=1}^{2^{n-1}} I_n^k$ where I_n^k is an open interval of length $\frac{1}{3^n}$ and, for fixed n, the sets I_n^k are indexed by their natural ordering in I. Let $\ell_\tau \in \Omega(\mathbb{H}, b_0)$ be the loop defined so that $\ell_\tau(\mathcal{C}) = b_0$ and $\ell_\tau := \ell_{2^{n-1}+k-1}$ on $\overline{I_n^k}$ (see Figure 1).

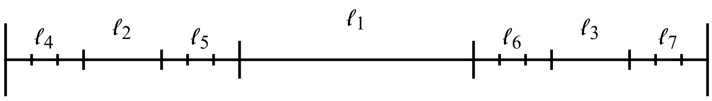

FIGURE 1. The tranfinite concatenation loop ℓ_τ

Definition 2.1. Suppose $\alpha_n \in \Omega(X, x)$ is a null-sequence and $f : \mathbb{H} \to X$ is the map with $f \circ \ell_n = \alpha_n$. The *transfinite concatenation* of the sequence α_n is the loop $f \circ \ell_\tau$, which we typically write as $\prod_\tau \alpha_n$.

For any set $A \subseteq \mathbb{N}$, let $\mathbb{H}_A = \bigcup_{n\in A} C_n$. Note that if A is finite, then $\pi_1(\mathbb{H}_A, b_0)$ is freely generated by the elements $[\ell_n]$, $n \in A$. The Hawaiian earring group was first correctly characterized in [13]. A characterizing property of the Hawaiian earring group needed for the current paper is that for every finite set $F \subseteq \mathbb{N}$, the inclusions $\mathbb{H}_F \to \mathbb{H}$ and $\mathbb{H}_{\mathbb{N}\backslash F} \to \mathbb{H}$ induce an isomorphism $\pi_1(\mathbb{H}_F, b_0) * \pi_1(\mathbb{H}_{\mathbb{N}\backslash F}, b_0) \to \pi_1(\mathbb{H}, b_0)$ (see [5]). Hence any $[\alpha] \in \pi_1(\mathbb{H}, b_0)$ factors uniquely as a finite product of homotopy classes alternating between $\mathbb{H}_F$ and $\mathbb{H}_{\mathbb{N}\backslash F}$.

Definition 2.2. A space X is

(1) *infinitely π_1-commutative at $x \in X$* if for every null sequence $\alpha_n \in \Omega(X, x)$ and bijection $\phi : \mathbb{N} \to \mathbb{N}$, we have $[\prod_{n=1}^{\infty} \alpha_n] = [\prod_{n=1}^{\infty} \alpha_{\phi(n)}]$ in $\pi_1(X, x)$.

(2) *transfinitely π_1-commutative at $x \in X$* if for every null sequence $\alpha_n \in \Omega(X,x)$ and bijection $\phi : \mathbb{N} \to \mathbb{N}$, we have $[\prod_\tau \alpha_n] = [\prod_\tau \alpha_{\phi(n)}]$ in $\pi_1(X,x)$.

As pointed out in [1], the transfinitely π_1-commutative property "at x" is not a local property, but rather is a property of the pair (X,x), which is invariant under basepoint-relative homotopy equivalence. The most practical characterization of this property makes use of the infinite torus $\mathbb{T} = \prod_{n\in\mathbb{N}} S^1$. Let $\eta : \mathbb{H} \to \mathbb{T}$ be the canonical embedding onto the subspace $\bigvee_{n=1}^{\infty} S^1$ of $\mathbb{T}$. Using η, we may identify $\mathbb{H}$ as a subspace of $\mathbb{T}$ so that b_0 is the identity element of $\mathbb{T}$. The fundamental group $\pi_1(\mathbb{T}, b_0)$ is isomorphic to the Specker group $\mathbb{Z}^{\mathbb{N}}$ where $[\eta\circ\ell_n]$ is identified with the unit vector $\mathbf{e}_n \in \mathbb{Z}^{\mathbb{N}}$, which has 1 in the n-th coordinate and 0's elsewhere. It is well-known that the induced homomorphism $\eta_\# : \pi_1(\mathbb{H}, b_0) \to \pi_1(\mathbb{T}, b_0)$ is surjective and $[\alpha] \in \ker(\eta_\#)$ if and only if α has winding number 0 around C_n for all $n \in \mathbb{N}$.

Theorem 2.3. [1] *For any space X and $x \in X$, the following are equivalent:*

(1) *for every map $f : (\mathbb{H}, b_0) \to (X,x)$, there exists a homomorphism $g : \pi_1(\mathbb{T}, x_0) \to \pi_1(X,x)$ such that $f_\# = g \circ \eta_\#$,*
(2) *X is transfinitely π_1-commutative at $x \in X$,*
(3) *X is infinitely π_1-commutative at $x \in X$.*

3. Pre-Δ-monoids and their fundamental groups

A *monoid* is a set M equipped with a monoid operation, i.e. an associative, binary operation $\mu : M \times M \to M$, $\mu(a,b) = a * b$, which has an identity element $e \in M$. We utilize the following notation:

- If $f, g : X \to M$ are functions, let $f * g : X \to M$ denote the function $(f * g)(x) = f(x) * g(x)$.
- If $f : X \to M$ is a function and $a \in M$, we let $a*f$ and $f*a$ denote the functions $X \to M$ given by $x \mapsto a * f(x)$ and $x \mapsto f(x) * a$ respectively.
- More generally, we write $\ast_{i=1}^{n} a_i$ to denote iterated products of elements $a_i \in M$. We also use this notation if one or more of the a_i are functions $X \to M$ from a fixed set X, e.g. if $a_1, a_3 \in M$ and $a_2, a_4 : X \to M$ are functions, then $\ast_{i=1}^{4} a_i : X \to M$ is the function $x \mapsto a_1 a_2(x) a_3 a_4(x)$.
- If $A_1, A_2, \dots, A_n \subseteq M$, then $\ast_{i=1}^{n} A_i$ denotes the product

$$\mu_n\left(\prod_{i=1}^{n} A_i\right) = \{\ast_{i=1}^{n} a_i \mid a_i \in A_i\}$$

where $\mu_n : M^n \to M$, $\mu_n(a_1, a_2, \dots, a_n) = \ast_{i=1}^{n} a_i$ is the n-ary multiplication function. In particular, we write $A \ast B$ for $\{a \ast b \mid a \in A, b \in B\}$.

If M is also equipped with a topology such that μ is continuous, then we call M a *topological monoid*. As mentioned in the introduction, there are many natural situations where a monoid operation $M \times M \to M$ is not jointly continuous, but is "continuous enough" to ensure that all of our results apply. To formalize this idea, we recall the notion of a Δ-generated space.

Definition 3.1. A topological space X is Δ*-generated* if $U \subseteq X$ is open if and only if $\alpha^{-1}(U)$ is open in I for all paths $\alpha : I \to X$.

Note that a space X is Δ-generated if and only if it is the topological quotient of a disjoint sum of copies of I. Hence, every Peano continuum (connected, locally path-connected, compact metric space) is Δ-generated and every Δ-generated space is a locally path-connected k-space. For any space X, let $\Delta(X)$ be the space with the same underlying set as X but with the topology generated by the sets $U \subseteq X$ such that for all paths $\alpha : I \to X$ the preimage $\alpha^{-1}(U)$ is open in I. Certainly, this new topology is finer than the original topology on X and is equal to the topology on X if and only if X is already Δ-generated. Moreover, $\Delta(X)$ is Δ-generated and is characterized by the universal property that if $f : D \to X$ is a map from a Δ-generated space D, then $f : D \to \Delta(X)$ is also continuous. In particular, the identity function $\Delta(X) \to X$ is a weak homotopy equivalence. We refer to [2, 8] for more on Δ-generated spaces.

Definition 3.2. A *pre-*Δ*-monoid* consists of a space M and a monoid operation $\mu : M \times M \to M$, $\mu(a, b) = a \ast b$ with the property that for any paths $\alpha, \beta : I \to M$, the product path $\alpha \ast \beta : I \to M$ is continuous. If M is also a group, then we refer to M as a *pre-*Δ*-group*. If M is Δ-generated, then we call M a Δ*-monoid*.

Every topological monoid is a pre-Δ-monoid but not every pre-Δ-monoid is a topological monoid.

Example 3.3. Consider the James reduced product $J(\mathbb{Q})$ over the rational numbers with basepoint $0 \in \mathbb{Q}$, i.e. the free monoid on $\mathbb{Q} \backslash \{0\}$ topologized naturally as the quotient of $\coprod_{n \in \mathbb{N}} \mathbb{Q}^n$ (where 0 becomes the identity element). To construct $J(\mathbb{Q})$, the monoid of finite words with letters in $\mathbb{Q} \backslash \{0\}$, we only need the topological structure of $\mathbb{Q}$; we do not consider an operation on $\mathbb{Q}$ itself. The proof of [7, Theorem 4.1] may be used to show that $J(\mathbb{Q}) \times J(\mathbb{Q}) \to J(\mathbb{Q})$ is not continuous. For a path-connected example, one may consider the James reduced product of the cone $C\mathbb{Q} = \frac{\mathbb{Q} \times I}{\mathbb{Q} \times \{1\}}$ with the image of $(0, 0)$ as the basepoint e.

In a sequel paper, the authors give the James reduced product a detailed treatment. In particular, a tedious but elementary argument is required to prove that $J(X)$ is a pre-Δ-monoid for any Hausdorff space X.

The following proposition illustrates the sense in which a pre-Δ-monoid is "almost" a topological monoid.

Proposition 3.4. *For any monoid M with topology and operation $\mu : M^2 \to M$, the following are equivalent:*

(1) *M is a pre-Δ-monoid,*
(2) *$\Delta(M)$ is a Δ-monoid,*
(3) *the k-ary operation $\mu_k : \Delta(M^k) \to M$, $\mu_k(a_1, a_2, \dots a_k) = \ast_{i=1}^{k} a_i$ is continuous for all $k \geqslant 2$,*
(4) *The operation $M^D \times M^D \to M^D$, $(f, g) \mapsto f * g$ is well-defined for any Δ-generated space D.*

Proof. (1) $\Rightarrow$ (2) If M is a pre-Δ-monoid and $\alpha, \beta : I \to \Delta(M)$ are paths, then $\alpha, \beta : I \to M$ are also continuous. Thus $\alpha * \beta : I \to M$ is continuous by assumption. By the universal property of $\Delta(M)$, $\alpha * \beta : I \to \Delta(M)$ is continuous.

(2) $\Rightarrow$ (3) Suppose $\Delta(M)$ is a Δ-monoid and let $U \subseteq M$ be open. Let $f = (\alpha_1, \alpha_2, \dots, \alpha_k) : I \to M^k$ be any path. Then $\alpha_i : I \to \Delta(M)$ is continuous for each i and we have that $\alpha = \mu_k \circ f = \ast_{i=1}^{k} \alpha_i : I \to \Delta(M)$ is continuous by assumption. Since $\Delta(M)$ has a finer topology than M, $\alpha : I \to M$ is continuous. Thus $\alpha^{-1}(U) = f^{-1}(\mu_k^{-1}(U))$ is open in I. This shows that $\mu_k^{-1}(U)$ is in the topology of $\Delta(M^k)$, proving the continuity of μ_k.

(3) $\Rightarrow$ (4) Suppose $\mu : \Delta(M^2) \to M$ is continuous, D is Δ-generated, and $f, g : D \to M$ are maps. Then $(f, g) : D \to M^2$ is continuous and, since D is Δ-generated, so is $(f, g) : D \to \Delta(M^2)$. Thus $f * g = \mu \circ (f, g)$ is continuous as a function $D \to M$.

(4) $\Rightarrow$ (1) is obvious since I is Δ-generated. $\square$

Within the category of Δ-generated spaces, the categorical product of two Δ-generated spaces X, Y is $\Delta(X \times Y)$. Hence, condition (3) in Proposition 3.4 indicates that M is a pre-Δ-monoid if and only if $\Delta(M)$ is a monoid-object internal to the category of Δ-generated spaces. Since the category of Δ-generated spaces is Cartesian closed, it follows from the previous proposition that if M is a Δ-monoid and $q : M \to N$ is a monoid homomorphism and a topological quotient map, then N is also a Δ-monoid. Hence, Δ-monoids are closed under forming quotient monoids with the quotient topology, whereas the same is not true for topological monoids.

For the remainder of the paper, M will represent a given path-connected pre-Δ-monoid with identity $e \in M$. To begin formulating our proof of Theorem 1.1, we expand upon some familiar results on fundamental groups of topological monoids. We take special care to control the size of homotopies being used whenever possible. The strictness of both identity and associativity in M will be used implicitly and this appears to be crucial to the argument. First, we recall the usual *distributive law* for paths, which is a property of any pre-Δ-monoid M: given paths $\alpha_{i,j} : I \to M$, $1 \leqslant i,j \leqslant n$ satisfying $\alpha_{i,j}(1) = \alpha_{i+1,j}(0)$, we have $\circledast_{j=1}^{n} \prod_{i=1}^{n} \alpha_{i,j} = \prod_{i=1}^{n} \circledast_{j=1}^{n} \alpha_{i,j}$, which is a path from $\circledast_{j=1}^{n}\alpha_{1,j}(0)$ to $\circledast_{j=1}^{n}\alpha_{n,j}(1)$ in M. We remind the reader here that $\circledast$ represents the monoid operation of M and $\prod$ denotes path-concatenation.

For the current paper, we only require a special case of the next lemma. However, we must use it in full generality in future work. Since the more general statement is no more difficult to prove, we give it here.

Lemma 3.5. *Let $\alpha_1, \alpha_2, \dots, \alpha_n : I \to M$ be paths and $\phi : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ be any bijection. For $j, k \in \{1, 2, \dots, n\}$, let*

$$\beta_{j,k} := \begin{cases} \alpha_k(0), & \text{if } \phi(k) \in \{j+1, j+2, \dots, n\} \\ \alpha_k, & \text{if } \phi(k) = j \\ \alpha_k(1), & \text{if } \phi(k) \in \{1, 2, \dots, j-1\}. \end{cases}$$

Then $\circledast_{k=1}^{n}\alpha_k \simeq \prod_{j=1}^{n} \left(\circledast_{k=1}^{n}\beta_{j,k}\right)$ by a homotopy with image in the product set $\circledast_{k=1}^{n} Im(\alpha_k)$.

Proof. Replace $\alpha_k(0)$ and $\alpha_k(1)$ in the definition of $\beta_{j,k}$ with the constant paths $c_{\alpha_k(0)}$ and $c_{\alpha_k(1)}$ respectively. The value of $\prod_{j=1}^{n} \left(\circledast_{k=1}^{n}\beta_{j,k}\right)$ is the same as in the statement of the lemma and the distributive law gives $\prod_{j=1}^{n} \left(\circledast_{k=1}^{n}\beta_{j,k}\right) = \circledast_{k=1}^{n} \left(\prod_{j=1}^{n} \beta_{j,k}\right)$. For each $k \in \{1, 2, \dots, n\}$, every factor $\beta_{j,k}$ of the concatenation $\prod_{j=1}^{n} \beta_{j,k}$ is constant except for $\beta_{\phi(k),k} = \alpha_k$. Hence, there is a path-homotopy $H_k : I^2 \to M$ from $\prod_{j=1}^{n} \beta_{j,k}$ to α_k with image in $Im(\alpha_k)$. Since M is a pre-Δ-monoid and I^2 is Δ-generated, the function product $\circledast_{k=1}^{n} H_k : I^2 \to M$ is continuous and gives a path-homotopy from $\circledast_{k=1}^{n} \prod_{j=1}^{n} \beta_{j,k}$ to $\circledast_{k=1}^{n}\alpha_k$ with image in $\circledast_{k=1}^{n} Im(\alpha_k)$. □

Example 3.6. In the case that $n = 2$, Lemma 3.5 states that for any paths $\alpha, \beta : I \to M$, the three paths $\alpha * \beta$, $(\alpha * \beta(0)) \cdot (\alpha(1) * \beta)$, and $(\alpha(0) * \beta) \cdot (\alpha * \beta(1))$, are all homotopic to each other by homotopies in $Im(\alpha) * Im(\beta)$. For a more complicated example, consider paths $\alpha_k : I \to M$, $1 \leqslant k \leqslant 4$ and the bijection ϕ given by $\phi(1) = 2$, $\phi(2) = 3$, $\phi(3) = 1$, $\phi(4) = 4$. Then the product $\circledast_{k=1}^{4}\alpha_k$ is homotopic (relative to the endpoints $\circledast_{k=1}^{4}\alpha_k(0)$ and $\circledast_{k=1}^{4}\alpha_k(1)$) to the concatenation

$$(\alpha_1(0) * \alpha_2(0) * \alpha_3 * \alpha_4(0)) \cdot (\alpha_1 * \alpha_2(0) * \alpha_3(1) * \alpha_4(0)) \cdot$$
$$(\alpha_1(1) * \alpha_2 * \alpha_3(1) * \alpha_4(0)) \cdot (\alpha_1(1) * \alpha_2(1) * \alpha_3(1) * \alpha_4)$$

by a homotopy in $\circledast_{k=1}^{4} Im(\alpha_k)$.

Corollary 3.7. *For all $\alpha, \beta \in \Omega(M, e)$, the loops $\alpha * \beta$, $\alpha \cdot \beta$, and $\beta \cdot \alpha$ are all homotopic to each other by homotopies in $Im(\alpha) * Im(\beta)$. Moreover, $\beta * \alpha$ is homotopic to each of these by a homotopy in $(Im(\alpha) * Im(\beta)) \cup (Im(\beta) * Im(\alpha))$.*

Proof. Since e is a strict identity for the operation of M, we have $\alpha * \beta(0) = \alpha = \alpha * \beta(1)$ and $\alpha(1) * \beta = \beta = \alpha(0) * \beta$. Applying Lemma 3.5 in the case $n = 2$ gives that $\alpha * \beta$, $\alpha \cdot \beta$, and $\beta \cdot \alpha$ are all homotopic to each other by homotopies in $Im(\alpha) * Im(\beta)$. By switching the roles of α and β, we also have that $\beta * \alpha$ is homotopic to $\beta \cdot \alpha$ by a homotopy in $Im(\beta) * Im(\alpha)$. $\square$

Corollary 3.8. *If M is a pre-Δ-monoid, then $\pi_1(M, e)$ is commutative and $[\alpha \cdot \beta] = [\alpha * \beta]$ for all $\alpha, \beta \in \Omega(M, e)$.*

Lemma 3.9. *If $\{\alpha_n^i\}_{n \in \mathbb{N}}$, $1 \leqslant i \leqslant k$ is a finite collection of null-sequences in $\Omega(M, e)$, then for any open neighborhood U of e in M, we have that $\circledast_{i=1}^{k} Im(\alpha_n^i) \subseteq U$ for all but finitely many n. In particular, $\{\circledast_{i=1}^{k} \alpha_n^i\}_{n \in \mathbb{N}}$ is a null sequence.*

Proof. For $1 \leqslant i \leqslant k$, let $f_i : \mathbb{H} \to M$ be the maps such that $f_i \circ \ell_n = \alpha_n^i$ for all $n \in \mathbb{N}$. Since M is a pre-Δ-monoid, the k-ary operation $\mu_k : \Delta(M^k) \to M$ is continuous (recall Proposition 3.4). Since $\mathbb{H}^k$ is a Δ-generated space, $f = \prod_{i=1}^{k} f_i : \mathbb{H}^k \to \Delta(M^k)$ is continuous. We have $\mu_k(f(b_0, b_0, \ldots, b_0)) = e \in U$ and so there exists an open neighborhood V of b_0 such that $\mu_k(f(V^k)) \subseteq U$. Recall that C_n denotes the n-th circle of $\mathbb{N}$. Find $N \in \mathbb{N}$ such that $C_n \subseteq V$ for all $n \geqslant N$. Then $\circledast_{i=1}^{k} Im(\alpha_n^i) = \mu_k(f((C_n)^k)) \subseteq \mu_k(f(V^k)) \subseteq U$ for all $n \geqslant N$, completing the proof of the first statement. Since $Im(\circledast_{i=1}^{k} \alpha_n^i) \subseteq \circledast_{i=1}^{k} Im(\alpha_n^i)$ for all $n \in \mathbb{N}$, the second statement is an immediate consequence of the first statement. $\square$

The next lemma is the key technical step in the proof of Theorem 1.1.

Lemma 3.10 (Infinite Shuffle)**.** *If $\alpha_n \in \Omega(M, e)$ is a null-sequence and $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection, then $\prod_{n=1}^{\infty} \alpha_n \simeq \prod_{n=1}^{\infty} \alpha_{\phi(n)}$ by a homotopy in $\bigcup_{n \in \mathbb{N}} Im(\alpha_n) * \bigcup_{n \in \mathbb{N}} Im(\alpha_n)$.*

Proof. Let $\alpha_n \in \Omega(M, e)$ be a null sequence and $\phi : \mathbb{N} \to \mathbb{N}$ be a bijection. Note that $\{\ell_{\phi(n)}\}$ and $\{\alpha_{\phi(n)}\}$ are null sequences in $\Omega(\mathbb{H}, b_0)$ and $\Omega(M, e)$ respectively. For convenience, we reparameterize the infinite concatenations $\ell_\infty = \prod_{n=1}^{\infty} \ell_n$ and $L = \prod_{n=1}^{\infty} \ell_{\phi(n)}$ in $\mathbb{H}$ to be defined respectively as the loops ℓ_n and $\ell_{\phi(n)}$ on $\left[1 - 1/2^{n-1}, 1 - 1/2^n\right]$ and $\ell_\infty(1) = L(1) = b_0$.

Find a map $f : \mathbb{H} \to M$ for which $\alpha_n = f \circ \ell_n$. We construct a path-homotopy $H : I^2 \to M$ from $f \circ L = \prod_{n=1}^{\infty} \alpha_{\phi(n)}$ to $f \circ \ell_\infty = \prod_{n=1}^{\infty} \alpha_n$ by recursively defining H piecewise on a sequence of shrinking rectangles in I^2.

We begin the definition of H as follows: set $H(s, 0) = f \circ L(s)$, $H(s, 1) = f \circ \ell_\infty(s)$, $H(0, t) = H(1, t) = e$, and for all (s, t) in the rectangle $\left[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}\right] \times \left[0, \frac{1}{2^n}\right]$, we set $H(s, t) = f \circ L(s)$. We complete the definition of H on the rectangles $R_n = \left[1 - \frac{1}{2^{n-1}}, 1\right] \times \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$ by constructing individual path-homotopies $H_n : I^2 \to M$ and pre-composing with the canonical affine homeomorphism $R_n \to I^2$.

Set $A_n = \{\phi(1), \ldots, \phi(n)\}$ and $B_n = \mathbb{N} \backslash A_n$ and consider the subspaces $\mathbb{H}_{A_n}$ and $\mathbb{H}_{B_n}$ of $\mathbb{H}$. The first is a finite wedge of circles and the second is homeomorphic to $\mathbb{H}$. Since $\pi_1(\mathbb{H}, b_0) = \pi_1(\mathbb{H}_{A_n}, b_0) * \pi_1(\mathbb{H}_{B_n}, b_0)$, there are unique (up to reparameterization) loops $a_1, b_1 \in \Omega(\mathbb{H}_{B_1}, b_0)$ such that $\ell_\infty \equiv a_1 \cdot \ell_{\phi(1)} \cdot b_1$. Applying the same argument inductively, we may find unique (up to reparameterization) loops $a_n, b_n \in \Omega(\mathbb{H}_{B_n}, b_0)$ such that $a_{n-1} \cdot b_{n-1} \equiv a_n \cdot \ell_{\phi(n)} \cdot b_n$. Observe that both $\{a_n\}$ and $\{b_n\}$ are null sequences.

Set $\beta_0 = f \circ \ell_\infty$ and for each $n \in \mathbb{N}$, set $\delta_n = f \circ a_n$, $\epsilon_n = f \circ b_n$, and $\beta_n = \delta_n \cdot \epsilon_n$. The continuity of f and the loop-concatenation operation ensures that $\{\delta_n\}$, $\{\epsilon_n\}$, and $\{\beta_n\}$ are all null-sequences in $\Omega(M, e)$.

For every $n \in \mathbb{N}$, β_{n-1} is a reparameterization of $\delta_n \cdot \alpha_{\phi(n)} \cdot \epsilon_n$. Thus $\beta_{n-1} \simeq \alpha_{\phi(n)} \cdot \delta_n \cdot \epsilon_n \equiv \alpha_{\phi(n)} \cdot \beta_n$. In particular, we may choose a homotopy $\delta_n \cdot \alpha_{\phi(n)} \cdot \epsilon_n \simeq \alpha_{\phi(n)} \cdot \delta_n \cdot \epsilon_n$, which

- is the constant homotopy of ϵ_n on $[2/3, 1] \times I$,
- and commutes δ_n and $\alpha_{\phi(n)}$ on the domain $[0, 2/3] \times I$ by a path-homotopy with image in $Im(\alpha_{\phi(n)}) * Im(\delta_n)$ (recall Corollary 3.7).

We conclude that there is a path-homotopy $H_n : I^2 \to M$ with $H_n(s, 0) = (\alpha_{\phi(n)} \cdot \beta_n)(s)$, $H_n(s, 1) = \beta_{n-1}(s)$, and such that $Im(H_n) \subseteq (Im(\alpha_{\phi(n)}) * Im(\delta_n)) \cup Im(\epsilon_n)$. This completes the definition of H (See Figure 2). For all $t \in I$ and $n \in \mathbb{N}$, we have $\alpha_{\phi(n)}(t), \delta_n(t), \epsilon_n(t) \in Im\left(\prod_{n=1}^{\infty} \alpha_n\right)$, making it clear that $Im(H) \subseteq Im\left(\prod_{n=1}^{\infty} \alpha_n\right) * Im\left(\prod_{n=1}^{\infty} \alpha_n\right) = \bigcup_{n \in \mathbb{N}} Im(\alpha_n) * \bigcup_{n \in \mathbb{N}} Im(\alpha_n)$.

The continuity of H is clear at every point except for $(1, 0) \in I^2$. However, all three sequences $\alpha_{\phi(n)}$, δ_n, and ϵ_n are null-sequences based at e. By Lemma 3.9, every neighborhood U of e in M contains all but finitely many of the sets $(Im(\alpha_{\phi(n)}) * Im(\delta_n)) \cup Im(\epsilon_n)$. Therefore, all but finitely many of the sets $Im(H_n) \cup Im(\alpha_{\phi(n)})$ lie in U. Continuity at $(1, 0)$ is immediate from this observation. $\square$

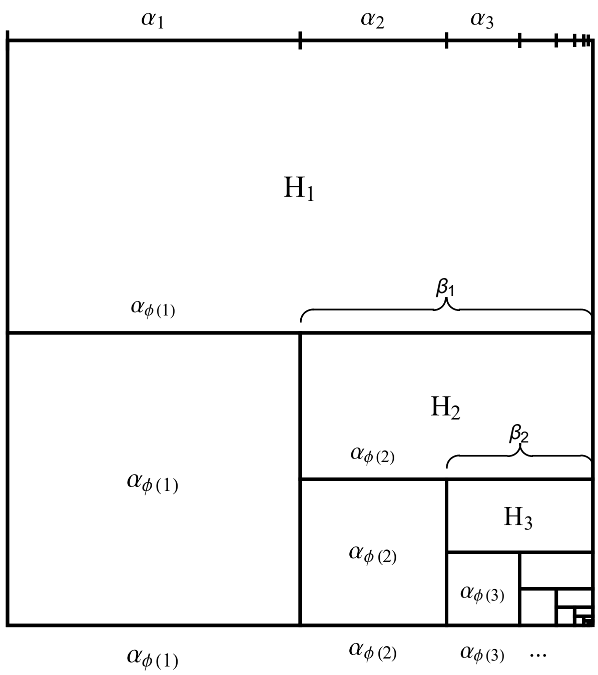

FIGURE 2. The homotopy constructed in Lemma 3.10.

Proof of Theorem 1.1. By Lemma 3.10, M is infinitely π_1-commutative at the identity $e \in M$. By Theorem 2.3, M is also transfinitely π_1-commutative at e. $\square$

Remark 3.11. The homotopy H in Lemma 3.10 is obtained by gluing together constant homotopies and the homotopies H_n, which themselves represent an Eckmann-Hilton-type shuffle. We also note that one may replace $\prod_{n=1}^{\infty}\alpha_n$ with $\prod_{\tau}\alpha_n$ in Lemma 3.10 and a slightly modified argument will show that for any bijection $\phi : \mathbb{N} \to \mathbb{N}$, we have $\prod_{\tau}\alpha_n \simeq \prod_{n=1}^{\infty}\alpha_{\phi(n)}$ by a homotopy in $A = Im(\prod_{\tau}\alpha_n) * Im(\prod_{\tau}\alpha_n) = Im(\prod_{n=1}^{\infty}\alpha_n) * Im(\prod_{n=1}^{\infty}\alpha_n)$. Applying the same argument with ϕ^{-1} gives $\prod_{\tau}\alpha_{\phi(n)} \simeq \prod_{n=1}^{\infty}\alpha_n$ in A. These two homotopies combined with $\prod_{n=1}^{\infty}\alpha_n \simeq \prod_{n=1}^{\infty}\alpha_{\phi(n)}$ from Lemma 3.10 gives a homotopy $\prod_{\tau}\alpha_n \simeq \prod_{\tau}\alpha_{\phi(n)}$ with image in A. Hence, with some repeated use of the constructions in the proof Lemma 3.10, one may verify that M is transfinitely π_1-commutative at $e \in M$ without appealing to Theorem 2.3.

Remark 3.12. In the prequel [1], the authors show that every loop space $\Omega(X,x)$ (including higher loop spaces) is transfinitely π_1-commutative at the constant loop. However, the proof given depends entirely on the highly technical work in [6] on the n-th homotopy group of the n-dimensional Hawaiian earring. Using our main result, we formulate an argument that depends on three much simpler results: It is well-known that any based loop space $\Omega(X,x)$ is homotopy equivalent (rel. basepoint, the constant loop) to the topological monoid $\Omega^*(X,x)$ of Moore loops based at x [14]. Therefore, $\Omega^*(X,x)$ is transfinitely π_1-commutative at the constant Moore loop at x by Theorem 1.1. Since the transfinitely π_1-commutative property is an invariant of basepoint-relative homotopy equivalence [1, Corollary 3.12], we conclude that $\Omega(X,x)$ is transfinitely π_1-commutative at the constant loop.

Example 3.13. Let $J(\mathbb{H},b_0)$ be the James reduced product on the Hawaiian earring space. Since $\mathbb{H}$ is compact Hausdorff, standard topological arguments show that $J(\mathbb{H},b_0)$ is the free topological monoid generated by the space $(\mathbb{H},b_0)$ (where b_0 becomes the identity element). The inclusion $i:\mathbb{H}\to J(\mathbb{H},b_0)$ of generators is continuous. Since the topological monoid $J(\mathbb{H},b_0)$ is transfinitely π_1-commutative at b_0, Theorem 2.3 guarantees the existence of a homomorphism $g:\pi_1(\mathbb{T},b_0)\to\pi_1(J(\mathbb{H},b_0),b_0)$ such that $g\circ\eta_{\#}=i_{\#}$. Moreover, since $\mathbb{T}$ is a topological group, the embedding $\eta:\mathbb{H}\to\mathbb{T}$ extends uniquely to a continuous homomorphism $\widetilde{\eta}:J(\mathbb{H},b_0)\to\mathbb{T}$ such that $\widetilde{\eta}\circ i=\eta$. Since $\widetilde{\eta}_{\#}\circ g\circ\eta_{\#}=\widetilde{\eta}_{\#}\circ i_{\#}=\eta_{\#}$ where $\eta_{\#}$ is surjective, it follows that g is a section of $\widetilde{\eta}_{\#}$. Thus $\pi_1(\mathbb{T},b_0)\cong\mathbb{Z}^{\mathbb{N}}$ is naturally a summand of $\pi_1(J(\mathbb{H},b_0),b_0)$. Per the introduction, the authors will prove in a sequel paper that the inclusion $i:\mathbb{H}\to J(\mathbb{H},b_0)$ induces a surjection $i_{\#}$ on fundamental groups. We note that once the surjectivity of $i_{\#}$ is established, the equality $g\circ\widetilde{\eta}_{\#}\circ i_{\#}=g\circ\eta_{\#}=i_{\#}$ implies that g and $\widetilde{\eta}_{\#}$ are inverse isomorphisms, and thus $\pi_1(J(\mathbb{H},b_0),b_0)\cong\mathbb{Z}^{\mathbb{N}}$.

In a pre-Δ-monoid M, transfinite π_1-commutativity is only guaranteed at the identity. We show how this result is strengthened when M is a pre-Δ-group.

Theorem 3.14. *A pre-Δ-group is transfinitely π_1-commutative at all of its points.*

Proof. Let G be a pre-Δ-group. For given $g\in G$, we verify the factorization property given in Theorem 2.3. Let $f:(\mathbb{H},b_0)\to(G,g)$ be a map and pick any path $\alpha:(I,0,1)\to(G,e,g)$. Consider the map $g^{-1}*f:(\mathbb{H},b_0)\to(G,e)$, which is continuous since G is a pre-Δ-group. The map $H:\mathbb{H}\times I\to G$, $k(x,t)=\alpha(t)*g^{-1}*f(x)$ is also continuous and defines a free homotopy from $H(x,0)=g^{-1}*f(x)$ to $H(x,1)=f(x)$

along the path $H(b_0, t) = \alpha(t)$. Thus if $\varphi_\alpha : \pi_1(G, g) \to \pi_1(G, e)$ is the basepoint-change isomorphism $\varphi_\alpha([\beta]) = [\alpha \cdot \beta \cdot \alpha^-]$, we have $(g^{-1} * f)_\# = \varphi_\alpha \circ f_\#$. Since $g^{-1} * f : (\mathbb{H}, b_0) \to (G, e)$ is based at the identity and G is a pre-Δ-monoid, Theorems 1.1 and 2.3 guarantee the existence of a unique homomorphism $F_\alpha : \pi_1(\mathbb{T}, x_0) \to (G, e)$ such that $F_\alpha \circ \eta_\# = (g^{-1} * f)_\# = \varphi_\alpha \circ f_\#$. Now $\lambda_\alpha = \varphi_\alpha^{-1} \circ F_\alpha$ is a homomorphism satisfying $\lambda_\alpha \circ \eta_\# = f_\#$. By Theorem 2.3, we conclude that G is transfinitely π_1-commutative at g. □

Example 3.15. The free topological group $F_G(X, e)$ in the sense of Graev [9] on a based path-connected space (X, e) is a path-connected topological group. Little seems to be known about the fundamental group $\pi_1(F_G(X, e), e)$ although there are some known results when $F_G(X, e)$ is coreflected into the category of k-spaces and X is assumed to have some local "niceness" conditions [10]. The topological group $F_G(X, e)$ is always transfinitely π_1-commutative at all of its points by Theorem 3.14. Since $\mathbb{T}$ is a topological group, the analogous arguments used in 3.13 show that $\pi_1(\mathbb{T}, b_0) = \mathbb{Z}^{\mathbb{N}}$ is naturally a summand of $\pi_1(F_G(\mathbb{H}, b_0))$. Although it seems probable that these groups are isomorphic, the authors do not currently have a proof of the surjectivity of the homomorphism $i_\# : \pi_1(\mathbb{H}, b_0) \to \pi_1(F_G(\mathbb{H}, b_0))$ induced by inclusion $i : \mathbb{H} \to F_G(\mathbb{H}, b_0)$.

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