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by

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HEREDITARILY INDECOMPOSABLE SUBCONTINUA OF THE PRODUCT OF TWO HAUSDORFF ARCS ARE METRIC

MICHEL SMITH

Abstract. We show that if each of A and B is a Hausdorff arc, then every hereditarily indecomposable subcontinuum of $A \times B$ is a planar metric continuum.

1. Introduction

The author is interested in a generalization of Bing's result to non-metric spaces concerning the existence of hereditarily indecomposable continua in higher dimensional continua [1]. He showed that there exist non-metric continua arbitrary products of which do not contain non-degenerate hereditarily indecomposable continua. On the other hand in the case of a disc $[0,1] \times [0,1]$ there are lots of hereditarily indecomposable continua contained therein. In [8] and [9] the author showed that for a Hausdoff arc X that any hereditarily indecomposable subcontinua of $X \times X$ must be metric in the case where X is a long line type of arc and the case where X is a Souslin line respectively. Furthermore this results for the lexicographic arc follows from the stronger result of Greiwe, Smith and Stone [4]. Thus it appears that in certain non-metric spaces adding the condition of hereditary indecomposability on subcontinua of the space yields metric continua. In general one would think that the condition of hereditary indecomposability would have nothing to do with metrizability, but indeed it does as these results show.

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In this paper we generalize these results and show that if each of A and B is a Hausdorff arc then every hereditarily indecomposable subcontinuum of $A \times B$ is metric. Once this is established it is relatively easy to see that such a non-degenerate continuum must also sit inside a copy of the unit disc inside $A \times B$.

2. Definitions and Background

By a continuum we mean a compact and connected Hausdorff space. By a Hausdorff arc is meant a continuum with exactly two non-cut points.

For a Hausdorff arc X we use the symbol $<_X$ to denote the order that induces the topology of X; if the space X under consideration is understood, the subscript will be omitted. If $a, b \in X$, we use the follow notation for specific subsets of X:

$$\begin{array}{rcl} (a,b) & = & \{x | a < x < b\} \\ [a,b] & = & \{x | a \leq x \leq b\} \\ [a,b) & = & \{x | a \leq x < b\} \\ \vdots & & \vdots \end{array}$$

For a space S under consideration and $H \subset S$, we use the symbol $\overline{H}$ to denote the closure of H in S.

Suppose that each of X and Y is a topological space. In order to avoid confusion with this notation, we will denote a point p of a product space $X \times Y$ as $p = \langle x, y \rangle$ where $x \in X, y \in Y$. The functions π_1 and π_2 denote the projection maps with domain $X \times Y$ and are defined as follows: For each $\langle x, y \rangle \in X \times Y$:

$$\begin{array}{rcl} \pi_1(\langle x, y \rangle) & = & x, \\ \pi_2(\langle x, y \rangle) & = & y. \end{array}$$

Theorem 2.1. *Suppose that each of A and B is a Hausdorff arc, $X = A \times B$ and $M \subset X$ is a hereditarily indecomposable continuum so that $\pi_1(M) = A$ and $\pi_2(M) = B$. Then the arcs A and B are metric arcs.*

We will prove the theorems by proving a sequence of lemmas which together yield a proof of the theorem. One should observe that each of the lemmas hold with the interchange of A and B. We will frequently use this fact without comment.

Background Theorem 1. *Suppose that A is a Hausforff arc and $\{a_i\}_{i=1}^{\infty}$ is a sequence of distinct points of A. Then there is a subsequence of $\{a_i\}_{i=1}^{\infty}$ that is either increasing or decreasing.*

Background Theorem 2. *Suppose that M is a hereditarily indecomposable continuum; let H and K be two disjoint closed subsets of M and let U and V be open set containing H and K respectively. Then M is the union of three closed sets D, E, F so that:*

$$\begin{aligned} M &= D \cup E \cup F \\ H &\subset D \\ K &\subset F \\ D \cap F &= \emptyset \\ D \cap E &\subset V - K \\ E \cap F &\subset U - H. \end{aligned}$$

Background Theorem 3. *[Boundary Bumping Theorem] Suppose that M is a continuum, U is an open set which does not contain M and C is a component of $M \cap U$. Then $\overline{C}$ intersects the boundary of U. [See Nadler [7].]*

3. Lemmas and Proof of the Theorem.

In the following we suppose that each of A and B is a Hausdorff arc with endpoints $r_A < s_A$ and $r_B < s_B$ respectively and that M is a hereditarily indecomposable subcontinuum of $A \times B$ so that $\pi_1(M) = A$ and $\pi_2(M) = B$.

Lemma 3.1. *Suppose that $b_1, b_2 \in B$ with $b_1 < b_2$ and C is a subcontinuum of $A \times B$ so that C intersects both $A \times \{b_1\}$ and $A \times \{b_2\}$; furthermore suppose D is a subcontinuum of C irreducible between $A \times \{b_1\}$ and $A \times \{b_2\}$. Then $D \subset A \times [b_1, b_2]$.*

Proof. Assume the hypothesis of the theorem and that there is a subcontinuum D of C so that D is irreducible between $A \times \{b_1\}$ and $A \times \{b_2\}$ but D is not a subset of $A \times [b_1, b_2]$. Then there is no subcontinuum of $D \cap A \times [b_1, b_2]$ intersecting both $A \times \{b_1\}$ and $A \times \{b_2\}$. (Note: here and throughout the text, expressions such as $D \cap A \times [b_1, b_2]$ should be interpreted as $D \cap (A \times [b_1, b_2])$ since another grouping doesn't make sense in the current spaces under consideration.) Therefore we have $D \cap A \times [b_1, b_2]$ is the union of two disjoint compact sets H and K so that $D \cap A \times \{b_1\} \subset H$ and $D \cap A \times \{b_2\} \subset K$. But then $H \cup (D \cap A \times [r_B, b_1])$ and $K \cup (D \cap A \times [b_2, s_B])$ are two disjoint compact sets whose union is D which is a contradiction. □

Lemma 3.2. *Suppose $a \in A$ is such that $r_A < a < s_A$. Then $(\{a\} \times B) \cap M$ is uncountable.*

Proof. Let G denote the components of $M \cap [r_A, a] \times B$ and J denote the components of $M \cap [a, s_A] \times B$. Then since $M \cap (r_A, a) \times B$ is nonempty, it follows that each component of $M \cap [a, s_A]$ is a proper subcontinuum of M and so is nowhere dense. So by the Baire Category Theorem, J is uncountable. Since each component of $M \cap [a, s_A] \times B$ intersects $\{a\} \times B$ it follows that $M \cap \{a\} \times B$ is uncountable. □

Lemma 3.3. *Suppose $a \in A$ is such that $r_A < a < s_A$, G° denote the nondegenerate components of $M \cap [r_A, a) \times B$ and J° denote the components of $M \cap (a, s_A] \times B$. Then each of G° and J° is uncountable.*

Proof. Since each proper subcontinuum of M is nowhere dense in M and $M \cap [r_A, a) \times B$ is a locally compact Hausdorff space, the uncountability of G° follows from the Baire Category Theorem; similarly for J°. □

The lemma applies equally well to the arc B. Let us use the following notation:

$G^\circ_{A,a}$ denotes the collection of non-degenerate components of $M \cap [r_A, a] \times B$;

$J^\circ_{A,a}$ denotes the collection of non-degenerate components of $M \cap [a, s_A] \times B$.

And similarly for the $b \in B$ with $r_B < b < s_B$:

$G^\circ_{B,b}$ denotes the collection of non-degenerate components of $M \cap A \times [r_b, b]$;

$J^\circ_{B,b}$ denotes the collection of non-degenerate components of $M \cap A \times [b, s_B]$.

And we observe that Lemma 3.3 implies that each of these collections is uncountable.

Lemma 3.4. *Suppose $\langle a, b\rangle \in M$ is such that $r_A < a < s_A$ and $r_B < b < s_B$. Then, if $x < a < y$ some element of of $G^\circ_{B,b}$ or $J^\circ_{B,b}$ intersects $(x, y) \times \{b\}$.*

Proof. The set $M \cap (A \times \{b\})$ is totally disconnected so without loss of generality, by shrinking (x, y) if necessary, we can assume $\langle x, b\rangle \notin M$ and $\langle y, b\rangle \notin M$. So there is $u < b < v$ so that $\{x\} \times [u, v] \cap M = \emptyset$ and $\{y\} \times [u, v] \cap M = \emptyset$ but $(x, y) \times [u, v]$ contains $\langle a, b\rangle$ so the component C of $M \cap (x, y) \times [u, v]$ containing $\langle a, b\rangle$ intersects the boundary of $(x, y) \times [u, v]$. Thus C intersects $(x, y) \times \{u\}$ or $(x, y) \times \{v\}$. If C intersects $(x, y) \times \{u\}$ then consider the subcontinuum D of C irreducible from $(x, y) \times \{u\}$ to $(x, y) \times \{b\}$. Then, by lemma 3.1, $D \subset (x, y) \times [u, b]$ so the element of $G^{\circ}_{B,b}$ that is the component of $M \cap A \times [r_B, b]$ containing D is the required set. The case where C intersects $(x, y) \times \{v\}$ is similar. □

Lemma 3.5. *Let $H \subset A \times B$ be a continuum. Suppose that $\{a_i\}_{i=1}^{\infty}$ is an increasing (or decreasing) sequence of distinct points in A and there is a point $b' > b$ in B so that a component of $H \cap [a_i, a_{i+1}] \times [b, b']$ intersects both $[a_i, a_{i+1}] \times \{b\}$ and $[a_i, a_{i+1}] \times \{b'\}$. Then H contains an arc.*

Proof. Since $\{a_i, \}_{i=1}^{\infty}$ is an increasing sequence, let a be the limit point of the sequence $\{a_i, \}_{i=1}^{\infty}$. Let C_i denote the component of $H \cap [a_i, a_{i+1}] \times [b, b']$ that intersects $[a_i, a_{i+1}] \times \{b\}$ and $[a_i, a_{i+1}] \times \{b'\}$. Then for each $p \in [b, b']$ there is a point $\langle x_i, p\rangle \in C_i \cap ([a_i, a_{i+1}] \times \{p\})$. Then since $a_i \leq x_i \leq a_{i+1}$, $\langle a, p\rangle$ is a limit point of $\{\langle x_i, p\rangle\}_{i=1}^{\infty}$. Thus $\langle a, p\rangle \in H$ for all $p \in [b, b']$ and so $\{a\} \times [b, b'] \subset H$, so H contains an arc. The argument is similar in the case that $\{a_i, \}_{i=1}^{\infty}$ is decreasing. □

Definition. Suppose that A is a Hausdorff arc, then the point $p \in A$ is called a p-point from below iff whenever $\{x_i\}_{i=1}^{\infty}$ is sequence of points of A with $x_i < p$ then there is a point $x \in A$ so that $x_i < x < p$ for all $i \in \mathbb{N}$; the point $p \in A$ is called a p-point from above iff whenever $\{x_i\}_{i=1}^{\infty}$ is sequence of points of A with $p < x_i$ then there is a point $x \in A$ so that $p < x < x_i$ for all $i \in \mathbb{N}$.

Lemma 3.6. *Suppose that $\langle a, b\rangle \in M$ is such that $r_A < a < r_B$. Then a is not a p-point from below in A.*

Proof. Suppose that the lemma is not true and $a \in A$ is a lower p-point such that $r_A < a < s_A$. By Lemma 3.4, $G^{\circ}_{A,a}$ is uncountable. So we can construct inductively an infinite set $\{C_i\}_{i=1}^{\infty}$ of disjoint components of $M \cap [r_A, a] \times B$, an infinite set of points $\{\langle a, x_i\rangle\}_{i=1}^{\infty}$ with $\langle a, x_i\rangle \in C_i$ and an infinite sequence $\{(u_i, v_i)\}_{i=1}^{\infty}$ of disjoint open sets in B so that $x_j \in (u_i, v_i)$ if and only if $i = j$ and so that $\langle a, u_i\rangle \notin M, \langle a, v_i\rangle \notin M$;

furthermore $\{x_i,\}_{i=1}^{\infty}$ can be constructed so that it is either increasing or decreasing.

For each i there is an element $a_i \in A$ and a subcontinuum $L_i \subset C_i$ lying in $[a_i, a] \times (u_i, v_i)$ so that $\langle a, x_i \rangle \in L_i$ and $L_i \cap \{a_i\} \times (u_i, v_i) \neq \emptyset$ (in fact a component of $C_i \cap [a_i, a] \times [u_i, v_i]$ intersecting $\{a\} \times (u_i, v_i)$ will witness this). By the lower p-point property there is an element $\hat{a} \in A$ so that $a_i < \hat{a} < a$ for all i. Then there is a subcontinuum $\hat{L}_i$ of L_i irreducible from $\{\hat{a}\} \times (u_i, v_i)$ to $\{a\} \times (u_i, v_i)$ (e.g. a subcontinuum of L_i irreducible from $\{\hat{a}\} \times [u_i, v_i]$ to $\{a\} \times [u_i, v_i]$). This satisfies the hypothesis of Lemma 3.5 so this implies that M contains an arc which is a contradiction. $\square$

Applying the above lemma to both A and B for the portions both below and above cut points, we have the following:

Corollary 1. *The arcs A and B are first countable at their cut points.*

Lemma 3.7. *Suppose $r_A < r < a < s_A$. Let H be a component of $M \cap ([r_A, a] \times B)$ that intersects both $\{r_A\} \times B$ and $\{a\} \times B$; let U be open with $H \subset U$. Then there exist $a' \in A$, $a < a'$ and a continuum $C \subset M \cap ([r, a'] \times B)$ so that $C \subset U$, C intersects both $\{r\} \times B$ and $\{a'\} \times B$ and $C \cap H = \emptyset$.*

Proof. Let V be open so that $H \subset V \subset \overline{V} \subset U$ and so that $V \cap (M \cap [r_A, a] \times B) = \overline{V} \cap (M \cap [r_A, a] \times B)$. Let $\hat{H}$ be the component of $M \cap \overline{V}$ that contains H. Since $\hat{H}$ intersects $Bd(V)$ and H is a component of $M \cap ([r_A, a] \times B)$ it follows that $\hat{H}$ contains a point $\langle a', y \rangle$ for some $a' > a$ and $y \in B$. Since H is nowhere dense in $\hat{H}$, $\hat{H}$ contains a point $\langle r', y' \rangle$ not in H for some $r' < r$ and $y' \in B$. So $\hat{H}$ contains a continuum C irreducible from $\{r'\} \times B$ to $\{a'\} \times B$. So $C \subset [r', a'] \times B$. Since H contains a point of $\{r_A\} \times B$ and no point of $\{a'\} \times B$ and C contains a point of $\{a'\} \times B$ and no point of $\{r_A\} \times B$ is follows that $C \cap H = \emptyset$. $\square$

Lemma 3.8. *The arcs A and B are first countable at their endpoint.*

Proof. Suppose not; without loss of generality suppose that A is not first countable at r_A. Let $r_A < r < s_A$ be such that $\langle r, r_B \rangle \notin M$ and $\langle r, s_B \rangle \notin M$; let H be a component of $M \cap ([r_A, r] \times B)$ intersecting $\{r_A\} \times B$. Let $P \in H \cap \{r\} \times B$ and let $Q \in M \cap (r_A, r) \times (r_B, s_B) - H$. By Corollary 1, X is first countable at P so let $\{U_i\}_{i=1}^{\infty}$ be a countable local basis at P so that $\overline{U_{i+1}} \subset U_i$. Construct a sequence of subcontinua $\{C_i\}_{i=1}^{\infty}$ of M so that C_i is irreducible from Q to $\overline{U_i}$. By the hereditary indecomposability of M, $C_i \subset C_{i+1}$. We consider the following two cases.

Case 1: There is a sequence of integers $\{n_i\}_{i=1}^{\infty}, n_{i+1} > n_i$ and a sequence of distinct points $\{\langle r_A, b_i\rangle\}_{i=1}^{\infty}, \langle r_A, b_i\rangle \in C_{n_i}$ so that some point of H is a limit point of the sequence.

There is a subsequence of $\{b_i\}_{i=1}^{\infty}$ that is either increasing or decreasing in B and has sequential limit b. Without loss of generality, for ease of notation, suppose $n_i = i$ and $\{b_i\}_{i=1}^{\infty}$ is increasing. Then let $\{(u_i, v_i)\}_{i=1}^{\infty}$ be a sequence of disjoint segments in B so that $b_i \in (u_i, v_i)$ and for some $a_i \in A$ with $a_i < \pi_1(Q)$ the continuum M intersects neither $\{u_i\} \times [r_A, a_i]$ nor $\{v_i\} \times [r_A, a_i]$. Then for each i there is a component of $C_i \cap [r_A, a_i] \times [u_i, v_i]$ intersecting $\{r_A\} \times [u_i, v_i]$ and $\{a_i\} \times [u_i, v_i]$. Since A is not first countable at r_A, there is an element $\hat{a} \in A$ so that $r_A < \hat{a} < a_i$ for all i. But then by lemma 3.5, M contains an arc, which is a contradiction.

Case 2. There is a point $\langle r_A, y\rangle$ of H and an open set (u, v) containing y so that no point of $\{r_A\} \times [u, v]$ intersects any C_i. Then for each i there is a point $a_i \in A$ so that $[r_A, a_i) \times (u, v)$ misses C_i. Then there is an $\hat{a} \in A$ so that $\hat{a} < a_i$ for all i. Then $[r_A, \hat{a}) \times (u, v)$ misses C_i for all i. Then $\overline{\cup_{i=1}^{\infty} C_i}$ is a continuum containing P and Q and not $\langle r_A, y\rangle$ and H is a continuum containing P and $\langle r_A, y\rangle$ and not Q; this contradicts the hereditary indecomposablity of M. $\square$

Lemma 3.9. *Suppose $r_A < a < s_A$ and G is the collection of components of $\{a\} \times B - M$. Then G is countable.*

Proof. Suppose not and that for some $r_A < a < s_A$ the collection of components of $\{a\} \times B - M$ is not countable. Then let $G = \{\{a\} \times (u_\alpha, v_\alpha)\}_{\alpha \in I}$ for some uncountable index set I be this uncountable collection of components. Since A is first countable at a there is a countable sequence of segments $\{(w_i, z_i)\}_{i=1}$ with $w_i < w_{i+1} < a < z_{i+1} < z_i$ and $\{a\} = \cap_{i=1}^{\infty}(w_i, z_i)$. For each $\alpha \in I$ let $\langle a, x_\alpha\rangle$ be a point of $\{a\} \times (u_\alpha, v_\alpha)$; for each α there exist (u'_α, v'_α) with $u_\alpha < u'_\alpha < x_\alpha < v'_\alpha < v_\alpha$ and an integer n_α so that $(w_{n_\alpha}, z_{n_\alpha}) \times (u'_\alpha, v'_\alpha)$ does not intersect M.

Since I is uncountable, there is an integer k so that $n_\alpha = k$ for uncountably many α. Then, without loss of generality, we can assume that there is an infinite increasing sequence $\{x_{\alpha_i}\}_{i=1}^{\infty}$ where $n_{\alpha_i} = k$. Then for each i, observe that neither $(w_k, z_k) \times \{u'_{\alpha_i}\}$ nor $(w_k, z_k) \times \{v'_{\alpha_i}\}$ intersect M but since there is a point of $M \cap \{a\} \times B$ in $\{a\} \times (v'_{\alpha_i}, u'_{\alpha_{i+1}})$ it follows that there is a subcontinuum of M intersecting $\{a\} \times (v'_{\alpha_i}, u'_{\alpha_{i+1}})$ that must intersect either $\{w_k\} \times (v'_{\alpha_i}, u'_{\alpha_{i+1}})$ or $\{z_k\} \times (v'_{\alpha_i}, u'_{\alpha_{i+1}})$. In either case, the hypothesis of lemma 3.4 is satisfied and M contains an arc, which is a contradiction. $\square$

Observation regarding Lemma 3.9: Assume the hypothesis and conclusion of the lemma. Suppose that $G = \{(u_i, v_i)\}_{i=1}^{\infty}$ is the collection of components of $(\{a\} \times B) - M$. Then $\mathcal{B} = \{M \cap \{a\} \times [u_i, v_j] | i, j \in \mathbb{N}\}$ is a countable basis for $(\{a\} \times B) \cap M$ and so $(\{a\} \times B) \cap M$ is a metric space.

Lemma 3.10. *Suppose r and s are such that $r_A < r < s < s_A$. Then $[r, s]$ is separable.*

Proof. By lemma 3.9, the collection of components of $\{r\} \times B - M$ is countable. These components will be in the form $\{r\} \times (u, v)$, $\{r\} \times [r_B, v)$ or $\{r\} \times (u, s_B]$. Let $\{r\} \times (u_i, v_i)$ denote those components, where for notational ease we let $(u_{-\infty}, v_i)$ denote $[r_B, v_i)$ and (u_i, v_∞) denote $(u_i, s_B]$ when necessary. Let $x_i \in (u_i, v_i)$, again for notational ease we will use the notation $(x_{-\infty}, x_i)$ to denote $[r_B, x_i)$ and (x_i, x_∞) to denote $(x_i, s_B]$. Then the collection of open basis elements $\{(x_i, x_j)\}_{i,j \in \mathbb{N} \cup \{-\infty, \infty\}}$ is countable. For each pair, $i, j \in \mathbb{N}$, let $\ell_{i,j} = lub\{\pi_1(C) | C$ is a component of $M \cap [r, s] \times B$ that intersects $\{r\} \times (x_i, x_j)\}$.

Suppose that $\{\ell_{i,j}\}_{i,j \in \mathbb{N}}$ is not dense in $[r, s]$ then there exists $r < r' < s' < s$ so that no point of $\{\ell_{i,j}\}_{i,j \in \mathbb{N}}$ lies in (r', s'). Let $r_A < t < r$; then $M \cap [r_A, t] \times B$ and $M \cap [s', s_A] \times B$ are two closed subset of M lying respectively in the two open sets $M \cap [r_A, r) \times B$ and $M \cap (r', s_A] \times B$. So M is the union of three closed sets D, E, F so that:

$$
\begin{aligned}
M \cap [r_A, t] \times B &\subset D, \\
M \cap [s', s_A] \times B &\subset F, \\
D \cap F &= \emptyset, \\
D \cap E &\subset M \cap (r', s_A) \times B - M \cap [s', s_A] \times B \\
&\qquad = M \cap (r', s') \times B, \\
F \cap E &\subset M \cap [r_A, r) \times B - M \cap [r_A, t] \times B \\
&\qquad = M \cap (t, r) \times B.
\end{aligned}
$$

Let C be a component of $M \cap [r, s] \times B$ that intersects $E \cap D$. Then C cannot intersect F (since it contains no points of $E \cap F \subset [r_A, r) \times B$). So C intersects either $D \cap \{r\} \times B$ or $E \cap \{r\} \times B$.

Case 1. C intersects $D \cap \{r\} \times B$ at some point $\langle r, y \rangle$. There are x_i, x_j with $x_i < y < x_j$ so that $\{r\} \times (x_i, x_j) \cap M \subset D$. (In the case $j = \infty$ we would have $x_i < y \leq s_B$ and similarly for $i = -\infty$ we would have $r_B \leq y < x_j$.) So every component C' of $M \cap [r, s] \times B$ intersecting $\{r\} \times (x_i, x_j)$ does not intersect F and so must lie in $D \cup E$. Then for such a C', $lub\{\pi_1(C')) \leq s'$. But $lub\{\pi_1(C)) > r'$ so $r' < \ell_{i,j} < s'$ which contradicts the choice of r' and s'.

Case 2. C intersects $E \cap \{r\} \times B$. Let $\langle r, y\rangle \in C \cap (E \cap \{r\} \times B)$ be such a point. There are i, j so that $x_i < y < x_j$ and $\{r\} \times (x_i, x_j) \cap M \subset E$. (If $j = \infty$ then $x_i < y \leq s_B$ and similarly for $i = -\infty$.) So if C' is a component of $M \cap [r, s] \times B$ intersecting $\{r\} \times (x_i, x_j)$, then C' does not intersect F; so $C' \subset D \cup E$. Furthermore, $lub\{\pi_1(C')) \leq s'$. But $lub\{\pi_1(C)) > r'$ so $r' < \ell_{i,j} < s'$ which is a contradiction to the choice of r' and s'. □

Proof of the main theorem.

Proof. Since by lemma 3.10, for each $r_A < r < s < s_A$, $[r, s]$ is separable and A is first countable at its endpoints it follows that A is separable and therefore metric. Similarly for B. □

Corollary to the proof: If each of A and B is a Hausdorff arc and M is a hereditarily indecomposable subcontinuum of $A \times B$ then each of $\pi_1(M)$ and $\pi_2(M)$ is a metric arc and M sits in a metric disc $\pi_1(M) \times \pi_2(M)$ lying in $A \times B$.

4. Questions

Question 1. *If each of A, B and C is a Hausdorff arc, then is every hereditarily indecomposable subcontinuum of $A \times B \times C$ metric?*

The next step in this area of research would be to determine whether or not the theorem holds for the three (or more) Hausdorff arcs. In her dissertation, Regina Greiwe Jackson, proved that the theorem does indeed hold for the product of three arcs for various species of Lexicographic arcs. [5] In another direction, Smith showed for a large class of Hausdoff arcs that include the compactified long-line, that hereditarily indecomposable subcontinua of finite products of these arcs are metric [8]. Also in [10] he constructed Hausdorff arcs arbitrary products of which do not contain non-degenerate hereditarily indecomposable continua; thus in these cases the generalized theorem is true vacuously.

Question 2. *If X is a Hausdorff continuum that contains no non-degenerate hereditarily indecomposable continua, then is every hereditarily indecomposable continuum of $X \times X$ metric.*

We conjecture that the answer is "no" and that the non-metric perfectly normal hereditarily indecomposable continuum constructed by Greiwe, Smith and Stone [3] is imbeddable in the product of two copies of the 'non-metric solenoid' that was described in [11] [12]. In any case this leads to the following general question.

Question 3. *What are the characteristics of Hausdorff continua with the property that all hereditarily indecomposable subcontinua of products with itself must be metric.*

References

[1] R.H. Bing, *Higher dimensional hereditarily indecomposable continua*, Trans, Amer. Math. Soc. **71** (1951), 267–273.

[2] R. H. Bing, *concerning hereditarily indecomposable continua*, Pacific J. Math., **1** (1951), 43–51.

[3] R. Greiwe, M. Smith and J. Stone, *A nonmetric perfectly normal hereditarily indecomposable continuum*, Topology Appl. **159** (2012), no. 5, 1277–1287.

[4] ———, *Every Non-Metric Indecomposable Subcontinuum of the Square of the Lexicographic Arc Contains an Arc*, Houston Journal of Mathematics, v. **40** (2014), no. 4, 1271–1284.

[5] R. Greiwe, *Properties of non-metric hereditarily indecomposable subcontinua of finite products of lexicographic arcs*, Dissertation, Auburn University, 2009

[6] Krasinkiewicz and Minc, *Mappings onto indecomposable continua*, Bulletin de L'Academie Polonaise des Sciences **25** (1977), 675–680.

[7] Sam Nadler *Continuum Theory*, CRC Press, (1992) 348 pp.

[8] M. Smith, *On Non-Metric Indecomposable and Hereditarily Indecomposable Subcontinua of Products of Long Hausdorff Arcs*, Aportaciones Matematicas, special issue in honor of 60th birthday of David Bellamy (2006).

[9] ———, *Hereditarily Indecomposable Subcontinua of the product of two Souslin arcs are Metric*, Topology Appl. **156** (2009), no. 3, 522–524.

[10] ———, *Hausdorff hypercubes which do not contain arcless continua*, Pro AMS **95** No.1 (1985), 109–114.

[11] M. Smith, J. Stone, *On Non-Metric Continua that Support Whitney Maps*, Topology Appl. **170** (2014), 63–85.

[12] Jennifer W. Stone, *Non-Metric Continua that Supports Whitney Maps*, Dissertation, Auburn University, May 2007.

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