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Daron Anderson

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INDECOMPOSABLE CONTINUUM WITH A STRONG NON-CUT POINT

DARON ANDERSON

Abstract. We construct an indecomposable continuum with exactly one strong non-cut point. The method is an adaptation of Bellamy [1]. We start with an ω_1-chain of indecomposable metric continua and retractions. The inverse limit is an indecomposable continuum with exactly two composants. Our example is formed by identifying a point in each composant.

1. Introduction

Every point p of an indecomposable metric continuum M is a weak cut point. That means there are distinct $x, y \in M - p$ such that each subcontinuum $K \subset M$ with $\{x, y\} \subset K$ has $p \in K$. The proof follows from M having more than one composant; and the composant-by-composant version of the result fails. Namely some $q \in M$ might fail to weakly cut its composant $\kappa(q)$. In that case we call q a *strong non-cut point* of $\kappa(q)$. For example consider the endpoint c of the Knaster buckethandle. It is easy to see $\kappa(c) - c$ is even arcwise connected. Hence c has only trivial reasons for being a weak cut point.

There exist indecomposable non-metric continua with exactly one composant − henceforth called *Bellamy continua*. Each Bellamy continuum is simultaneously an indecomposable continuum and a composant of an

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indecomposable continuum. Bellamy continua resemble indecomposable metric continua in being compact. In this paper we show they resemble composants of indecomposable metric continua in that they can have strong non-cut points.

2. Terminology and Notation

Throughout X is a continuum. That means a nondegenerate compact connected Hausdorff space. For $a, b \in X$ we say X is *irreducible about* $\{a, b\}$ or *irreducible from* a *to* b to mean no proper subcontinuum of X contains $\{a, b\}$. The subspace $A \subset X$ is called a *semicontinuum* to mean for each $a, c \in A$ some subcontinuum $K \subset A$ has $\{a, c\} \subset K$. Every subspace $A \subset X$ is partitioned into maximal semicontinua called the *continuum components* of A. For background on metric continua see [5] and [7]. The results cited here have analagous proofs for non-metric continua.

For a subset $S \subset X$ denote by S° and $\overline{S}$ the interior and closure of S respectively. By *boundary bumping* we mean the principle that, for each closed $E \subset X$, each component C of E meets $\partial E = \overline{E} \cap \overline{X - E}$. For the non-metric proof see §47, III Theorem 2 of [5]. One corollary of boundary bumping is that the point $p \in X$ is in the closure of each continuum component of $X - p$.

For $b \in X$ we omit the curly braces and write $X - b$ instead of $X - \{b\}$. For distinct $a, b, c \in X$ we say b *weakly cuts* a from c and write $[a, b, c]_X$ to mean a and c have different continuum components in $X - b$. When there is no confusion we just write $[a, b, c]$. We say $b \in X$ *weakly cuts* the semicontinuum $A \subset X$ to mean $[a, b, c]$ for some $a, c \in A$ and call b a *weak cut point* to mean it weakly cuts X and a *strong non-cut point* otherwise.

We define the *interval* $[a, c]_X = \big\{b \in X : [a, b, c]_X\big\}$. Again we often write $[a, c]$ without confusion. Note $[a, c]$ is not in general connected as the interval notation suggests. In case $[a, c]$ is connected and $b \in [a, c]$ we have $[a, b] \cup [b, c] = [a, c]$. Moreover $[a, b, c]_{[a,c]}$ for each $b \in [a, c] - \{a, c\}$. Clearly the weak cut structure is topologically invariant. That means $[a, b, c]_X \iff [h(a), h(b), h(c)]_Y$ for each $a, b, c \in X$ and homeomorphism $h : X \to Y$.

We say X is *indecomposable* to mean it is not the union of two proper subcontinua. Equivalently each proper subcontinuum is nowhere dense. The *composant* $\kappa(x)$ of the point $x \in X$ is the union of all proper subcontinua that have x as an element. Indecomposable metric continua are partitioned into $\mathfrak{c}$ many pairwise disjoint composants [6]. In case $\kappa(x) \neq \kappa(y)$ then X is irreducible about $\{x, y\}$. There exist indecomposable non-metric continua [1, 2, 9] with exactly one composant, henceforth called *Bellamy continua.*

We call X *hereditarily unicoherent* to mean it has some $-$ and therefore all $-$ of the equivalent properties:

(I) The intersection of any two subcontinua of X is connected.
(II) $[a, b, c]_X \iff [a, b, c]_L$ for each subcontinuum $L \subset X$ with $a, b, c \in L$.
(III) $[a, c]_X = [a, c]_L$ for each subcontinuum $L \subset X$ with $a, c \in L$.
(IV) Whenever $a, c \in X$ the set $[a, c]$ is a subcontinuum.

The continuous function $f : X \to Y$ is called *monotone* to mean each $f^{-1}(y) \subset X$ is connected for $y \in Y$. Theorem 6.1.28 of [4] says this implies $f^{-1}(K) \subset X$ is a continuum for $K \subset Y$ a continuum. The function f is called *proper* to mean $f(L) \subset Y$ is proper whenever $L \subset X$ is a proper subcontinuum.

The partition $\mathcal{P}$ of X into closed subsets is called *upper semicontinuous* to mean the following: For each $P \in \mathcal{P}$ and open $U \subset X$ containing P there is open $V \subset U$ with $P \subset V$ and V a union of elements of $\mathcal{P}$. Upper semicontinuity of the partition is equivalent to the quotient space $X/\mathcal{P}$ being a continuum.

Throughout $K \subset \mathbb{R}^2$ is the *quinary Cantor set*. That means the points in $[0, 1] \times \{0\}$ whose x-coordinate can be expressed in base-5 without the digits 1 or 3. We write K_1 for the middle third of K; K_2 for the left-most two thirds of the portion of K right of K_1 ; K_3 for the leftmost two thirds of the portion of K right of K_2 and so on. Formally $K_1 = K \cap [2/5, 3/5]$, $K_2 = K \cap [4/5, 1 - 2/25]$ and $K_n = K \cap [1 - 1/5^n, 1 - 2/5^{n+1}]$ for each $n > 1$. Let each $k(n) = 1 - 2/5^n$ be the right endpoint of K_n. Let $G : K \to K$ be the unique linear order-reversing isometry and define each $P_n = G(K_{n+1})$ and $p(n) = G(k(n + 1))$. Clearly $K = \{0\} \cup \left(\bigcup_n P_n\right) \cup \left(\bigcup_n K_n\right) \cup \{1\}$ is a disjoint union.

We write $Q' \subset \mathbb{R}^2$ for the union of all semicircles in the upper half-plane with centre $(1 - 7/10^n, 0)$ for some $n \in \mathbb{N}$ and endpoints in K. We write $R' \subset \mathbb{R}^2$ for the union of all semicircles in the lower half-plane with centre $(7/10^n, 0)$ for some $n \in \mathbb{N}$ and endpoints in K. We write Q for the set $\{(x, y + 1) \in \mathbb{R}^2 : (x, y) \in Q'\}$ and R for the set $\{(x, y - 1) \in \mathbb{R}^2 : (x, y) \in R'\}$.

Throughout $B = Q' \cup R'$ is the *quinary Knaster buckethandle*. B is a metric indecomposable hereditarily unicoherent continuum. It is easy to see $[x, y]$ is an arc whenever x, y share a composant and $[x, y] = B$ otherwise. Let a_0 be the point $(3/10, 3/10) \in B$. We write C_0 for the composant of the left endpoint $c_0 = (0, 0)$ and D_0 for the composant of the right endpoint $d_0 = (1, 0)$.

Throughout $\omega = \{0, 1, 2, \dots\}$ is the first infinite ordinal and ω_1 the first uncountable ordinal. Every initial segment of ω_1 is countable and every countable subset has an upper bound. For the ordered set Ω we say $\Psi \subset \Omega$ is *cofinal* to mean it has no upper bound in Ω. Every countable ordinal has a cofinal subset order-isomorphic to ω. We say $\Psi \subset \Omega$ is *terminal* to mean $\Omega - \Psi$ has an upper bound.

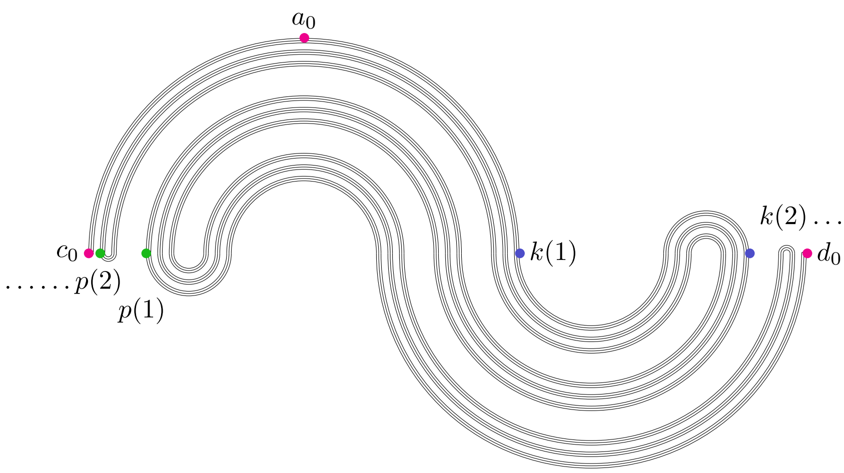

FIGURE 1. The *quinary Knaster buckethandle*

The poset Ω is said to be *directed* to mean for each $\gamma, \beta \in \Omega$ there is $\alpha \in \Omega$ with $\gamma, \beta \le \alpha$. Note most authors require $\gamma, \beta < \alpha$. This prohibits maximal elements. In this paper it is convenient to allow a directed set to have maximal elements.

An *inverse system* over the directed set Ω consists of the following data: (1) a family of topological spaces $T(\alpha)$ for each $\alpha \in \Omega$ and (2) a family of continuous maps $f^\alpha_\beta : T(\alpha) \to T(\beta)$ for each $\beta \le \alpha$ such that (3) we have $f^\beta_\gamma \circ f^\alpha_\beta = f^\alpha_\gamma$ whenever $\gamma \le \beta \le \alpha$. The property (3) is called *commutativity of the diagram*. The *inverse limit* T of the system is the space

$$\varprojlim\{T(\alpha); f^\alpha_\beta : \alpha, \beta \in \Omega\} = \Big\{(x_\alpha) \in \prod_{\alpha \in \Omega} T(\alpha) : f^\alpha_\beta(x_\alpha) = x_\beta \ \forall\, \beta \le \alpha\Big\}.$$

We often suppress the index set and write for example $\varprojlim\{T(\alpha); f^\alpha_\beta\}$. The functions f^α_β are called the *bonding maps*. Write $\pi_\beta : T \to T(\beta)$ for the restriction of the projection $\prod_\alpha T(\alpha) \to T(\beta)$.

The inverse limit X of a system $\{X(\alpha); f^\alpha_\beta\}$ of continua is itself a continuum. If moreover each bonding map is surjective then so is each π_β. In that case we call the inverse system (limit) *surjective.*

For any subcontinuum $L \subset X$ we have $L = \varprojlim\{\pi_\alpha(L); f^\alpha_\beta\}$ where each f^α_β is restricted to $\pi_\alpha(L)$. Note commutativity implies f^α_β has range $\pi_\beta(L)$ hence the subsystem is well defined. For cofinal $\Psi \subset \Omega$ the map $(x_\alpha)_{\alpha\in\Omega} \mapsto (x_\alpha)_{\alpha\in\Psi}$ is an homeomorphism between X and the inverse limit $\varprojlim\{X(\alpha); f^\alpha_\beta : \alpha, \beta \in \Psi\}$ over Ψ.

3. The Successor Stage

We use transfinite recursion to construct the eponymous indecomposable continuum as the inverse limit of a system $\{X(\alpha); f^\alpha_\beta : \alpha, \beta < \omega_1\}$ of metric continua and retractions. This section shows how to construct each $X(\beta + 1)$ from $X(\beta)$. The following section deals with limit ordinals.

To begin let $X(0) = B$ be the quinary buckethandle. Let the composants $C_0, D_0 \subset B$ and points $a_0, c_0 \in C_0$ be as described in the Introduction. Define the following two sequences (p^n_0) and (q^n_0) in $X(0)$: Choose an homeomorphism $[0, 1] \mapsto [a_0, c_0]$ with $0 \mapsto a_0$ and $1 \mapsto c_0$ and let each p^n_0 be the image of $1 - 1/n$. Let each q^n_0 be the point $k(n) \in B$ as defined in the Introduction. Observe the pair of sequences $\big((p^n_0), (q^n_0)\big)$ satisfies the following definition.

Definition 1. For $a \in X$ and $c \in \kappa(a)$ we define a *tail from a to c* as an ordered pair $T = \big((p^n), (q^n)\big)$ of sequences in $\kappa(a)$ with the properties:

(1) For each $n \in \mathbb{N}$ we have $a \in [p^n, q^n]$ and $a \in [c, q^n]$.
(2) For each $n \in \mathbb{N}$ we have $q^n \notin [c, a]$.
(3) $[p^1, a] \subsetneq [p^2, a] \subsetneq \dots$.
(4) $[a, q^1] \subsetneq [a, q^2] \subsetneq \dots$.
(5) $\bigcup\big\{[p^n, q^n] : n \in \mathbb{N}\big\} = \kappa(a) - c$.
(6) $\bigcup\big\{[p^n, a] : n \in \mathbb{N}\big\} = [c, a] - c$.
(7) For each $n \in \mathbb{N}$ and $x \in X - [c, a]$ we have either $[c, q^n] \subset [c, x]$ or $[c, x] \subset [c, q^n]$.

The notion of a tail is pivotal to our example. Indeed as part of the construction we will at stage $\alpha < \omega_1$ choose a tail $T^\alpha = \big((p^n_\alpha), (q^n_\alpha)\big)$ on $X(\alpha)$ so that the tails behave nicely with respect to the bonding maps. This is made precise below.

In the next definition and throughout when we write for example $a_\beta \mapsto a_\gamma$ the map in question is understood to be the bonding map f^β_γ. Similarily for subsets $B \subset X(\beta)$ and $C \subset X(\gamma)$ we write $B \to C$ to mean $f^\beta_\gamma(B) = C$.

Definition 2. Suppose $\{X(\beta); f_\gamma^\beta : \gamma, \beta < \alpha\}$ is an inverse system and each $X(\beta)$ has a distinguished pair of points (a_β, c_β) and pair of sequences $\big((p_\beta^n), (q_\beta^n)\big)$. We say the system is *coherent* to mean $a_\beta \mapsto a_\gamma$ and $c_\beta \mapsto c_\gamma$ and each $p_\beta^n \mapsto p_\gamma^n$, $q_\beta^n \mapsto q_\gamma^n$, $[p_\beta^n, q_\beta^n] \to [p_\gamma^n, q_\gamma^n]$, $[p_\beta^n, a_\beta] \to [p_\gamma^n, a_\gamma]$ and $[c_\beta, a_\beta] \to [c_\gamma, a_\gamma]$ for $\gamma, \beta < \alpha$.

In practice the distinguished points and sequences in Definition 2 will always come about from a tail. However in Section 4 we already have an inverse system, and most of the work goes into showing a given pair of sequences is indeed a tail. Thus the definition is given in slightly more generality.

At stage $\alpha < \omega_1$ we have already constructed the coherent inverse system $\{X(\beta); f_\gamma^\beta : \beta, \gamma < \alpha\}$ of indecomposable hereditarily unicoherent metric continua and retractions. We assume the following objects have been specified for each $\beta < \alpha$:

(i) Distinct composants $C(\beta), D(\beta) \subset X(\beta)$
(ii) Points $a_\beta, c_\beta \in C(\beta)$
(iii) A tail $T^\beta = \big((p_\beta^n), (q_\beta^n)\big)$ from a_β to c_β

We also assume for each $\gamma, \delta < \alpha$ the three conditions hold:

(a) $\bigcup \big\{X(\delta) : \delta < \gamma\big\} \subset D(\gamma)$.
(b) $\bigcup \big\{f_\delta^\gamma\big(X(\gamma) - X(\delta)\big) : \gamma > \delta\big\} = C(\delta)$.
(c) $\big\{x \in [c_\gamma, a_\gamma] : f_\delta^\gamma(x) \in [p_\delta^n, a_\delta]\big\} = [p_\gamma^n, a_\gamma]$ for each $n \in \mathbb{N}$.

Conditions (a) and (b) come straight from [1] and will ensure the limit has exactly two composants. Condition (c) is needed to make the resulting point (c_β) not weakly cut the composant.

For an illustration of what Condition (c) means consider the system of retractions $[0, 1] \leftarrow [0, 2] \leftarrow [0, 3] \leftarrow \ldots$ where the bonding map $[0, n] \to [0, m]$ collapses $[m, n]$ to the point $m \in [0, m]$. For $X(\beta) = [0, \beta]$ and $a_\beta = \beta$ and each $p_\beta^n = 1/n$ we see the system has Condition (c). Indeed our initial $[c_0, a_0] \leftarrow [c_1, a_1] \leftarrow [c_2, a_2] \leftarrow \ldots$ will turn out to be a copy of this simpler system, and so the example should be kept in mind throughout.

We are ready to begin the succesor step. Suppose $\alpha = \beta+1$ is a successor ordinal. We will construct an indecomposable hereditarily unicoherent metric continuum $X(\beta+1)$ and retraction $f_\beta^{\beta+1} : X(\beta+1) \to X(\beta)$. Then we can define the bonding maps $f_\gamma^{\beta+1} = f_\beta^{\beta+1} \circ f_\gamma^\beta$. We will specify the objects (i), (ii) and (iii) when β is replaced by $\beta+1$. Finally we will check the enlarged system is coherent and Condition (a), (b) and (c) hold for all $\gamma, \delta \le \beta + 1$.

We use terminology from the Introduction. Identify $Q \cup R \subset \mathbb{R}^2$ with the subspace $(Q \cup R) \times \{a_\beta\}$ of $\mathbb{R}^2 \times X(\beta)$. Define $M \subset \mathbb{R}^2 \times X(\beta)$ as follows.

$$\widetilde{P}_n = P_n \times \{-1,1\} \times [p^n_\beta, a_\beta] \qquad\qquad \widetilde{K}_n = K_n \times \{-1,1\} \times [a_\beta, q^n_\beta]$$
$$M = \left(\bigcup_{n\in\mathbb{N}} \widetilde{P}_n\right) \cup \left(\bigcup_{n\in\mathbb{N}} \widetilde{K}_n\right)$$

Properties (4) and (5) of the tail imply $\overline{M} = \widetilde{P}_\infty \cup M \cup \widetilde{K}_\infty$ for

$$\widetilde{P}_\infty = \{0\} \times \{-1,1\} \times [c_\beta, a_\beta] \qquad \widetilde{K}_\infty = \{1\} \times \{-1,1\} \times X(\beta)$$

The set $\overline{M} \cup (Q \cup R)$ is compact since $(Q \cup R)$ is closed and bounded and $\overline{M}$ is contained in the product $K \times \{-1,1\} \times X(\beta)$ of compact sets. It is also metric since $\mathbb{R}^2$, K, $\{-1,1\}$ and by assumption $X(\beta)$ are metric spaces.

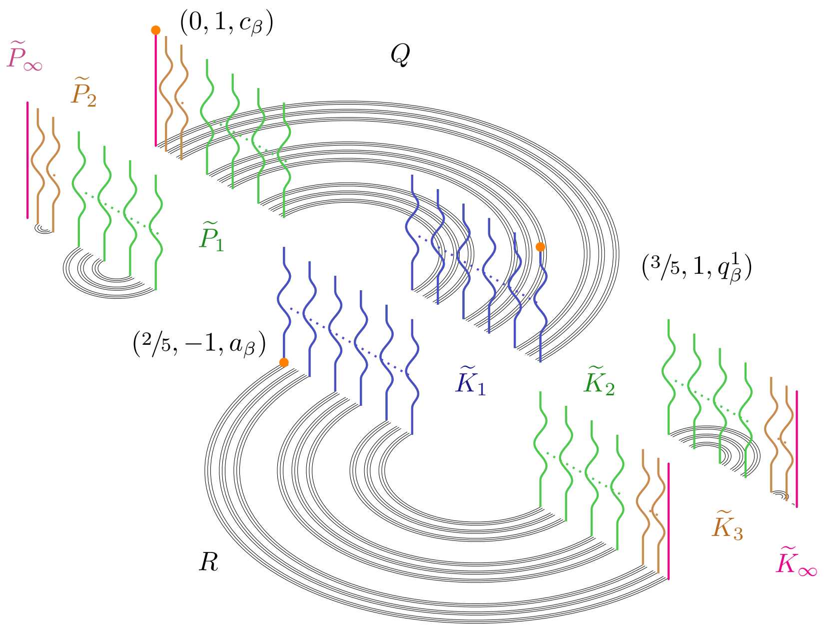

FIGURE 2. Schematic for $\overline{M}$. For example the set $\widetilde{K}_1$ is a family of copies of $[a_\beta, q^1_\beta]$ attached to either half of the bucket handle at the endpoint a_β with the endpoints q^1_β free.

To obtain $X(\beta+1)$ first make for each $n \in \mathbb{N}$ and $k \in \bigcup P_n$ the identification $(k,-1,p^n_\beta) \sim (k,1,p^n_\beta)$ and for each $k \in \bigcup K_n$ make the identification $(k,-1,q^n_\beta) \sim (k,1,q^n_\beta)$. Then for each $x \in [c_\beta, a_\beta]$ make the identification $(0,-1,x) \sim (0,1,x)$ and for each $x \in X(\beta)$ make the identification $(1,-1,x) \sim (1,1,x)$. It is straightforward to verify the decomposition is upper semicontinuous. Hence $X(\beta+1)$ is a continuum and [7] Lemma 3.2 says $X(\beta+1)$ is metric.

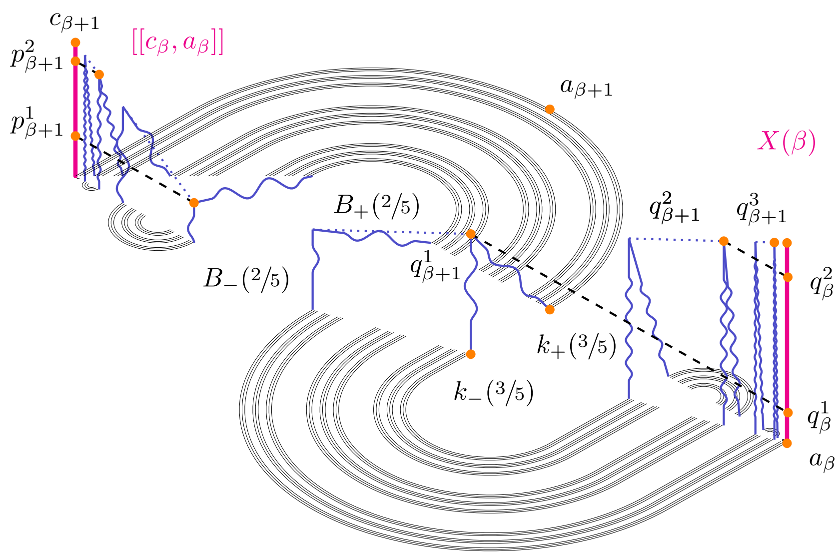

FIGURE 3. The continuum $X(\beta+1)$ is obtained by identifying the free endpoints of opposite pairs of subcontinua, and by identifying the two copies of $[c_\beta, a_\beta]$ and $X(\beta)$ that make up $\widetilde{P}_\infty$ and $\widetilde{K}_\infty$ respectively. Dashed lines indicate projection onto $X(\beta)$ or $[[c_\beta, a_\beta]]$.

For $k \in K$ write $B(k)$ for the quotient space of $\{(x, y, z) \in M : x = k\}$. For $k \notin \{0, 1\}$ clearly $B(k)$ is homeomorphic to two copies of some $[p^n_\beta, a_\beta]$ or $[a_\beta, q^n_\beta]$ joined at the points corresponding to p^n_β or q^n_β respectively. Hence $B(k)$ is a continuum irreducible from $(k, -1, a_\beta)$ to $(k, 1, a_\beta)$. $B(1)$ is a copy of $X(\beta)$. Henceforth identify that copy with $X(\beta)$. $B(0)$ is a copy of $[c_\beta, a_\beta]$. Denote that copy by $[[c_\beta, a_\beta]]$ and write $\widetilde{x} \in [[c_\beta, a_\beta]]$ for the point corresponding to $x \in [c_\beta, a_\beta]$.

Define a surjection $g : X(\beta + 1) \to B$ onto the buckethandle.

$$g(x, y, z) = \begin{cases} (x, y - 1) & \text{for } x \in Q \\ (x, y + 1) & \text{for } x \in R \\ (k, 0) & \text{for } x \in B(k) \\ (1, 0) & \text{for } x \in X(\beta) \\ (0, 0) & \text{for } x \in [[c_\beta, a_\beta]] \end{cases}$$

For each $x \in B$ the fibre $g^{-1}(x)$ is either a singleton, $X(\beta)$, $[c_\beta, a_\beta]$ or some $B(k)$. Thus all fibres are subcontinua and g is monotone. Therefore subcontinua pull back to subcontinua. In particular $g^{-1}(B) = X(\beta+1)$ is a continuum. Define the points $c_{\beta+1} = \widetilde{c_\beta}$ and $a_{\beta+1} = g^{-1}(a_0)$ and each $p^n_{\beta+1} = \widetilde{p^n_\beta}$. The definition of $q^n_{\beta+1}$ will be given later in the construction.

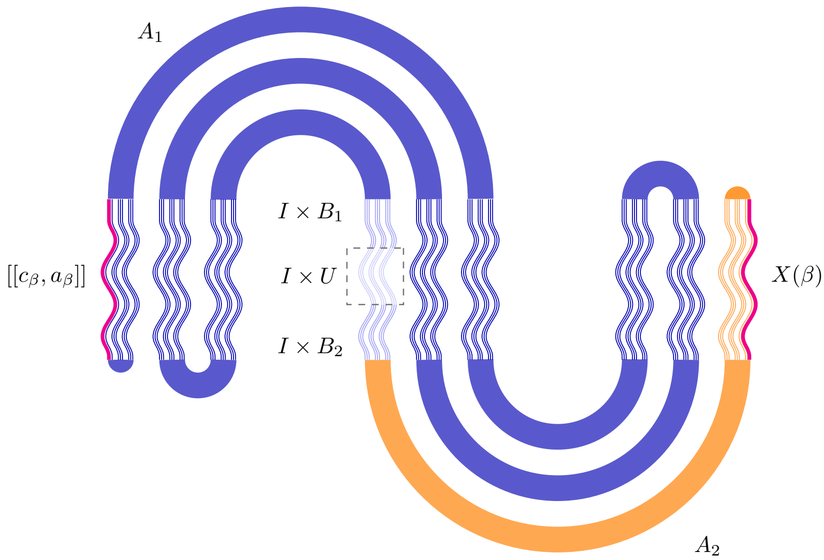

FIGURE 4. Schematic of Claim 1 for I an interval of $K(2)$

Claim 1. The function $g : X(\beta+1) \to B$ is proper.

Proof. Identify the plane $\mathbb{R}^2$ with the subspace $\mathbb{R}^2 \times \{a_\beta\}$ of $\mathbb{R}^2 \times X(\beta)$. For each $n \in \mathbb{N}$ let $K(n)$ be the nth stage of construction of the quinary Cantor set. Let $Q(n) \subset \mathbb{R}^2$ be the union of all upper semicircles with centre $(1 - 7/10^m, 1)$ for some $m \in \mathbb{N}$ and endpoints in $K(n) \times \{1\}$. Let $R(n) \subset \mathbb{R}^2$ be the union of all lower semicircles with centre $(7/10^m, -1)$ for some $m \in \mathbb{N}$ and endpoints in $K(n) \times \{-1\}$. Define the continuum $Y(n) = Q(n) \cup R(n) \cup X(\beta+1)$.

Recall $K(n)$ consists of 3^n intervals of length $1/5^n$. For any such interval I with $0, 1 \notin I$ it is a straightforward exercise to verify the closure of $Y(n) - \bigcup\{B(k) : k \in I\}$ has exactly two components A_1 containing $I \times \{1\} \times \{a_\beta\}$ and A_2 containing $I \times \{-1\} \times \{a_\beta\}$.

Suppose the subcontinuum $L \subset X(\beta+1)$ has $g(L) = B$. Clearly L contains the interior of $Q \cup R$. Now assume L is proper. Then $X(\beta+1) - L$ contains a basic open subset of the quotient space of some $P_m \times \{\pm 1\} \times [p^m_\beta, a_\beta]$ or $K_m \times \{\pm 1\} \times [a_\beta, q^m_\beta]$. Without loss of generality that set is $V = I \times \{1\} \times U$ for some open $I \subset K_m$ and $U \subset [a_\beta, q^m_\beta]$. We can shrink I further to make it an interval of some $K(n)$ and shrink U to get $a_\beta \notin U$.

The subcontinua $B(k)$ are homeomorphic for $k \in I$ and $\bigcup\{B(k) : k \in I\}$ is just $I \times B(k)$. Treat U as a subset of $B(k)$. Since $B(k)$ is irreducible from $(k, 1, a_\beta)$ to $(k, -1, a_\beta)$ we know $B(k) - U$ is the disjoint union $B_1 \cup B_2$ of two clopen sets that include $(k, 1, a_\beta)$ and $(k, -1, a_\beta)$ respectivly. Hence

$$\bigcup\{B(k) : k \in I\} - V = (I \times B_1) \cup (I \times B_2)$$

is the disjoint union of two clopen sets containing $I \times \{1\} \times \{a_\beta\}$ and $I \times \{-1\} \times \{a_\beta\}$ respectively. It follows that

$$Y(m) - V = \big((I \times B_1) \cup A_1\big) \cup \big((I \times B_2) \cup A_2\big)$$

is the disjoint union of two clopen sets containing $I \times \{1\} \times \{a_\beta\}$ and $I \times \{-1\} \times \{a_\beta\}$ respectively.

Recall L is contained $X(\beta + 1) - V \subset Y(m) - V$. Since the closed set L contains the interior of $Q \cup R$ it meets both clopen sets which contradicts how L is connected. We conclude $g(L) = B$ if and only if $L = X(\beta + 1)$. □

Claim 2. The composants of $X(\beta + 1)$ are $\{g^{-1}(E) : E \subset B$ is a composant$\}$. Hence $X(\beta + 1)$ is indecomposable.

Proof. Since g is monotone and proper each $g^{-1}(E)$ is contained in some composant of $X(\beta + 1)$. Now suppose the subcontinuum $L \subset X(\beta + 1)$ meets two distinct $g^{-1}(E_1)$ and $g^{-1}(E_2)$. It follows $g(L) = B$. Since g is proper we must have $L = X(\beta + 1)$. The result follows. □

Claim 2 tells us $g^{-1}(C_0)$ and $g^{-1}(D_0)$ are distinct composants of $X(\beta + 1)$. Define $C(\beta + 1) = g^{-1}(C_0)$ and $D(\beta + 1) = g^{-1}(D_0)$. From the construction $X(\beta) \subset D(\beta + 1)$. The next claim follows.

Claim 3. Condition (a) holds for all $\gamma, \delta \leq \beta + 1$.

Observe each subcontinuum of B is contained in some union $J_1 \cup J_2 \cup \ldots \cup J_N$ of arcs where $J_i = [a_i, b_i]$ are alternately contained in Q' or R' and $J_i \cap J_j = \{b_n\} \iff \{i, j\} = \{n, n + 1\}$ and $J_i \cap J_j = \varnothing$ otherwise. Let each J_i be identified with the corresponding arc in Q or R.

The monotone surjection g witnesses how each subcontinuum $L \subset X(\beta + 1)$ is contained in some

$$(\dagger) \qquad J_1 \cup B(x^1) \cup J_2 \cup B(x^2) \cup \ldots \cup J_N \cup B(x^{N+1}) \cup J_{N+1}$$

for each $J_n \cap B(x^n) = (x^n, \pm 1, a_\beta)$ and $B(x^n) \cap J_{n+1} = (x^n, \mp 1, a_\beta)$ and all other intersections are empty.

Observe each summand of (†) is hereditarily unicoherent and irreducible between two endpoints − and meets the previous factor at exactly one endpoint and the next factor at the other endpoint. It follows by induction the union is hereditarily unicoherent.

Claim 4. $X(\beta+1)$ is hereditarily unicoherent.

Proof. Suppose the proper subcontinua $L_1, L_2 \subset X(\beta+1)$ have nonempty intersection. Since $X(\beta+1)$ is indecomposable $L_1 \cup L_2 \neq X$. Hence $L_1 \cup L_2$ is contained in a union of the form (†). Since the union is an hereditarily unicoherent subcontinuum $L_1 \cap L_2$ is connected. □

Let $\{y^0, y^1, y^2 \ldots\} \subset B$ be the set $C_0 \cap K$ linearly ordered by $x \leq y \iff [c, x] \subset [c, y]$. We must have $y^0 = c_0$. For each $n \in \mathbb{N}$ write $y^n_\pm = (y^n, \pm 1, a_\beta)$. Note for $n = 0$ we have $y^n_+ = y^n_-$ as elements of $X(\beta+1)$. For each $n > 0$ write $B(n)$ instead of $B(y^n)$. Write $I(n)$ for the arc in Q or R corresponding to the arc $[y^n, y^{n+1}] \subset B$. By (†) each subcontinuum of $C(\beta+1)$ is contained in some

$$(\dagger\dagger) \qquad [[c_\beta, a_\beta]] \cup I(0) \cup B(1) \cup \ldots \cup I(N-1) \cup B(N) \cup I(N).$$

For example let $x \in B(n)$ and $y \in I(m)$ be arbitrary with $n < m$. It follows from hereditary unicoherence that

$$(\ddagger) \qquad [x, y] = [x, y^n_\pm]_{B(n)} \cup I(n) \cup B(n+1) \cup \ldots \cup B(m) \cup [y^m_\pm, y]_{I(m)}$$

for exactly one of the four choices of $\pm$ indices. The other intervals of $[x, y]$ have a similar form based on whether each endpoint is in some $B(n)$ or $I(m)$.

For each $n \in \mathbb{N}$ write $k_\pm(n) = \big(k(n), \pm 1, a_\beta\big)$. By definition $k(n)$ is the right endpoint of K_n. Thus we see $B\big(k(n)\big)$ is the quotient space of the set $\{k(n)\} \times \{-1, 1\} \times [a_\beta, q^n]$. Write $B_\pm\big(k(n)\big)$ for the quotient space of the set $\{k(n)\} \times \{\pm 1\} \times [a_\beta, q^n]$. Then the continua $B_+\big(k(n)\big)$ and $B_-\big(k(n)\big)$ meet at the point $\big(k(n), -1, q^n\big) \sim \big(k(n), 1, q^n\big)$. Define each $q^n_{\beta+1} \in X(\beta+1)$ as that point. The expression (‡) gives the equalities.

$$[p^n_{\beta+1}, q^n_{\beta+1}] = [[p^n_\beta, a_\beta]] \cup I(0) \cup B(1) \cup \ldots \cup I(k(n)-1) \cup B_+(k(n))$$

$$[a_{\beta+1}, q^n_{\beta+1}] = [a_{\beta+1}, k_+(1)] \cup B(1) \cup \ldots \cup I(k(n)-1) \cup B_+(k(n))$$

$$[c_{\beta+1}, q^n_{\beta+1}] = [[c_\beta, a_\beta]] \cup I(0) \cup B(1) \cup \ldots \cup I(k(n)-1) \cup B_+(k(n))$$

$$[p^n_{\beta+1}, a_{\beta+1}] = [[p^n_\beta, a_\beta]] \cup [\widetilde{a}_\beta, a_{\beta+1}]$$

$$[c_{\beta+1}, a_{\beta+1}] = [[c_\beta, a_\beta]] \cup [\widetilde{a}_\beta, a_{\beta+1}]$$

Claim 5. $T^{\beta+1} = \big((p^n_{\beta+1}), (q^n_{\beta+1})\big)$ is a tail from $a_{\beta+1}$ to $c_{\beta+1}$.

Proof. Properties $(1)-(4)$ follow from the equalities above. Property (6) follows from the equalities, the definition of $[[c_\beta, a_\beta]]$ as a copy of $[c_\beta, a_\beta]$, and how T^β has Property (6).

For Property (5) first observe each $c_{\beta+1} \notin [p^n_{\beta+1}, q^n_{\beta+1}]$. Then recall $\kappa(a_{\beta+1}) = C(\beta+1) = g^{-1}(C_0)$. First suppose $x \in B(0) = [[c_\beta, a_\beta]] \subset [c_{\beta+1}, a_{\beta+1}]$. Then Property (6) implies x is some $[p^N_{\beta+1}, a_{\beta+1}] \subset [p^N_{\beta+1}, q^N_{\beta+1}]$.

Otherwise (‡‡) says x is an element of some $B(n)$ or $I(n)$. It is easy to verify the set $\{k(n) : n \in \mathbb{N}\}$ is cofinal in $\{y^0, y^1, y^2, \ldots\}$. Thus $y^n + 1 < k(N)$ for some $N \in \mathbb{N}$. Then the above equalities say $x \in [p^N_{\beta+1}, q^N_{\beta+1}]$.

To prove Property (7) for $T^{\beta+1}$ observe each $[c_{\beta+1}, x]$ has one of the forms

$$[[c_\beta, a_\beta]] \cup I(0) \cup B(1) \cup \ldots \cup I(N-1) \cup B(N) \cup [y^N_\pm, x]_{I(N)}$$

$$[[c_\beta, a_\beta]] \cup I(0) \cup B(1) \cup \ldots \cup B(N-1) \cup I(N-1) \cup [y^N_\pm, x]_{B(N)}$$

depending on whether $x \in I(N)$ or $B(N)$ for some $N \in \mathbb{N}$. For $k(n) < N$ the expression for $[c_\beta, q^n_{\beta+1}]$ says $[c_\beta, q^n_{\beta+1}] \subset [c_\beta, x]$. Likewise $[c_\beta, x] \subset [c_\beta, q^n_{\beta+1}]$ for $N < k(n)$.

Finally assume $N = k(n)$. For $x \in I(N)$ compare the two expressions to see $[c_\beta, x] \subset [c_\beta, q^n_{\beta+1}]$. For $x \in B(N)$ we need only compare the final two summands $[y^N_\pm, x]_{B(N)}$ and $B_+\big(k(n)\big)$. Observe both summands are sub-continua of $B\big(k(n)\big)$ and include the point $k_+(n)$. For $x \in B_+\big(k(n)\big)$ clearly $[y^N_\pm, x]_{B(N)} \subset B_+\big(k(n)\big)$ and so $[c_\beta, x] \subset [c_\beta, q^n_{\beta+1}]$.

Otherwise $x \in B_-\big(k(n)\big)$. Recall that $B_+\big(k(n)\big) = [k_+(n), q^n_{\beta+1}]$ and $B_-\big(k(n)\big) = [q^n_{\beta+1}, k_-(n)]$ and $B_+\big(k(n)\big) \cap B_-\big(k(n)\big) = \{q^n_{\beta+1}\}$. It follows $[y^N_\pm, x]_{B(N)}$ includes $q^n_{\beta+1}$ hence contains $B_+\big(k(n)\big)$. We conclude that $[c_\beta, q^n_{\beta+1}] \subset [c_\beta, x]$. □

Recall we built $X(\beta+1)$ from the subset $\overline{M} \cup (Q \cup R)$ of $\mathbb{R}^2 \times X(\beta)$ by making some identifications. Since the projection $\mathbb{R}^2 \times X(\beta) \to X(\beta)$ onto the third coordinate respects those identifications, it induces a continuous map $f^{\beta+1}_\beta : X(\beta+1) \to X(\beta)$.

Claim 6. The map $f^{\beta+1}_\beta : X(\beta+1) \to X(\beta)$ is a retraction.

Proof. Write $\pi : \mathbb{R}^2 \times X(\beta) \to X(\beta)$ for the projection onto the third coordinate. Recall the subset $M_1 = \big\{(x, y, z) \in M : x = 1\big\}$ is the disjoint union $\{1\} \times \{-1, 1\} \times X(\beta)$ of two copies of $X(\beta)$. In forming $X(\beta+1)$ from $\overline{M} \cup (Q \cup R)$ we identify $(1, -1, x) \sim (1, 1, x)$ for each $x \in X(\beta)$. Hence the restriction of $f^{\beta+1}_\beta$ to the quotient space of M_1 is an homeomorphism onto $X(\beta)$. But recall we have identified $X(\beta)$ with the quotient space of M_1. □

Claim 7. Condition (b) holds for all $\gamma, \delta \leq \beta + 1$.

Proof. Recall we have defined each $f_\gamma^{\beta+1} = f_\beta^{\beta+1} \circ f_\gamma^\beta$. Hence by induction and commutativity of the diagram it is enough to show $f_\beta^{\beta+1}\big(X(\beta+1) - X(\beta)\big) = C(\beta)$. To that end recall we have

$$X(\beta+1) = Q \cup R \cup [[c_\beta, a_\beta]] \cup \Big(\bigcup\big\{B(k) : k \in K - \{0,1\}\big\}\Big) \cup X(\beta).$$

Consider the image of each factor under the projection π onto the third coordinate. Both Q and R map onto the singleton $a_\beta \in C(\beta)$. By definition $[[c_\beta, a_\beta]]$ is a copy of $[c_\beta, a_\beta]$ and the restriction of π is a homeomorphism $[[c_\beta, a_\beta]] \to [c_\beta, a_\beta] \subset C(\beta)$.

For $k \in K - \{0,1\}$ each $B(k)$ is homeomorphic to two copies of some $[p_\beta^n, a_\beta]$ or $[a_\beta, q_\beta^n]$ joined at the points corresponding to p_β^n or q_β^n respectively; and the restriction of π projects each copy onto the subset $[p_\beta^n, a_\beta]$ or $[a_\beta, q_\beta^n]$ of $C(\beta)$ respectively.

Conversely the construction ensures that for any given $n \in \mathbb{N}$ there are $k_1, k_2 \in K - \{0,1\}$ with $B(k_1)$ and $B(k_2)$ formed from two copies of $[p_\beta^n, a_\beta]$ and $[a_\beta, q_\beta^n]$ respectively as described above. From this we see the set $f_\beta^{\beta+1}\big(X(\beta+1) - X(\beta)\big)$ equals

$$\begin{aligned}
&\{a_\beta\} \cup [c_\beta, a_\beta] \cup \Big(\textstyle\bigcup_n [p_\beta^n, a_\beta]\Big) \cup \Big(\textstyle\bigcup_n [a_\beta, q_\beta^n]\Big) \\
&= [c_\beta, a_\beta] \cup \Big(\textstyle\bigcup_n [p_\beta^n, q_\beta^n]\Big) = [c_\beta, a_\beta] \cup \big(\, \kappa(a_\beta) - c_\beta \,\big) \\
&= \kappa(a_\beta) = C(\beta).
\end{aligned}$$

The first equality follows from hereditary unicoherence and how each $a_\beta \in [p_\beta^n, q_\beta^n]$; the second from Property (5) of the tail; and the last from the definition of $C(\beta)$. $\square$

Claim 8. The system $\{X(\gamma); f_\delta^\gamma : \gamma, \delta \leq \beta + 1\}$ is coherent.

Proof. The choice of $a_{\beta+1}$ and $c_{\beta+1}$ and $f_\beta^{\beta+1}$ makes it clear $a_{\beta+1} \mapsto a_\beta$ and $c_{\beta+1} \mapsto c_\beta$. To see $f_\beta^{\beta+1}\big([c_{\beta+1}, a_{\beta+1}]\big) = [c_\beta, a_\beta]$ recall that $[c_{\beta+1}, a_{\beta+1}] = [[c_\beta, a_\beta]] \cup [\,\widetilde{a}_\beta, a_{\beta+1}]$. The bonding map projects $[[c_\beta, a_\beta]]$ onto $[c_\beta, a_\beta]$ and sends $[\,\widetilde{a}_\beta, a_{\beta+1}] \subset I(0)$ to the point $a_\beta \in [c_\beta, a_\beta]$. Likewise $f_\beta^{\beta+1}$ projects each $[p_{\beta+1}^n, a_{\beta+1}] = [[p_\beta^n, a_\beta]] \cup [\,\widetilde{a}_\beta, a_{\beta+1}]$ onto $[p_\beta^n, a_\beta]$.

Since the subcontinuum $B_+\big(k(n)\big) \subset [p_{\beta+1}^n, q_{\beta+1}^n]$ projects onto $[p_\beta^n, q_\beta^n]$ we have $[p_\beta^n, q_\beta^n] \subset f_\beta^{\beta+1}\big([p_{\beta+1}^n, q_{\beta+1}^n]\big)$. To get equality observe $[p_{\beta+1}^n, q_{\beta+1}^n] = [p_{\beta+1}^n, a_{\beta+1}] \cup [a_{\beta+1}, q_{\beta+1}^n]$. The previous paragraph shows $[p_{\beta+1}^n, a_{\beta+1}]$ maps onto $[p_\beta^n, a_\beta] \subset [p_\beta^n, q_\beta^n]$.

Now we treat the remainder $[a_{\beta+1}, q^n_{\beta+1}]$. Recall we define $\big[a_0, k(n)\big]$ as the unique arc in B with endpoints a_0 and $k(n)$. Now observe $[a_{\beta+1}, q^n_{\beta+1}] \subset g^{-1}\big(\big(a_0, k(n)\big]\big)$.

Consider the intersection $K \cap \big[a_0, k(n)\big]$ of the arc with the Cantor set. Inspecting the buckethandle we see $p(n) \le x \le k(n)$ for each $x \in K \cap \big[a_0, k(n)\big]$. Therefore $\big\{k \in K : [a_{\beta+1}, q^n_{\beta+1}] \text{ meets } B(k)\big\}$ is contained in the interval $\big[p(n), k(n)\big] \subset K$.

By construction all $f^{\beta+1}_\beta\big(B(k)\big)$ for $1/5 \le k \le k(n)$ are contained in $f^{\beta+1}_\beta\big(B(k(n)\big) = [p^n_\beta, q^n_\beta]$ and all $f^{\beta+1}_\beta\big(B(k)\big)$ for $p(n) \le k \le 1/5$ are contained in $f^{\beta+1}_\beta\big(B(p(n)\big) = [p^n_\beta, a_\beta] \subset [p^n_\beta, q^n_\beta]$. We conclude that $f^{\beta+1}_\beta\big([p^n_{\beta+1}, q^n_{\beta+1}]\big) \subset [p^n_\beta, q^n_\beta]$ as required. □

Claim 9. Condition (c) holds for all $\gamma, \delta \le \beta + 1$.

Proof. Let $h : [c_{\beta+1}, a_{\beta+1}] \to [c_\beta, a_\beta]$ be the restriction of $f^{\beta+1}_\beta$. By induction we need only show each $h^{-1}\big([p^n_\beta, a^n_\beta]\big) = [p^n_{\beta+1}, a^n_{\beta+1}]$. To that end recall $[c_{\beta+1}, a_{\beta+1}] = [[c_\beta, a_\beta]] \cup [\widetilde{a}_\beta, a_{\beta+1}]$. Since $[\widetilde{a}_\beta, a_{\beta+1}] \subset Q$ the map h takes $[\widetilde{a}_\beta, a_{\beta+1}]$ to a_β and sends each $\widetilde{x} \in [[c_\beta, a_\beta]]$ to the corresponding $x \in [c_\beta, a_\beta]$. It follows $h^{-1}\big([x, a_\beta]\big) = [\widetilde{x}, a_{\beta+1}]$ for each $x \in [c_\beta, a_\beta]$. Taking $x = p^n_\beta$ we see $h^{-1}\big([p^n_\beta, a_\beta]\big) = [\widetilde{p}^n_\beta, a_{\beta+1}] = [p^n_{\beta+1}, a_{\beta+1}]$ as required. □

Claim 9 completes the discussion of $\alpha = \beta + 1$ a successor ordinal.

4. The Limit Stage

This section deals with the limit stage of our construction. Henceforth assume $\alpha \le \omega_1$ is a limit ordinal and $\{X(\beta); f^\beta_\gamma : \beta, \gamma < \alpha\}$ a coherent system of indecomposable hereditarily unicoherent metric continua and retractions. For all $\beta, \gamma, \delta < \alpha$ we assume the objects (i), (ii) and (iii) from Section 3 have been specified and Conditions (a), (b) and (c) hold.

We define $X(\alpha) = \varprojlim\{X(\beta); f^\beta_\gamma\}$ and each f^α_β as the projection from the inverse limit onto its factors. For each $\gamma < \alpha$ we identify $X(\gamma)$ with the subset $\big\{x \in X(\alpha) : x_\beta = x_\gamma \text{ for all } \beta > \gamma\big\}$ of $X(\alpha)$.

Straightforward modifications of [8] Theorem 3.1 and [3] Corollary 1 show $X(\alpha)$ is both indecomposable and hereditarily unicoherent. To see $X(\alpha)$ is metric we observe by definition that α is a countable ordinal. The product $\prod_{\beta<\alpha} X(\beta)$ of countably many metric spaces is itself a metric space. The inverse limit $X(\alpha)$ is by definition a subset of that product and therefore a metric space.

It remains to show the enlarged system is coherent; to check Conditions (a), (b) and (c) hold for the enlarged system; and to specify the data (i), (ii) and (iii) for $X(\alpha)$. Much of the effort will go into proving the pair of sequences given for (iii) is indeed a tail. To that end we first prove some general facts about tails.

Recall at stage 0 we defined the tail $\big((p_0^n),(q_0^n)\big)$ on the quinary buckethandle. Observe the sequence (p_0^n) tends to c_0 and the intervals $[p_0^n, a_0]$ increase to be dense in $[c_0, a_0]$ while the sequence (q_0^n) has no limit and the intervals $[a_0, q_0^n]$ increase to be dense in the whole space. The next lemma show this holds for general tails.

Lemma 10. Suppose the indecomposable and hereditarily unicoherent continuum X has a tail $T = ((p^n),(q^n))$ from a to c. Then $\bigcup_n [p^n, a]$ is nowhere dense and $\bigcup_n [a, q^n]$ is dense.

Proof. For the first statement Property (6) says $\bigcup_n [p^n, a] \subset [c, a]$. By hereditary unicoherence $[c, a]$ is a subcontinuum, and it is nowhere dense by indecomposability. We conclude the first union is nowhere dense.

For the second statement Property (1) says each $a \in [p^n, q^n]$. Hence $[p^n, q^n] = [p^n, a] \cup [a, q^n]$. Now observe by Property (5) that

$$\kappa(a) - c = \bigcup_{n\in\mathbb{N}} [p^n, q^n] = \bigcup_{n\in\mathbb{N}} \Big([p^n, a] \cup [a, q^n]\Big) =$$
$$\Big(\bigcup_{n\in\mathbb{N}} [p^n, a]\Big) \cup \Big(\bigcup_{n\in\mathbb{N}} [a, q^n]\Big) = \Big([c, a] - c\Big) \cup \Big(\bigcup_{n\in\mathbb{N}} [a, q^n]\Big).$$

The left-hand-side is dense in X. The right hand-side is the union of two sets. We have already showed the right-hand-side is nowhere dense. It follows the second set is dense. This completes the proof. □

One can observe Section 3 made no explicit reference to strong non-cut points. In fact these points were mentioned explicitly when we talked about tails. The next lemma is needed to obtain the strong non-cut point mentioned in the title.

Lemma 11. Suppose the indecomposable and hereditarily unicoherent continuum X has a tail $T = (p^n, q^n)$ from a to c. Then c is the only point of $\kappa(a)$ to not weakly cut the composant.

Proof. Property (5) says $\kappa(a) - c$ is the union of a chain of subcontinua hence is a semicontinuum. This is the definition of c not weakly cutting $\kappa(a)$. Now let $b \in \kappa(a) - c$ be arbitrary. Property (6) says b is an element of some $[p^n, q^n]$. Define $L = [p^n, q^n]$ and $P = [p^{n+1}, q^{n+1}]$.

First suppose $b \notin \{p^n, q^n\}$. Then $[p^n, b, q^n]_L$ and thus $[p^n, b, q^n]_X$ by hereditary unicoherence. Now suppose $b = p^n$. Then Properties (2) and (3) imply $p^n \neq p^{n+1}$ and $q^n \neq p^{n+1}$ respectively. Thus $b \notin \{p^{n+1}, q^{n+1}\}$ which implies $[p^{n+1}, b, q^{n+1}]_P$ which in turn implies $[p^{n+1}, b, q^{n+1}]_X$ by hereditary unicoherence. The case for $b = q^n$ is similar. We conclude each $b \in \kappa(a) - c$ weakly cuts the composant. □

We have assumed $\{X(\beta); f^\beta_\gamma : \beta, \gamma < \alpha\}$ is coherent. Hence each $a_\beta \mapsto a_\gamma$, $c_\beta \mapsto c_\gamma$, $p^n_\beta \mapsto p^n_\gamma$, $q^n_\beta \mapsto q^n_\gamma$ and it follows the inverse limit $X(\alpha)$ has points $a_\alpha = (a_\beta)_{\beta<\alpha}$, $c_\alpha = (c_\beta)_{\beta<\alpha}$, $p^n_\alpha = (p^n_\beta)_{\beta<\alpha}$ and $q^n_\alpha = (q^n_\beta)_{\beta<\alpha}$. For ease of notation we suppress the subscript and write a, c, p^n, q^n instead of $a_\alpha, c_\alpha, p^n_\alpha, q^n_\alpha$ respectively.

To see $T^\alpha = \big((p^n), (q^n)\big)$ is a tail from a to c we use Corollary 1 and the facts $[p^n_\beta, q^n_\beta] \to [p^n_\gamma, q^n_\gamma]$, $[p^n_\beta, a_\beta] \to [p^n_\gamma, a_\gamma]$ and $[c_\beta, a_\beta] \to [c_\gamma, a_\gamma]$ respectively, from the definition of coherence, to get the three expressions.

$$[p^n, q^n] = \varprojlim \big\{[p^n_\beta, q^n_\beta]; f^\beta_\gamma; \gamma, \beta < \alpha\big\} \tag{I}$$

$$[p^n, a] = \varprojlim \big\{[p^n_\beta, a_\beta]; f^\beta_\gamma; \gamma, \beta < \alpha\big\} \tag{II}$$

$$[c, a] = \varprojlim \big\{[c_\beta, a_\beta]; f^\beta_\gamma; \gamma, \beta < \alpha\big\} \tag{III}$$

The three expressions show the expanded system $\{X(\beta); f^\beta_\gamma : \beta, \gamma \le \alpha\}$ is coherent. Expressions (I), (III) and (II) respectively imply T^α has Properties (1), (2) and (3) of being a tail. Property (4) is slightly more complicated.

Claim 12. T^α has Property (4) of being a tail from a to c.

Proof. We first show each $q^{n+1}_\beta \notin [p^n_\beta, q^n_\beta]$. Property (1) for T^β says $a_\beta \in [p^n_\beta, q^n_\beta]$. Hence $[p^n_\beta, q^n_\beta] = [p^n_\beta, a_\beta] \cup [a_\beta, q^n_\beta]$. Property (6) says $[p^n_\beta, a_\beta] \subset [c_\beta, a_\beta]$ thus $q^{n+1}_\beta \notin [p^n_\beta, a_\beta]$ by Property (2). Property (4) says $q^{n+1}_\beta \notin [a_\beta, q^n_\beta]$. We conclude $q^{n+1}_\beta \notin [p^n_\beta, q^n_\beta]$.

Now we show $q^{n+1} \notin [a, q^n]$ which proves Property (4) for T^α . Just like before we have $[p^n, q^n] = [p^n, a] \cup [a, q^n]$. In particular $[a, q^n]$ is contained in $[p^n, q^n]$. We have already shown $q^{n+1}_\beta \notin [p^n_\beta, q^n_\beta]$. Thus the expression (I) implies $q^{n+1} \notin [p^n, q^n]$ as required. □

To show T^α has Properties (5) and (6) we introduce some notation to measure how far a given subcontinuum extends along the tail.

Notation 1. For each $\beta < \alpha$ and subcontinuum $L \subset X(\beta)$ define

$$\|L\| = \max\big\{n \in \mathbb{N} : [c_\beta, q^n_\beta] \subset L\big\}$$

where we allow the value $\|L\| = \infty$ in case all $[c, q^n] \subset L$.

Lemma 10 says each $\bigcup_n [a_\beta, q_\beta^n]$ is dense. Therefore $\|L\|$ is well defined whenever $L \subset X(\beta)$ is proper. In case $L = X(\beta)$ we define $\|L\| = \infty$. Clearly $\|L\| \leq \|P\|$ for all $L \subset P$ and $\|L\| = 0$ when either $c_\beta \notin L$ or $L \subset X(\beta) = \kappa(a_\beta) = X(\beta) - C(\beta)$.

Claims 13 and 14 will be used to show T^α has Property (5).

Claim 13. We have $\|L\| \leq \|f_\gamma^\beta(L)\|$ for each $\gamma, \beta < \alpha$ and subcontinuum $L \subset X(\beta)$.

Proof. In case $\|L\| = 0$ the result is obvious. Hence assume $\|L\| > 0$. That means $[c_\beta, q_\beta^n] \subset L$ for some $n > 0$. Property (1) for T^β says $a_\beta \in [c_\beta, q_\beta^n]$ hence $[c_\beta, q_\beta^n] = [c_\beta, a_\beta] \cup [a_\beta, q_\beta^n]$. Property (6) says each $p_\beta^m \in [c_\beta, a_\beta]$ hence $[c_\beta, q_\beta^n] = [c_\beta, p_\beta^m] \cup [p_\beta^m, a_\beta] \cup [a_\beta, q_\beta^n]$. We conclude each $[p_\beta^m, a_\beta] \subset L$.

To show $[c_\gamma, q_\gamma^n] \subset f_\gamma^\beta(L)$ first observe $c_\beta \in L$ and so $c_\gamma \in f_\gamma^\beta(L)$ by coherence. Now suppose $x \in [c_\gamma, a_\gamma] - c_\gamma$. Property (6) for $X(\gamma)$ says $x \in [p_\gamma^m, a_\gamma]$ for some $m > 0$. We have already shown $[p_\beta^m, a_\beta] \subset L$. Therefore $f_\gamma^\beta(L)$ contains $f_\gamma^\beta\big([p_\beta^m, a_\beta]\big) = [p_\gamma^m, a_\gamma]$ by coherence hence $x \in f_\gamma^\beta(L)$.

Now suppose $x \in [c_\gamma, q_\gamma^n] - [c_\gamma, a_\gamma]$. By the first paragraph we have $[c_\gamma, q_\gamma^n] = [c_\gamma, a_\gamma] \cup [a_\gamma, q_\gamma^n]$ hence $x \in [a_\gamma, q_\gamma^n]$. The first paragraph proves $[c_\beta, q_\beta^n] = [c_\beta, p_\beta^n] \cup [p_\beta^n, a_\beta] \cup [a_\beta, q_\beta^n]$ which equals $[c_\beta, p_\beta^n] \cup [p_\beta^n, q_\beta^n]$ since $a_\beta \in [p_\beta^n, q_\beta^n]$. Therefore $[p_\beta^n, q_\beta^n] \subset [c_\beta, q_\beta^n]$. Finally observe $x \in [a_\gamma, q_\gamma^n] \subset [p_\gamma^n, q_\gamma^n] = f_\gamma^\beta\big([p_\beta^n, q_\beta^n]\big) \subset f_\gamma^\beta\big([c_\beta, q_\beta^n]\big) \subset f_\gamma^\beta(L)$. We conclude $x \in f_\gamma^\beta(L)$ as required. Taking $n = \|L\|$ gives the result. □

Claim 14. Suppose $x \in \kappa(a) - [c, a]$. Then $x \in [p^n, q^n]$ for some $n \in \mathbb{N}$.

Proof. By Lemma 23 the set $\Psi = \big\{\beta < \alpha : x_\beta \notin [c_\beta, a_\beta]\big\}$ is terminal. Replace α with Ψ and hence assume all $x_\beta \notin [c_\beta, a_\beta]$. This does not change $X(\alpha)$ or whether $x \in [p^n, q^n]$.

We deal with two cases separately. First assume $\big\{\big\|[c_\beta, x_\beta]\big\| : \beta \in \Omega\}$ is bounded for some terminal $\Omega \subset \alpha$. Like before we can assume without loss of generality $\alpha = \Omega$. Hence there is $N \in \mathbb{N}$ with each $[c_\beta, q_\beta^N] \not\subset [c_\beta, x_\beta]$. Property (7) for β says each $[c_\beta, x_\beta] \subset [c_\beta, q_\beta^N]$. In particular $x_\beta \in [c_\beta, q_\beta^N] = [c_\beta, a_\beta] \cup [a_\beta, q_\beta^N]$ and so $x_\beta \in [a_\beta, q_\beta^N] \subset [p_\beta^N, q_\beta^N]$. By (I) we conclude $(x_\beta) \in [p^N, q^N]$.

Now assume no such Ω exists. By induction we can select an increasing sequence $\beta(1) < \beta(2) < \dots$ with $\big\|[c_{\beta(n)}, x_{\beta(n)}]\big\| > n$ for each $n \in \mathbb{N}$. Lemma 24 says $\pi_0\big([c, x]\big) = \overline{\bigcup\big\{f_0^\beta\big([c_\beta, x_\beta] : \beta < \alpha\big)\big\}}$. In particular $\pi_0\big([c, x]\big)$ contains each $f_0^{\beta(n)}\big([c_{\beta(n)}, x_{\beta(n)}]\big)$. Thus we have $\big\|\pi_0\big([c, x]\big)\big\| \geq \big\|f_0^{\beta(n)}\big([c_{\beta(n)}, x_{\beta(n)}]\big)\big\| \geq \big\|[c_{\beta(n)}, x_{\beta(n)}]\big\|$ by Claim 13 and so $\big\|\pi_0\big([c, x]\big)\big\| > n$.

Since $n \in \mathbb{N}$ was arbitrary we conclude $\big\|\pi_0\big([c,x]\big)\big\| = \infty$. This means $\pi_0\big([c,x]\big) = X(0)$. Replacing 0 with any $\gamma < \alpha$ we see each $\pi_\gamma\big([c,x]\big) = X(\gamma)$ hence $[c,x] = X(\alpha)$. That means $X(\alpha)$ is irreducible from c to x. In other words $x \notin \kappa(c)$. Since $\kappa(c) = \kappa(a)$ this contradicts the assumption. □

Claim 15. T^α has Property (7) of being a tail from a to c.

Proof. Suppose $x \notin [c,a]$. It follows from (III) and Lemma 23 the set $\big\{\beta < \alpha : x_\beta \notin [c_\beta, a_\beta]\big\}$ is terminal. Without loss of generality assume all $x_\beta \notin [c_\beta, a_\beta]$. Now suppose $[c, q^n] \not\subset [c,x]$. It follows from Lemma 24 we cannot have $[c_\beta, q^n_\beta] \subset [c_\beta, x_\beta]$ cofinally. Therefore $[c_\beta, q^n_\beta] \not\subset [c_\beta, x_\beta]$ terminally. Property (6) for β says $[c_\beta, x_\beta] \subset [c_\beta, q^n_\beta]$ terminally. Thus $[c,x] \subset [c, q^n]$ by Lemma 24. □

To deal with Property (5) of being a tail, we use Condition (c) from the construction.

Claim 16. T^α has Property (6) of being a tail from a to c.

Proof. Let $x \in [c,a] - c$ be arbitrary. By (III) each $x_\beta \in [c_\beta, a_\beta]$. Let $\gamma < \alpha$ be fixed and observe $\Psi = \{\beta < \alpha : \gamma \le \beta\}$ is cofinal. Property (6) for γ says $x_\gamma \in [p^n, a_\gamma]$ for some $n \in \mathbb{N}$. Let $g : [c_\beta, a_\beta] \to [c_\gamma, a_\gamma]$ be the restriction of f^β_γ.

Then $x_\gamma \in [p^n, a_\gamma]$ hence $x_\beta \in g^{-1}\big([p^n_\gamma, a_\gamma]\big)$ which equals $[p^n_\beta, a_\beta]$ by Condition (c). Thus $x_\beta \in [p^n_\beta, a_\beta]$ and $x \in \varprojlim \big\{[p^n_\beta, a_\beta] : \beta \in \Psi\big\}$. The expression (II) says the set on the right-hand-side equals $[p^n, a]$ and so $x \in [p^n, a]$.

Since $x \in [c,a] - c$ is arbitrary we see $\bigcup \big\{[p^n, a] : n \in N\big\}$ contains $[c,a] - c$. Property (6) for β says each $c_\beta \notin [p^n_\beta, a_\beta]$ thus $c \notin \bigcup \big\{[p^n, a] : n \in \mathbb{N}\big\}$. We conclude $\bigcup \big\{[p^n, a] : n \in \mathbb{N}\big\} = [c,a] - c$. □

Claim 17. T^α is a tail from a to c.

Proof. The expressions (I), (III) and (II) imply T^α has Properties (1), (2) and (3) respectively. Properties (4), (6) and (7) follow from Claims 12, 16 and 15 imply T^α respectively.

Claim 14 says that $\bigcup_n \big[p^n, q^n\big]$ contains each $x \in \kappa(a) - [c,a]$. Combined with Property (6) and how $[p^n, q^n] = [p^n, a] \cup [a, q^n]$ we see $\bigcup_n \big[p^n, q^n\big]$ contains $\kappa(a) - c$. Since it is disjoint from $D(\alpha)$ it is a proper semicontinuum. Hence the only possibilities are $\bigcup_n \big[p^n, q^n\big] = \kappa(a) - c$ or $\bigcup_n \big[p^n, q^n\big] = \kappa(a)$. To see the former observe Property (5) for each β says $c_\beta \notin [p^n_\beta, q^n_\beta]$ thus $c \notin \bigcup_n \big[p^n, q^n\big]$. This shows Property (5) for T^α. □

We have already chosen the points $a = a_\alpha$ and $c = c_\alpha$. Claim 17 shows the pair T^α of sequences is indeed a tail from a_α to c_α. Thus we have specified the objects (ii) and (iii) for stage α. It remains to select the composants (i) and prove Conditions (a), (b) and (c) hold for all $\gamma, \delta \leq \alpha$. Finally we must show each $X(\beta)$ occurs as a subspace of $X(\alpha)$ and how f^α_β is a retraction.

For (i) we must choose $C(\alpha) = \kappa(a_\alpha)$. Hereditary unicoherence along with (III) shows $C(\alpha) = \kappa(c_\alpha)$ as required. For each $\gamma < \alpha$ the identification $X(\gamma) = \big\{x \in X(\alpha) : x_\beta = x_\gamma \text{ for all } \beta > \gamma\big\}$ makes it clear the $X(\gamma)$ are nested and the bonding maps are retractions. Hence $\bigcup\{X(\beta) : \beta < \alpha\}$ is a semicontinuum of $X(\alpha)$ and thus contained in some composant $D(\alpha)$ of $X(\alpha)$. Then condition (a) follows from the definition of $D(\alpha)$.

Claim 18. The composants $D(\alpha)$ and $C(\alpha)$ of $X(\alpha)$ are distinct. Thus at stage α the condition on (i) holds.

Proof. Since X is indecomposable it is enough to show it is irreducible from $c \in C(\alpha)$ to some $x \in D(\alpha)$. Let $\gamma < \alpha$ and $x \in X(\gamma)$ be arbitrary and suppose $\{c, x\} \subset L$ for some subcontinuum $L \subset X(\alpha)$. We claim $\pi_\beta(L) = X(\beta)$ for all $\beta > \gamma$. Thus by surjectivity $L = X(\alpha)$.

Clearly $\{c_\beta, x_\beta\} \subset \pi_\beta(L)$. By definition of the embedding $X(\gamma) \to X(\alpha)$ we have $x_\beta = x_\gamma$ hence$\{c_\beta, x_\gamma\} \subset \pi_\beta(L)$. Since $x_\gamma = \pi_\gamma(x)$ we have $x_\gamma \in X(\gamma)$. By (a) for stage β we have $X(\gamma) \subset D(\beta)$. Hence $\pi_\beta(L)$ meets the distinct composants $C(\beta)$ and $D(\beta)$ of $X(\beta)$ and so $\pi_\beta(L) = X(\beta)$ as required. □

To prove Condition (b) for the expanded system we use the following claim.

Claim 19. Suppose $x_\gamma = x_{\gamma+1}$ for some $x \in X(\alpha)$ and $\gamma < \alpha$. Then $x \in X(\gamma)$.

Proof. We claim $x_\beta = x_{\gamma+1}$ for each $\beta > \gamma + 1$. If $x_\beta \in X(\gamma + 1)$ then since $f^\beta_{\gamma+1}$ is a retraction we have $x_\beta = x_{\gamma+1}$. Otherwise $x_\beta \in X(\beta) - X(\gamma+1)$ and Condition (b) says $f^\beta_{\gamma+1}(x_\beta) \in C(\gamma+1)$. But recall $f^\beta_{\gamma+1}(x_\beta) = f^\beta_{\gamma+1} \circ f^\alpha_\beta(x) = f^\alpha_{\gamma+1}(x) = x_{\gamma+1}$. By assumption $x_{\gamma+1} = x_\gamma$ and therefore $x_\gamma \in C(\gamma + 1)$. But $x_\gamma \in X(\gamma)$ and by Condition (a) we have $X(\gamma) \subset D(\gamma + 1)$. This is a contradiction since the composants $C(\gamma + 1)$ and $D(\gamma + 1)$ are disjoint. □

Claim 20. Condition (b) holds for all $\gamma, \delta \leq \alpha$.

Proof. We want to show $\bigcup\big\{f^\gamma_\delta\big(X(\gamma) - X(\delta)\big) : \alpha \geq \gamma > \delta\big\} = C(\delta)$ for each $\delta < \alpha$. First observe the above union can be written

$$f_\delta^\alpha\big(X(\alpha)-X(\delta)\big)\cup\Big(\bigcup\big\{f_\delta^\gamma\big(X(\gamma)-X(\delta)\big):\alpha>\gamma>\delta\big\}\Big).$$

By induction the second factor equals $C(\delta)$. So it is enough to show the first factor equals $C(\delta)$ as well.

To see $f_\delta^\alpha\big(X(\alpha)-X(\delta)\big)\subset C(\delta)$ let $x\in X(\alpha)-X(\delta)$ be arbitrary. Claim 19 implies $x_{\delta+1}\neq x_\delta$. Since $f_\delta^{\delta+1}$ is a retraction we have $x_{\delta+1}\in X(\delta+1)-X(\delta)$ and so $f_\delta^{\delta+1}(x_{\delta+1})\in C(\delta)$ by Condition (b) at stage $\delta+1$. But $f_\delta^{\delta+1}(x_{\delta+1})$ is just $x_\delta=f_\delta^\alpha(x)$ and so $f_\delta^\alpha(x)\in C(\delta)$.

To see $f_\delta^\alpha\big(X(\alpha)-X(\delta)\big)=C(\delta)$ recall T^α is a tail from a to c. Property (5) says $\{c\}\cup\Big(\bigcup_n[p^n,q^n]\Big)=C(\alpha)$. By coherence the image under f_γ^α is $\{c_\gamma\}\cup\Big(\bigcup_n[p_\gamma^n,q_\gamma^n]\Big)$ which equals $C(\gamma)$ by Property (5) of T^γ. □

Finally we prove (c) for the expanded system.

Claim 21. Condition (c) holds for all $\gamma,\delta\le\alpha$.

Proof. By commutativity and Condition (c) at earlier stages it is enough to consider the case $\gamma=\alpha$. We must show $\big\{x\in[c,a]:x_\delta\in[p_\delta^n,a_\delta]\big\}=[p^n,a]$.

Let $\gamma>\delta$ be arbitrary and consider the γ-coordinate of a point x of the left-hand-side. Since $x\in[c,a]$ and $[c,a]=\varprojlim\big\{[c_\delta,a_\delta];f_\delta^\gamma\big\}$ by (III) we have $x_\gamma\in[c_\gamma,a_\gamma]$. Since $x_\delta=f_\delta^\gamma(x_\gamma)$ we see x_γ is an element of $\big\{y\in[c_\gamma,a_\gamma]:f_\delta^\gamma(y)\in[p_\delta^n,a_\delta]\big\}$. Property (c) at earlier stages says this set equals $[p_\gamma^n,a_\gamma]$. Thus $x_\gamma\in[p_\gamma^n,a_\gamma]$. By (II) we know the set $\varprojlim\big\{[p_\delta^n,a_\delta];f_\delta^\gamma\big\}$ is well defined and equals $[p^n,a]$. Since $x_\gamma\in[p_\gamma^n,a_\gamma]$ for all $\gamma>\delta$ we conclude $\big\{x\in[c,a]:x_\delta\in[p_\delta^n,a_\delta]\big\}\subset[p^n,a]$.

For the other inclusion let $x\in[p^n,a]$ be arbitrary. By (II) each $x_\delta\in[p_\delta^n,a_\delta]$. By Property (6) at stage δ we have $[p_\delta^n,a_\delta]\subset[c_\delta,a_\delta]$. Hence $x_\delta\in[c_\delta,a_\delta]$ and by (III) we have $x\in[c,a]$. We conclude $[p^n,a]\subset\big\{x\in[c,a]:x_\delta\in[p_\delta^n,a_\delta]\big\}$. □

We are ready to prove the main theorem.

Theorem 22. There exists a Bellamy continuum with a strong non-cut point.

Proof. By Theorem 1 of [1] the limit $X=\varprojlim\{X(\alpha);f_\beta^\alpha:\beta,\alpha<\omega_1\}$ is indecomposable with at most two composants. The *trivial composant* $E\subset X$ is the set $\bigcup\{X(\alpha):\alpha<\omega_1\}$ of eventually constant ω_1-sequences. The *nontrivial composant* $X-E$ is equal to $\varprojlim\{C(\alpha);f_\beta^\alpha:\beta,\alpha<\omega_1\}$. The points $a=(a_\beta)$ and $c=(c_\beta)$ witness how $X-E$ is nonempty. Therefore X has exactly two composants.

Theorem 17 applied to $\{X(\alpha); f^\alpha_\beta : \beta, \alpha < \omega_1\}$ says there is a tail from $a = (a_\beta)$ to $c = (c_\beta)$. Lemma 11 says c does not weakly cut $X - E$.

Choose any two points $x \in E$ and $y \in X - E$ both distinct from c. Let $\widetilde{X}$ be obtained by treating $\{x, y\}$ as a single point. It follows $\widetilde{X}$ is an indecomposable continuum with exactly one composant and $c \in \widetilde{X}$ is not a weak cut point. $\square$

Finally we prove the lemmas cited throughout.

Lemma 23. Suppose $\{Y(\beta); f^\beta_\gamma : \gamma, \beta \in \Omega\}$ is an inverse system whose limit Y has points $p = (p_\beta)$ and $q = (q_\beta)$ and $a = (a_\beta)$. Suppose the set $\big\{\beta \in \Omega : [p_\beta, a_\beta, q_\beta]\big\}$ is cofinal. Then $[p, a, q]$.

Proof. First replace Ω with $\big\{\beta \in \Omega : a_\beta \in [p_\beta, a_\beta, q_\beta]\big\}$ hence assume each $[p_\beta, a_\beta, q_\beta]$. Now suppose $L \subset Y$ is a subcontinuum with $\{p, q\} \subset L$. Then each $\{p_\beta, q_\beta\} \subset \pi_\beta(L)$ and so $a_\beta \in \pi_\beta(L)$ since $\pi_\beta(L)$ is a subcontinuum. Now recall $L = \varprojlim\{\pi_\beta(L); f^\beta_\gamma : \gamma, \beta \in \Omega\}$. Since each $a_\beta \in \pi_\beta(L)$ we have $a \in L$. Since $L \subset Y$ is arbitrary we conclude $[p, a, q]$. $\square$

Corollary 1. Suppose $\{Y(\beta); f^\beta_\gamma : \gamma, \beta \in \Omega\}$ is an inverse system whose limit Y has points $p = (p_\beta)$ and $q = (q_\beta)$. Suppose $\big\{\beta \in \Omega : Y(\beta) = [p_\beta, q_\beta]\big\}$ is cofinal. Then $Y = [p, q]$.

Lemma 24. Suppose $\{X(\beta); f^\beta_\gamma : \gamma, \beta \in \Omega\}$ is an inverse system with points $a = (a_\beta)$ and $b = (b_\beta)$. Each $\pi_\gamma\big([a, b]\big) = \overline{\bigcup\big\{f^\beta_\gamma\big([a_\beta, b_\beta]\big) : \beta > \gamma\big\}}$.

Proof. Write $J_\gamma = \bigcup\big\{f^\beta_\gamma\big([a_\beta, b_\beta]\big) : \beta > \gamma\big\}$. We know $[a, b]$ has the form $\varprojlim\{I_\beta; f^\beta_\gamma : \gamma, \beta \in \Omega\}$ for some subcontinua $I_\beta \subset X(\beta)$. Since each I_β contains $\{a_\beta, b_\beta\}$ it contains $[a_\beta, b_\beta]$. By commutativity each $I_\gamma = f^\beta_\gamma(I_\beta)$ contains $f^\beta_\gamma\big([a_\beta, b_\beta]\big)$ for $\beta > \gamma$. hence I_γ contains J_γ and by closure contains $\overline{J_\gamma}$.

By commutativity each f^β_γ maps J_β onto J_γ. By continuity f^β_γ maps $\overline{J_\beta}$ onto $\overline{J_\gamma}$. Hence $J = \varprojlim\{\overline{J_\beta}; f^\beta_\gamma : \gamma, \beta \in \Omega\}$ is a well defined subcontinuum containing $\{a, b\}$. We have shown it is minimal with respect to containing $\{a, b\}$. We conclude $J = [a, b]$ and each $I_\beta = \overline{J_\beta}$ as required. $\square$

5. The Trivial Composant

We have already determined the weak cut structure of the nontrivial composant. Namely $c \in X - E$ is the only point to not weakly cut its composant.

This section examines the trivial composant $E \subset X$. We show E is weakly cut by its every point. Hence the continuum $\widetilde{X}$ from Theorem 1 has exactly one strong non-cut point.

To that end recall E is the set of eventually constant ω_1-sequences. The first claim is that the subcontinua of E share the property of being *eventually constant*.

Claim 25. Suppose the subcontinuum $L \subset E$ meets both $X(\beta)$ and $X - X(\beta+1)$ for some $\beta < \omega_1$. Then $X(\beta+1) \subset L$.

Proof. By assumption L meets $X(\alpha) - X(\beta+1)$ for some $\alpha > \beta + 1$. By hereditary unicoherence $X(\alpha) \cap L$ is a subcontinuum. Since

$$X(\alpha) \cap L = \big((X(\alpha) \cap L) \cap X(\beta+1)\big) \cup \big((X(\alpha) \cap L) - X(\beta+1)\big)$$

we know $f^{\alpha}_{\beta+1}\big(X(\alpha) \cap L\big)$ equals

$$f^{\alpha}_{\beta+1}\big((X(\alpha) \cap L) \cap X(\beta+1)\big) \cup f^{\alpha}_{\beta+1}\big((X(\alpha) \cap L) - X(\beta+1)\big).$$

Since the map is a retraction the first summand equals $L \cap X(\beta+1)$ which meets $X(\beta)$ by assumption and hence meets $D(\beta+1)$. By (b) the second summand is contained in $C(\beta+1)$.

Thus the subcontinuum $f^{\alpha}_{\beta+1}(X(\alpha) \cap L) \subset X(\beta+1)$ meets the distinct composants $C(\beta+1)$ and $D(\beta+1)$ hence equals $X(\beta+1)$. Since the second summand is contained in $C(\beta+1)$ the first summand $L \cap X(\beta+1)$ must contain $D(\beta+1)$. Since $L \cap X(\beta+1)$ is closed and $D(\beta+1)$ dense we have $L \cap X(\beta+1) = X(\beta+1)$ and so $X(\beta+1) \subset L$. $\square$

One consequence of Claim 25 is any subcontinuum that meets all $X(\alpha)$ must contain all $X(\alpha)$ and hence equal X. The next claim follows.

Claim 26. Each subcontinuum of E is contained in some $X(\alpha)$.

Claim 27. Each point of E weakly cuts its composant.

Proof. Let $b \in E$ be arbitrary. Then $b \in X(\beta)$ for some $\beta < \omega_1$. Choose any $a \in X(\beta) - b$ and $c \in X(\beta+2) - X(\beta+1)$. By Claim 25 each subcontinuum that includes a and c must contain $X(\beta+1)$. Hence the subcontinuum contains $X(\beta)$ and includes b. We conclude $[a, b, c]$. $\square$

Theorem 28. There exists a Bellamy continuum with exactly one strong non-cut point.

Proof. Let X be the continuum from Theorem 1 and $X \to \widetilde{X}$ the quotient map that treats $\{x, y\}$ as the single point $z \in \widetilde{X}$. We have already shown $\widetilde{c} \in \widetilde{X}$ is a strong non-cut point. Now suppose $b \in \widetilde{X} - \widetilde{c}$. For $b = z$ we have $[r, b, s]$ for each $r \in \widetilde{\kappa(x)} - z$ and $s \in \widetilde{\kappa(y)} - z$.

Now assume $b \neq z$. Then $b = \widetilde{d}$ for some unique $d \in X - \{x, y\}$. The composant $\kappa(d)$ is one of E or $X - E$. In the first case Claim 27 says $[r, d, s]$ for some pair $r, s \in \kappa(d)$. In the second case Lemma 11 says the same.

Boundary bumping implies each continuum component of $X - d$ contains a nondegenerate subcontinuum hence has infinitely many points. That means we can reselect r and s outside $\{x, y\}$ if necessary. Hence $\widetilde{r}, \widetilde{s} \neq z$. We claim $[\widetilde{r}, b, \widetilde{s}]$ thus b weakly cuts $\widetilde{X}$.

Suppose to reach a contradiction $\{\widetilde{r}, \widetilde{s}\} \subset L \subset \widetilde{X} - b$ for some sub- continuum $L \subset \widetilde{X}$. Clearly $z \in L$ as otherwise $L = \widetilde{P}$ for some subcontinuum $P \subset X - \{x, y\}$. Then P contradicts how $[r, d, s]$.

Let C_r and C_s be the continuum components of $\widetilde{r}$ and $\widetilde{s}$ in $L - z$ respectively. Then $C_r = \widetilde{D}_r$ and $C_s = \widetilde{D}_s$ for semicontinua $D_r, D_d \subset X - \{x, y\}$. It follows $D_r, D_s \subset \kappa(d) - \{x, y\}$. Boundary bumping says $z \in \overline{C}_r$ and $z \in \overline{C}_s$. The definition of the quotient topology implies $\overline{D}_r$ and $\overline{D}_s$ meet $\{x, y\}$.

Without loss of generality $\kappa(d) = \kappa(x) = E$. Recall D_r and D_s are mapped into L which is proper. Continuity of the quotient says $\overline{D}_r$ and $\overline{D}_s$ are mapped into L hence proper. We cannot have $y \in \overline{D}_r$ as then the proper subcontinuum $\overline{D}_r$ meets both composants of X. Likewise $y \notin \overline{D}_s$. We conclude both $x \in \overline{D}_r$ and $x \in \overline{D}_s$.

Hence $\overline{D}_r \cup \overline{D}_s$ is a subcontinuum of X. Since X is indecomposable each summand is nowhere dense, and the same follows for the union. Thus the subcontinuum $\overline{D}_r \cup \overline{D}_s$ contradicts how $[r, d, s]$. We conclude no such subcontinuum $L \subset \widetilde{X}$ exists and therefore $[\widetilde{r}, b, \widetilde{s}]$. □

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Department of Computer Science and Statistics ; Trinity College Dublin
Email address: `daronanderson@live.ie`