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SELECTIVITY PROPERTIES OF SPACES

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ABSTRACT. This paper addresses several questions of Feng, Gruenhage, and Shen which arose from Michael's theory of continuous selections from countable spaces. We construct an example of a space which is $(\omega+1)$-selective but not $\mathbb{Q}$-selective from $\mathfrak{d} = \omega_1$, and an $(\omega+1)$-selective space which is not selective for a P-point ultrafilter from the assumption of CH. We also produce ZFC examples of Fréchet spaces where countable subsets are first countable which are not $(\omega+1)$-selective.

1. Introduction

Suppose X, Y are topological spaces and $\varphi : Y \to \mathcal{P}(X) \setminus \{\emptyset\}$ is a map. A general question investigated in detail by Michael asks under what conditions it is possible to find a continuous $s : Y \to X$ so that $s(y) \in \varphi(y)$ for all $y \in Y$. It is natural to require some kind of continuity assumption for φ. So let $\mathcal{F}(X)$ be the set of all nonempty closed subsets of X equipped with the Vietoris topology generated by the sets

(1) $\{A \in \mathcal{F}(X) : A \cap W \neq \emptyset\}$
(2) $\{A \in \mathcal{F}(X) : A \subseteq W\}$

where W ranges through open subsets of X. A map $\varphi : Y \to \mathcal{F}(X)$ is called *lower semicontinuous* if it is continuous with respect to open sets of the first kind, i.e., if for every nonempty open $W \subseteq X$, the set $\{y \in Y : \varphi(y) \cap W \neq \emptyset\}$ is open in Y.

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A space X is Y-*selective* if for every lower semicontinuous $\varphi : Y \to \mathcal{F}(X)$ there is a continuous selection $s : Y \to X$ for φ. In particular, we will consider $(\omega+1)$-selective spaces, where $\omega+1$ is taken with the order topology, i.e., a convergent sequence. X is C-*selective* if it is Y-selective for any countable regular space Y.

Michael [4] proved that every first countable space is C-selective. In [3], Feng, Gruenhage and Shen improved this result:

Fact 1.1.

- Every GO-space is C-selective.
- Every W-space is C-selective.

For $(\omega+1)$-selective spaces, they showed:

Fact 1.2. Every $(\omega+1)$-selective space X has the α_1-property, i.e., for any point $x \in X$ and any countable family $\mathcal{S}$ of sequences converging to x, there is a single sequence converging to x which contains all but finitely many elements of each member of $\mathcal{S}$.

And furthermore, it is not difficult to show that if X is $(\omega+1)$-selective, and $A \subseteq X$ is countable with $x \in \overline{A}$, then there is $B \subseteq A$ converging to x (a slight weakening of the Fréchet property).

The present work addresses several questions of [3]. They asked whether it is consistent that every $(\omega+1)$-selective space is C-selective, and constructed an example of an $(\omega+1)$-selective but not $\mathbb{Q}$-selective space under the assumption $\mathfrak{p} = \mathfrak{c}$. In Section 3, we construct an example from $\mathfrak{d} = \omega_1$, an assumption which is in some sense orthogonal to $\mathfrak{p} = \mathfrak{c}$. In Section 4 we construct an $(\omega+1)$-selective space which is not selective for a P-point ultrafilter from the assumption of CH. This construction uses the notion of a tight gap in $[\omega]^\omega$.

The authors of [3] also asked whether it is consistent that every Fréchet space in which countable subspaces are first countable (CFC) is $(\omega+1)$-selective. This question was motivated on two fronts. Firstly, every GO-space and W-space is CFC; and secondly, there is a model of Dow and Steprans [2] where every countable Fréchet α_1-space is first countable and hence every $(\omega+1)$-selective space is CFC. They produced an example of a Fréchet CFC space which is not $(\omega+1)$-selective from $\mathfrak{p} = \omega_1$. In Section 5, we modify their example to waive the cardinal invariant assumption and provide a negative answer. Moreover, from an Aronszajn tree, we produce an example of size and character $\aleph_1$.

In this paper, the starred relations indicate that the relation is to be taken modulo finite, so for functions f, g with domain ω, $f <^* g$ if and only if $\{n : f(n) \geq g(n)\}$ is a finite subset of ω, and for sets A, B, $A \subseteq^* B$ if and only if $A \setminus B$ is finite. The cardinal $\mathfrak{d}$ is the least cardinality of a family of functions $\omega \to \omega$ which is cofinal in the $<^*$ ordering.

For a sequence $\langle B_\alpha : \alpha < \kappa \rangle$ of subsets of ω, a *pseudo-intersection* is a set $B \subset \omega$ so that $B \subseteq^* B_\alpha$ for all $\alpha < \kappa$. A *tower* is a sequence $\langle B_\alpha : \alpha < \kappa \rangle$ of subsets of ω so that $B_\beta \subseteq^* B_\alpha$ for $\alpha < \beta$, and there is no infinite pseudo-intersection, and $\mathfrak{p}$ is the least cardinality of a tower.

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2. Spaces obtained from an ultrafilter

To investigate the problem of constructing an $(\omega+1)$-selective not C-selective space, it is helpful to have on hand some candidates for countable regular spaces Y so that the space constructed is not Y-selective. A class of particularly simple spaces are those with a single nonisolated point. The collection of neighborhoods of the nonisolated point form a filter on the countable set of isolated points, so we call this space a *filter space*. Given a filter F on ω, let Y_F be the space with underlying set $\omega+1$ so that the points in ω are isolated and the neighborhoods of the point ω (which we call ∞ to avoid confusion with the set of isolated points) are given by F. We will use the abbreviation *F-selective* to denote the property of being Y_F-selective.

To satisfy F-selectivity, it suffices to consider only those lower semicontinuous functions φ so that $\varphi(\infty)$ is a singleton. To see this, take any lower semicontinuous function $\varphi : Y_U \to \mathcal{F}(X)$ and let $x \in \varphi(\infty)$. The function $\tilde{\varphi}$ defined so that $\tilde{\varphi}(n) = \varphi(n)$ for $n < \omega$ and $\tilde{\varphi}(\infty) = \{x\}$ remains lower semicontinuous, and a selection for $\tilde{\varphi}$ is also a selection for φ.

If F and G are filters on ω, then a map $f : \omega \to \omega$ extends to a continuous map $\hat{f} : Y_F \to Y_G$ so that $\hat{f}(\infty) = \infty$ precisely when f is a *Katětov reduction*, i.e., $f^{-1}[B] \in F$ for all $B \in G$, and we write $G \leq_K F$. The extension $\hat{f}$ is a quotient map precisely when f is a *Rudin–Keisler reduction*, i.e., $f^{-1}[B] \in F$ if and only if $B \in G$, and we write $G \leq_{RK} F$. The Rudin-Keisler and Katětov orders coincide on the collection of ultrafilters.

If U is an ultrafilter, then being U-selective can be interpreted in terms of the ultrapower. Let $M = \mathrm{Ult}(H(\theta), U)$ and $j : H(\theta) \to M$ be the ultrapower embedding, where θ is a regular cardinal sufficiently large so that $H(\theta)$ includes all involved topological spaces and their subsets.

A lower semicontinuous function $\varphi : Y_U \to \mathcal{F}(X)$ represents a closed set $[\varphi]$ in $j(X)$ so that for every open W intersecting $\varphi(\infty)$, $M \Vdash j(W) \cap [\varphi] \neq \emptyset$. A continuous selection is a point in $[\varphi] \cap \bigcap_W j(W)$, where W ranges over all open sets in V intersecting $\varphi(\infty)$.

The next proposition shows that lower semicontinuous functions from an ultrafilter space whose range consists of bounded finite sets admits a selection. It is motivated by the proof of Proposition 3.2 in [3] that $(\omega+1)$-selective spaces are α_1. In that proof, any space X which is not α_1 is shown to admit a lower semicontinuous function mapping $\omega+1$ (with the ordinal topology) to pairs of elements of X.

Proposition 1. *For every space X and ultrafilter U, if $\varphi : Y_U \to \mathcal{F}(X)$ is lower semicontinuous and* $\sup_{n<\omega} |\varphi(n)| < \omega$*, then φ has a continuous selection.*

Proof. For each n, enumerate $\varphi(n) = \{x_{n,0}, x_{n,1}, \ldots, x_{n,k_n}\}$. Denote $\sup_n |\varphi(n)|$ by N. Pick $x \in \varphi(\infty)$. For each open W neighborhood of x, let X_W be the set of $k \leq N$ so that $\{n : x_{n,k} \in W\} \in U$.

We claim that $\bigcap_W X_W \neq \emptyset$. Otherwise, for each $k \leq N$, there is some W_k so that $\{n : x_{n,k} \in W_k\} \notin U$. Now $\bigcap_{k \leq N} W_k$ is an open neighborhood of x, but $\{n : \varphi(n) \cap \bigcap_{k \leq N} W_k \neq \emptyset\} \notin U$, contradicting lower semicontinuity. $\square$

Let S_ω be the space on $(\omega \times \omega) \cup \{\infty\}$, thought of as ω-many spines $\{n\} \times \omega$, $n < \omega$, where neighborhoods of ∞ are those sets which are cofinite in each spine. In other words, a neighborhood basis for ∞ is given by cofinite sets and open sets W_f, $f \in {}^\omega\omega$, where $W_f = \{\infty\} \cup \{(m,n) : n > f(m)\}$. It is a typical example of a space which is Fréchet but not α_1, and hence it is not $(\omega+1)$-selective.

However, we will show that S_ω is U-selective for P-point ultrafilters U. Recall that U is a *P-point ultrafilter* if every function $\omega \to \omega$ is either constant or finite-to-one on a set in U. Alternatively, for any partition of ω into subsets P_n, $n < \omega$, where $P_n \notin U$, there are finite sets $p_n \subseteq P_n$ for each n so that $\bigcup_n p_n \in U$. This notion occurs in several different places here, and seems to be important for this study.

Proposition 2. *For every P-point ultrafilter U, S_ω is U-selective.*

Proof. Suppose $\varphi : Y_U \to \mathcal{F}(S_\omega)$ is a lower semicontinuous function. We may assume that $\varphi(\infty) = \{\infty\}$.

Let $M = \mathrm{Ult}(H(\theta), U)$ and $j : H(\theta) \to M$ be the ultrapower embedding. In the ultrapower, $j(S_\omega)$ can be described as $j(\omega)$ many spines, each of which is a copy of $j(\omega)$, together with $j(\infty)$.

Claim 2.1. The intersection of all $j(W)$, where $W \in V$ ranges over open neighborhoods of ∞, is the set $\{(n, \beta) : n < \omega \text{ and } \beta \text{ infinite}\}$.

Proof of Claim. If $W \in V$ and $n < \omega$, then W contains all points on the nth spine above some natural number m, so $j(W)$ does as well and in particular contains $\{(n, \beta) : \beta \text{ infinite}\}$. Suppose $(\alpha, \beta) \in j(S_\omega)$ with α infinite. Now α, β are represented by functions $a, b : \omega \to \omega$, respectively, and since U is a P-point, we may take a to be finite-to-one. Let $f : \omega \to \omega$ be defined as $f(i) = \max\{b(m) : i = a(m)\} + 1$, so that $f(a(n)) > b(n)$ for all n. Then $(\alpha, \beta) \notin j(W_f)$. This completes the proof of the claim. ©

If the set $[\varphi]$ contains a point (n, β), where $n < \omega$ is finite and $\beta \in j(\omega)$ is infinite, then this gives a continuous selection. Otherwise, by overspill—the principle that states that the set ω is not a member of M— $[\varphi]$ is bounded on $\{n\} \times \omega$ for each $n < \omega$, since if it were unbounded then it must also contain an infinite point in the nth spine. Let $h : \omega \to \omega$ be the function giving a bound for each n.

Again by overspill there is some infinite γ so that for every $\alpha < \gamma$ and every $\beta > j(h)(\alpha)$, $(\alpha, \beta) \notin [\varphi]$. Let $c : \omega \to \omega$ be a finite-to-one function representing γ in M.

Since $[\varphi]$ is closed, there is some $g : j(\omega) \to j(\omega)$ in M so that $[\varphi]$ intersects the αth spine only below $g(\alpha)$. If $\langle g_n : \omega \to \omega \rangle$ is a sequence of functions which represents g in M, then let $g' : \omega \to \omega$ be such that for each $n < \omega$, $g'(n)$ is greater than $h(n)$ and $g_m(n)$ for all m so that $c(m) \leq n$; there are only finitely many such m since c is finite-to-one.

Let $\alpha \in j(\omega)$. If $\alpha < \gamma$, then by the choice of γ, $[\varphi]$ intersects the αth spine only below $j(h)(\alpha)$. If $\alpha \geq \gamma$, then α is represented by a function $a \geq c$. Then $g_n(a(n)) \geq g'(a(n))$ (since $c(n) \leq a(n)$) and therefore $g(\alpha) < j(g')(\alpha)$.

In total, $\{n : \varphi(n) \cap W_{g'} \neq \emptyset\} \notin U$, contradicting lower semicontinuity of φ. □

3. $(\omega + 1)$-selective, not C-selective from a scale

Suppose $\mathfrak{d} = \omega_1$. This is exemplified by a sequence of functions $\langle f_\alpha : \alpha < \omega_1 \rangle$ so that $f_\alpha <^* f_\beta$ if $\alpha < \beta$, and for all $g \in {}^\omega\omega$ there is $\alpha < \omega_1$ so that $g <^* f_\alpha$.

Let $\{P_i : i < \omega\}$ be a partition of ω so that P_i is infinite for each i. Let

$$Z_\alpha = \{n < \omega : \exists i (n \in P_i \text{ and } n > f_\alpha(i))\}.$$

We define a space X whose underlying set is $(\omega \times \omega_1) \cup \{\infty\}$, where all points of $\omega \times \omega_1$ are isolated, and a local subbase at ∞ is generated by the following sets:

- $X \setminus (P_i \times \omega_1)$,
- $X \setminus (Z_\alpha \times \alpha)$.

Let $\pi_0 : X \to \omega$ and $\pi_1 : X \to \omega_1$ denote the projection onto the first and second coordinates, respectively.

Theorem 3 ($\mathfrak{d} = \omega_1$)**.** *Suppose U is a filter on ω so that $P_i \in I$ for all $i < \omega$ and $Z_\alpha \in I^+$ for all $\alpha < \omega_1$, where I is the dual ideal of U. Then X is Fréchet and $(\omega+1)$-selective but not U-selective.*

Proof. First, we show that X is not U-selective. Let $\psi : Y_U \to \mathcal{F}(X)$ be given by $\psi(n) = \{n\} \times \omega_1$ and $\psi(\infty) = \infty$. Note that $\psi(n)$ is closed for all $n < \omega$, since every n is contained in some P_i and then $\psi(n) \cap (X \setminus (P_i \times \omega_1)) = \emptyset$.

The function ψ is lower semicontinuous since for every neighborhood W of ∞, $\{n : (\{n\} \times \omega_1) \cap W \neq \emptyset\} \in U$. Suppose $s : Y_U \to X$ is a selection for ψ. Then there is $\alpha < \omega$ so that $\operatorname{ran}(s) \subseteq \alpha$. But $s^{-1}(X \setminus (Z_\alpha \times \alpha)) = \omega \setminus Z_\alpha \notin U$, so s is not continuous.

Now we show that X is $(\omega+1)$-selective. Suppose $\varphi : \omega+1 \to \mathcal{F}(X)$ is lower semicontinuous. We will define a continuous selection $s : \omega+1 \to X$. This is only nontrivial when $\infty \in \varphi(\infty)$ and in this case we will take $s(\infty) = \infty$. Furthermore, if $\infty \in \varphi(n)$ then we will take $s(n) = \infty$, so we assume that $\infty \notin \varphi(n)$ for all $n < \omega$. Let $M \prec (H(\theta); \in, <_\theta)$ be a countable elementary submodel with $\varphi, \{f_\alpha : \alpha < \omega_1\} \in M$ and let $\delta = \sup(M \cap \omega_1)$. Define

$$\pi_u(n) = \{i < \omega : \varphi(n) \cap \psi(i) \text{ is uncountable}\},$$

and

$$\alpha^* = \sup\{\alpha : (\exists n)(\exists i \notin \pi_u(n))\, (i, \alpha) \in \varphi(n)\}$$

and note that $\alpha^* \in M$, so $\alpha^* < \delta$.

Claim 3.1. Every countable subspace of X is first countable.

Proof of Claim. Suppose $A \subseteq X$ is countable. Then let $\gamma = \sup \pi_1[A]$. We claim that the sets $A \setminus (P_i \times \omega_1)$ and $A \setminus (Z_\beta \times \beta)$, where $\beta \leq \gamma$, form a local subbase at ∞ in A. For any α, we show that $A \setminus (Z_\alpha \times \alpha)$ contains a finite intersection of the basic open sets. This is clear for $\alpha \leq \gamma$, so assume that $\alpha > \gamma$. Since $Z_\gamma \subseteq^* Z_\alpha$, there is some j so that $Z_\gamma \setminus Z_\alpha \subseteq \bigcup_{i \leq j} P_i$. Then

$$A \setminus (Z_\alpha \times \alpha) = A \setminus (Z_\alpha \times \gamma) \supseteq (A \setminus (Z_\gamma \times \gamma)) \cap \left(A \setminus (\bigcup_{i \leq j} P_i \times \omega_1) \right).$$

This completes the proof of the claim. ⓒ

Continuing with the proof of $(\omega+1)$-selectivity, by using Claim 3.1 in M fix

$$\emptyset = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$$

closed subsets of $X \setminus (\omega \times [\alpha^*, \omega_1))$ whose complements form a neighborhood basis of ∞ in $X \setminus (\omega \times [\alpha^*, \omega_1))$. Let

$$V_j = (\bigcup_{i<j} P_i \times \omega_1) \cup F_j,$$

a closed subset of X. For each j, let $B_j = \{n : \varphi(n) \subseteq V_j\}$. By the lower semicontinuity of φ, B_j is finite for each j. Let $b(n)$ be the least j so that $n \in B_{j+1}$, if it exists, and undefined otherwise. Note that b is finite-to-one.

Each of the following functions is in M, and therefore $<^* f_\delta$:

- for each n, the function $d_n : i \mapsto \min(\pi_u(n) \cap P_i)$ (and we interpret the minimum as 0 if this set is empty).
- the function $c : i \mapsto \max\{\min(\pi_u(n) \cap P_i) : b(n) = i - 1\}$.

Let i^* be large enough so that $c(i) < f_\delta(i)$ for $i > i^*$, and i_n large enough so that $d_n(i) < f_\delta(i)$ for $i > i_n$. Let $\langle \delta_n : n < \omega \rangle$ be increasing and cofinal in δ. We will define a continuous selection s for φ. There are three cases:

Case 1: $b(n)$ exists and $P_{b(n)} \cap \pi_u(n) \neq \emptyset$. Choose $s(n) \in \varphi(n) \cap M$ with first coordinate equal to $\min(\pi_u(n) \cap P_{b(n)})$ and second coordinate above δ_n (and hence in the interval (δ_n, δ)), which exists by elementarity.

Case 2: $b(n)$ exists and $P_{b(n)} \cap \pi_u(n) = \emptyset$. Choose $s(n)$ to be a member of $\varphi(n) \setminus F_{b(n)}$ with second coordinate less than α^*. This exists since otherwise $\varphi(n) \subseteq F_{b(n)}$, so the choice of $b(n)$ would not have been minimal.

Case 3: $b(n)$ doesn't exist. In this case, $\varphi(n) \not\subseteq V_j$ for any $j < \omega$, so by the choice of α^* and the fact that the V_j restrict to a neighborhood basis of ∞ in $X \setminus (\omega \times [\alpha^*, \omega_1))$, we have that $\{i : \pi_u(n) \cap P_i \neq \emptyset\}$ is infinite. Choose $s(n) \in \varphi(n) \cap M$ with first coordinate equal to $\min(\pi_u(n) \cap P_i)$ for some $i > \max\{n, i_n\}$, and second coordinate above δ_n.

We now verify that s is a converging sequence. By the proof of the earlier claim, it suffices to check that each of $P_i \times \omega_1$ and $Z_\beta \times \beta$, $\beta \leq \delta$, contain $s(n)$ for at most finitely many n. Since $b(n)$ is finite-to-one, by the construction each subbasic set contains $s(n)$ for only finitely many n from Case 2, and sets of the form $P_i \times \omega_1$ only contain $s(n)$ for at most finitely many n from any case. Since in Cases 1 and 3 we chose $s(n)$ to have second coordinate greater than δ_n, $Z_\beta \times \beta$ contains $s(n)$ for only finitely many n if $\beta < \delta$.

It remains to show the result for $Z_\delta \times \delta$. Suppose $\pi_0(s(n)) \in Z_\delta$. This means that $\pi_0(s(n)) > f_\delta(i)$, where $\pi_0(s(n)) \in P_i$.

If n is in Case 1, $\pi_0(s(n)) = \min(\pi_u(n) \cap P_{b(n)})$. If $b(n) \geq i^*$, then $\min(\pi_u(n) \cap P_{b(n)}) < f_\delta(b(n))$ and so $b(n) < i^*$. Since b is finite-to-one, there are only finitely many such n.

If n is in Case 3, $\pi_0(s(n)) = \min(\pi_u(n) \cap P_i)$ for some $i > \max\{n, i_n\}$. By the choice of i_n, $\min(\pi_u(n) \cap P_i) = d_n(i) < f_\delta(i)$, a contradiction.

This finishes the proof that s is continuous.

Finally, we check that X is Fréchet. Suppose $\infty \in \overline{A}$. Let $\pi_u(A) = \{i : A \cap \psi(i) \text{ is uncountable}\}$ and $\alpha^* = \sup \pi_1[\{(i, \alpha) \in A : i \notin \pi_u(A)\}]$. If $\infty \in \overline{A \cap (\omega \times \alpha^*)}$ then we are done by Claim 3.1. Otherwise, by removing a countable set from A, we may assume that $\pi_u(A) = \pi_0[A]$. Let $\delta < \omega_1$ be such that:

- δ is a limit point of $\pi_1[A \cap \psi(i)]$ for every i,
- f_δ dominates the function $i \mapsto \min(\pi_0[A] \cap P_i)$ (where we interpret the minimum as 0 if this set is empty).

It is then straightforward to verify that any finite union of sets of the form $P_i \times \omega_1$ and $Z_\beta \times \beta$, where $\beta \leq \gamma$, do not contain all members of A. □

In [3] it was observed that every countable metrizable space embeds as a closed subspace of $\mathbb{Q}$, and therefore if a space is not selective for some countable metrizable space, then it is not $\mathbb{Q}$-selective.

Corollary 4 ($\mathfrak{d} = \omega_1$)**.** *X is not $\mathbb{Q}$-selective.*

Proof. Take U to be the dual filter of the ideal generated by $\{P_i : i < \omega\}$. Then Y_U is countable metrizable, and U satisfies the hypotheses of Theorem 3 and since X is not U-selective, it is not $\mathbb{Q}$-selective. □

Corollary 5 ($\mathfrak{d} = \omega_1$)**.** *X is not U-selective for any ultrafilter U on ω which is not a P-point.*

Proof. There is some partition $\{Q_i : i < \omega\}$ of ω so that each Q_i is infinite but not in U so that whenever $q_i \subseteq Q_i$ is finite, then $\bigcup_{i<\omega} q_i$ is not in U. By mapping Q_i bijectively to P_i, U is isomorphic to an ultrafilter which satisfies the hypotheses of Theorem 3. □

Corollary 6. *It is consistent relative to* ZFC *that there is an $(\omega + 1)$-selective space X which is not U-selective for any ultrafilter U.*

Proof. The model obtained by iterated Silver forcing gives an example. In that model, $\mathfrak{d} = \omega_1$, and there are no P-points by a recent result of Chodounský and Guzmán [1]. □

4. $(\omega + 1)$-selective, not C-selective from a tight gap

Gaps in $[\omega]^\omega$ have been used to find examples of interesting convergence properties starting with Nyikos [5]. In this section, we will use CH to construct $(\omega + 1)$-selective, not C-selective spaces based on a certain kind of gap. A tight gap is a sequence $\langle A_\alpha, B_\alpha : \alpha < \omega_1 \rangle$ so that:

- A_α, B_α are infinite subsets of ω,

- For all $\alpha < \beta$, $A_\alpha \subseteq^* A_\beta$ and $B_\beta \subseteq^* B_\alpha$ and $A_\beta \subseteq^* B_\alpha$,
- If $E \subseteq^* B_\alpha$ for all α, then there is β so that $E \subseteq^* A_\beta$.
- If $E \supseteq^* A_\alpha$ for all α, then there is β so that $E \supseteq^* B_\beta$.

A sequence $\langle A_\alpha, B_\alpha : \alpha < \gamma\rangle$ (where $\gamma \leq \omega_1$) which satisfies the first two conditions of the definition of a tight gap is called a *pre-gap*. We say that a set $E \subseteq \omega$ *interpolates* the pre-gap $\langle A_\alpha, B_\alpha : \alpha < \gamma\rangle$ if $A_\alpha \subseteq^* E \subseteq^* B_\alpha$ for every $\alpha < \gamma$. The final two conditions of the definition (whose conjunction is called "tightness") ensure that no set interpolates the tight gap.

Given a tight gap $\mathcal{G} = \langle A_\alpha, B_\alpha : \alpha < \omega_1\rangle$, we can construct a space $X_\mathcal{G}$ with underlying set $\omega + 1$, where the points of ω are isolated and the B_α generate the neighborhoods of ω. We denote the point ω by ∞ to avoid confusion with the set of isolated points.

Fact 4.1.
(1) An infinite set $E \subseteq \omega$ converges to ∞ if and only if there is some α so that $E \subseteq^* A_\alpha$.
(2) $X_\mathcal{G}$ is a Fréchet α_1 space.

Proof. (1) If $E \subseteq^* A_\alpha$ for some α then $E \subseteq^* B_\beta$ for all $\beta < \omega_1$, so E converges to ∞. Conversely, suppose E converges to ∞, so $E \subseteq^* B_\alpha$ for every $\alpha < \omega_1$. Then by tightness, we have that there is some α so that $E \subseteq^* A_\beta$.

(2) Suppose $\infty \in \overline{E}$ for some $E \subseteq \omega$. There must be $\alpha < \omega_1$ so that $E \cap A_\alpha$ is infinite, otherwise $\omega \setminus E \supseteq^* A_\alpha$ for all α, and by tightness there is a β so that $E \cap B_\beta$ is finite, contradicting $\infty \in \overline{E}$. By (1), $E \cap A_\alpha$ converges to ∞, and we conclude that $X_\mathcal{G}$ is Fréchet. If there are sequences E_n, $n < \omega$, all converging to ∞, then using (1) we have that for each n there is $\alpha(n)$ so that $E_n \subseteq A_{\alpha(n)}$. Letting $\alpha = \sup_n \alpha(n)$, we have that A_α is sequence converging to ∞ which almost contains every E_n. □

In Corollary 5, or by modifying Example 5.3 of [3], spaces which are $(\omega + 1)$-selective but not U-selective for an ultrafilter U are constructed. In those spaces, it seems essential that U is not a P-point. The tight gap allows us to construct an example for a P-point ultrafilter.

Theorem 7 (CH)**.** *There is a P-point ultrafilter U and a tight gap $\mathcal{G}$ so that $X_\mathcal{G}$ is $(\omega + 1)$-selective but not U-selective.*

Proof. By induction we will construct a tight gap $\langle A_\alpha, B_\alpha : \alpha < \omega_1\rangle$ and an ultrafilter U generated by a tower $\langle U_\alpha : \alpha < \omega_1\rangle$. Using CH, enumerate all infinite subsets of ω as $\langle E_\alpha : \alpha < \omega_1\rangle$. Let $\{\psi(i) : i < \omega\}$ be a partition of ω so that $\psi(i)$ is infinite for each i.

We will construct so that $A_\alpha \subseteq B_\alpha$ for all α. In the construction, we maintain the following:

(1) Either $U_\alpha \cap E_\alpha = \emptyset$ or $U_\alpha \subseteq E_\alpha$,
(2) For each i, $B_\alpha \cap \psi(i)$ is finite,
(3) For all k, $\{i : |B_\alpha \cap \psi(i)| > k\} \supseteq^* U_\alpha$,
(4) $\{i : A_\alpha \cap \psi(i) = \emptyset\} \supseteq U_\alpha$.

Condition (1) ensures that $\langle U_\alpha : \alpha < \omega_1\rangle$ generates an ultrafilter. (2) ensures that each $\psi(i)$ is closed, and holds for all α if it holds for $\alpha = 0$. (3) ensures that $\psi \cup \{(\infty, \infty)\}$ is lower semicontinuous as a function $Y_U \to \mathcal{F}(X_\mathcal{G})$. (4) is needed to make sure that no selection from ψ gives a converging sequence (but is not sufficient to show that $\psi \cup \{(\infty, \infty)\}$ does not admit a continuous selection).

At limit stages α, we will construct $A_\alpha \subseteq B_\alpha$, both of which interpolate the gap constructed so far.

Claim 4.1. Suppose γ is a countable limit ordinal and $\langle A_\alpha, B_\alpha : \alpha < \gamma\rangle$ is a pre-gap which satisfies the conditions (2)–(4). Then there are a pseudo-intersection U_γ and interpolations $A_\gamma \subseteq B_\gamma$ which satisfy the conditions as well.

Proof of Claim. It suffices to prove the claim for $\gamma = \omega$, as for arbitrary countable limit γ we can take $\langle \alpha_n : n < \omega\rangle$ an increasing sequence cofinal in γ, and an interpolation for $\langle A_{\alpha_n}, B_{\alpha_n} : n < \omega\rangle$ also interpolates the full pre-gap. By making finite modifications to each set in the pre-gap, we may also assume that $A_m \subseteq A_n \subseteq B_m \subseteq B_n$ for all $m < n < \omega$.

Define U_γ be a pseudo-intersection of $\{U_\beta : \beta < \gamma\}$ satisfying (1). The set U_γ is almost contained in $\{i : |B_n \cap \psi(i)| > k\}$ and in $\{i : A_n \cap \psi(i) = \emptyset\}$ for each $n, k < \omega$.

For each $n < \omega$, define inductively $g(n)$ so that $g(n)$ is the least natural number greater than or equal to $g(m)$ for all $m < n$ so that

$$U_\gamma \setminus g(n) \subseteq \{i : |B_n \cap \psi(i)| > n\} \cap \{i : A_n \cap \psi(i) = \emptyset\}$$

Now define

$$A_\omega = \bigcup_{n<\omega} (A_n \setminus \Psi(g(n)))$$

and

$$B_\omega = \bigcup_{n<\omega} (B_n \cap \Psi(g(n+1))),$$

where $\Psi(k) = \bigcup_{i \le k} \psi(i)$ for all k.

For any $n < \omega$, $A_n \setminus A_\omega \subseteq A_n \cap \Psi(g(n))$, which is finite by condition (2); so $A_\omega \supseteq^* A_n$. Furthermore, $B_\omega \setminus B_n \subseteq \Psi(g(n)) \cap B_0$, so $B_\omega \subseteq^* B_n$. Since $A_m \subseteq B_n$, for all m, n, we have $A_\omega \subseteq B_\omega$.

For any $k < \omega$, $n > k$, and $i \in U_\gamma$ at least $g(k)$, we have $|B_n \cap \psi(i)| > k$, so

$$\{i : |B_\omega \cap \psi(i)| > k\} \supseteq U_\gamma \setminus g(k).$$

For any $n < \omega$ and $i > g(n)$ we have $|A_n \cap \psi(i)| = \emptyset$, so

$$\{i : |A_\omega \cap \psi(i)| = \emptyset\} \supseteq U_\gamma.$$

This finishes the proof of the claim. ©

Let $\langle s_\alpha : \alpha < \omega_1 \rangle$ enumerate all selections of ψ and $\langle \varphi_\alpha : \alpha < \omega_1 \rangle$ enumerate all functions $\varphi : \omega \to \mathcal{P}(\omega) \setminus \emptyset$. At stage $\alpha + 1$, we construct $U_{\alpha+1}$, $A_{\alpha+1}, B_{\alpha+1}$ so that:

(5) there is $\beta \leq \alpha + 1$ so that $A_\beta \cap E_\alpha$ is infinite or $B_\beta \cap E_\alpha$ is finite.
(6) there is $\beta \leq \alpha + 1$ so that $E_\alpha \setminus A_\beta$ is finite or $E_\alpha \setminus B_\beta$ is infinite.
(7) there is $\beta \leq \alpha + 1$ so that either
- for all $k < \omega$, $\{i : (A_\beta \setminus k) \cap \varphi_\alpha(i) = \emptyset\}$ is finite, or
- there is $k < \omega$ so that $\{i : (B_\beta \setminus k) \cap \varphi_\alpha(i) = \emptyset\}$ is infinite.

(8) $\{i : s_\alpha(i) \in B_{\alpha+1}\} \cap U_{\alpha+1} = \emptyset$.

Conditions (5) and (6) ensure that the gap we construct at the end is tight. Condition (7) ensures that it is $(\omega + 1)$-selective. In the first case, let

$$k_i = \max\{k : (A_\beta \setminus k) \cap \varphi_\alpha(i) \neq \emptyset\},$$

and $i \mapsto \min((A_\beta \setminus k) \cap \varphi_\alpha(i))$ gives a continuous selection of φ_α defined on a cofinite set. In the second case, B_β witnesses that the extension of φ_α to $\omega + 1$ is not lower semicontinuous. Condition (8) ensures that s_α is not a continuous selection for ψ.

Now suppose we are at stage $\alpha + 1$. Let us take care of conditions (5)–(8). There are two cases corresponding to the different options in (7).

Case 1: There is an infinite set $Y \subseteq \omega$ so that

$$\sup\{|(B_\alpha \setminus \bigcup_{i \in Y} \varphi_\alpha(i)) \cap \psi(n)| : n \in U_\alpha\} = \omega.$$

In this case, let $B' = B_\alpha \setminus \bigcup_{i \in Y} \varphi_\alpha(i)$. Let $\langle n_k : k < \omega \rangle$ be a sequence of distinct natural numbers such that $n_k \in U_\alpha$ and $|B' \cap \psi(n_k)| > k$, and let $U' = \{n_k : k < \omega\}$.

Let $Z_0 = \emptyset$ if $B_\alpha \cap E_\alpha$ is finite, and otherwise let Z_0 be an infinite subset of $B_\alpha \cap E_\alpha$ so that $U' \setminus \psi^{-1}[Z_0]$ is infinite. Let $U'' = U' \setminus \psi^{-1}[Z_0]$. Let $Z_1 = \emptyset$ if $E_\alpha \setminus A_\alpha$ is finite, and otherwise let Z_1 be an infinite subset of $E_\alpha \setminus A_\alpha$ so that $U'' \setminus \psi^{-1}[Z_1]$ is infinite.

Take

$$A_{\alpha+1} = A_\alpha \cup Z_0,$$

$$B_{\alpha+1} = A_\alpha \cup (B' \setminus (s_\alpha[\omega] \cup Z_1))$$

and

$$U_{\alpha+1} = U'' \setminus \psi^{-1}[Z_1].$$

By the choice of Z_0, (4) holds. We check that (3) holds. By the choice of $U_{\alpha+1}$, $U_{\alpha+1} \subseteq \{n : Z_1 \cap \psi(n) = \emptyset\}$. Since s_α is a selection for ψ, for any k we have

$$\begin{aligned}\{n : |B_{\alpha+1} \cap \psi(n)| > k\} &\supseteq \{n \in U_{\alpha+1} : |B_\alpha \cap \psi(n)| > k+1\} \\ &= \{n_\ell \in U_{\alpha+1} : \ell > k+1\} \\ &\supseteq^* U_{\alpha+1}.\end{aligned}$$

Now (5), (6), (7), hold by the choices of Z_0, Z_1, and B', respectively. Finally, (8) holds since $s_\alpha[\omega]$ was subtracted off in the definition of $B_{\alpha+1}$ and so $\{n : s_\alpha(n) \in B_{\alpha+1}\} \cap U_{\alpha+1} \subseteq \{n \in U_\alpha : A_\alpha \cap \psi(n) \neq \emptyset\} = \emptyset$ by (4) at α.

Case 2: For every infinite set $Y \subseteq \omega$,

$$\sup\{|(B_\alpha \setminus \bigcup_{i \in Y} \varphi_\alpha(i)) \cap \psi(n)| : n \in U_\alpha\} < \omega.$$

Let $U' \subset U_\alpha$ be such that $|U'| = |U_\alpha \setminus U'| = \aleph_0$. Then for any $k < \omega$, the set

$$\{i : \varphi_\alpha(i) \cap (B_\alpha \setminus k) \subseteq \bigcup_{n \in U_\alpha \setminus U'} \psi(n)\}$$

is finite, otherwise there is some k so that $Y = \{i : \varphi_\alpha(i) \cap (B_\alpha \setminus k) \subseteq \bigcup_{n \in U_\alpha \setminus U'} \psi(n)\}$ is infinite. But then

$$\bigcup_{i \in Y} \varphi_\alpha(i) \cap B_\alpha \subseteq k \cup \bigcup_{n \in U_\alpha \setminus U'} \psi(n),$$

so $B_\alpha \setminus \bigcup_{i \in Y} \varphi_\alpha(i)$ contains $\psi(n) \cap (B_\alpha \setminus k)$ for all $n \in U'$. Therefore

$$\sup\{|(B_\alpha \setminus \bigcup_{i \in Y} \varphi_\alpha(i)) \cap \psi(n)| : n \in U_\alpha\} = \omega,$$

so we would have been in Case 1.

Let $A' = A_\alpha \cup (B_\alpha \cap \bigcup_{n \in U_\alpha \setminus U'} \psi(n))$. Define Z_0, Z_1, and U'' in exactly the same way as the previous case, except using the new definition of U'.

Take

$$A_{\alpha+1} = A' \cup Z_0,$$
$$B_{\alpha+1} = A_\alpha \cup (B_\alpha \setminus (s_\alpha[\omega] \cup Z_1))$$

and

$$U_{\alpha+1} = U'' \setminus \psi^{-1}[Z_1].$$

Similarly as in the previous case, all of our conditions are satisfied for these choices. $\square$

5. CFC, Fréchet, and not $(\omega+1)$-selective

In [3], the question of whether CFC Fréchet spaces are $(\omega+1)$-selective is posed. There, they use $\mathfrak{p}=\omega_1$ to construct an example of a CFC Fréchet space which is not $(\omega+1)$-selective, but the question of whether there is a ZFC example is left open. However, a slight modification of their example answers the question.

Let $\langle A_\alpha : \alpha < \mathfrak{p}\rangle$ be a tower. Define a space X on $(\omega\times\mathfrak{p})\cup\{\infty\}$, where all points of $\omega\times\mathfrak{p}$ are isolated, and a local subbase at ∞ is generated by the following sets:

- $X\setminus(\{n\}\times\mathfrak{p})$,
- $X\setminus(A_\alpha\times\alpha)$,
- $X\setminus(B\times\alpha)$, where $\mathrm{cf}\,(\alpha)>\omega$ and B is a pseudo-intersection of $\{A_\beta : \beta<\alpha\}$.

Theorem 8. *X is a* CFC*, Fréchet, and not $(\omega+1)$-selective space.*

Proof. Let $\varphi:\omega+1\to X$ be defined by $\varphi(\omega)=\infty$ and $\varphi(n)=\{n\}\times\mathfrak{p}$. Then φ is lower semicontinuous, since the neighborhood filter at ∞ is contained in the dual filter of the ideal Fin $\times$ Bounded, where Fin is the ideal of finite subsets of ω and Bounded is the ideal of bounded subsets of $\mathfrak{p}$. Suppose $s:\omega+1\to X$ is a selection for φ. Then there is $\alpha<\omega$ so that $\mathrm{ran}(s{\restriction}\omega)\subseteq\omega\times\alpha$. But $s^{-1}(X\setminus(A_\alpha\times\alpha))=\omega\setminus A_\alpha$ is co-infinite, so s is not continuous. Therefore, X is not $(\omega+1)$-selective.

Claim 5.1. Every countable subspace of X is first countable.

Proof of Claim. Suppose $A\subseteq X$ is countable, and to avoid trivialities we may assume that $\infty\in A$. Let a be the set of successors of elements of $\pi_1[A]$. We check that the sets $A\setminus(\{n\}\times\mathfrak{p})$ and $A\setminus(A_\alpha\times\alpha)$, where $\alpha\in\overline{a}$ (closure in the order topology on κ), form a neighborhood basis for ∞ in A.

To see this, we check that the intersection of A with any of the subbasic open neighborhoods of ∞ contains a finite intersection of sets of the form $A\setminus(\{n\}\times\mathfrak{p})$ and $A\setminus(A_\alpha\times\alpha)$, $\alpha\in\overline{a}$. The relevant cases are the neighborhoods V of the form $X\setminus(A_\beta\times\beta)$ and $X\setminus(B\times\beta)$, $\beta\geq\min(a)$, since if $\beta<\min(a)$ then $A\cap V=A$.

For a neighborhood $X\setminus(A_\beta\times\beta)$, let $\gamma\in\overline{a}$ be maximum so that $\gamma\leq\beta$. Then $A_\beta\subseteq^* A_\gamma$ and $A\cap(\omega\times\beta)=A\cap(\omega\times\gamma)$, so $A\setminus(A_\beta\times\beta)$ contains the intersection of $A\setminus(A_\gamma\times\gamma)$ with finitely many sets of the form $A\setminus(\{n\}\times\mathfrak{p})$.

Now consider a neighborhood $X\setminus(B\times\beta)$, where $\mathrm{cf}\,(\beta)>\omega$ and B is a pseudo-intersection of $\{A_\gamma : \gamma<\beta\}$. Let $\gamma\in\overline{a}$ be maximum so that $\gamma\leq\beta$. Since A is countable, $\overline{a}$ consists of ordinals which are either successors or have countable cofinality, so $\gamma<\beta$. By assumption,

$B \subseteq^* A_\gamma$, so $A \setminus (B \times \beta)$ contains the intersection of $A \setminus (A_\gamma \times \gamma)$ with finitely many sets of the form $A \setminus (\{n\} \times \mathfrak{p})$. Ⓒ

Claim 5.2. X is Fréchet.

Proof of Claim. Suppose $\infty \in \overline{A}$. By the previous claim, it suffices to find a countable subset of A which accumulates to ∞. We may assume that $\pi_1[A]$ has no maximum element and let $\alpha = \sup \pi_1[A]$. We may further assume that for every $\beta < \alpha$, $\infty \notin \overline{A \cap (\omega \times \beta)}$.

If $\mathrm{cf}\,(\alpha) = \omega$, let $\langle \alpha_n : n < \omega \rangle$ be a sequence cofinal in α. For each n, there must be some $x_n \in A$ such that $\pi_0(x_n) \notin n \cup A_\alpha$ and $\pi_1(x_n) \geq \alpha_n$, since $\infty \notin \overline{A \cap (\omega \times \alpha_n)}$. It can be easily checked that $\{x_n : n < \omega\}$ has finite intersection with any set of the form $\{n\} \times \mathfrak{p}$ or $B \times \beta$ for $\beta < \alpha$. And if $\beta \geq \alpha$, the closed sets of the form $B \times \alpha$ have $B \subseteq^* A_\alpha$, so again $B \times \alpha$ and $\{x_n : n < \omega\}$ have finite intersection, and we conclude that $\{x_n : n < \omega\}$ converges to ∞.

If $\mathrm{cf}\,(\alpha) > \omega$, then for every $\beta < \alpha$, $A \cap (\omega \times \beta)$ is covered by the union of $B \times \beta$ (where either $B =^* A_\alpha$ or B is a pseudo-intersection of $\{A_\delta : \delta < \beta\}$ so that $A_\beta \subseteq B$) together with finitely many basic closed sets of the form $B_\gamma \times \gamma$, $\gamma < \beta$. Let $\gamma(\beta)$ be the maximum of the finitely many γ which appear in this union. By Fodor's lemma, there is an unbounded set $S \subseteq \alpha$ so that $\gamma \restriction S$ is constant, say with value γ^*. Since $\infty \notin \overline{A \cap (\omega \times \gamma^*)}$, we may remove $\omega \times \gamma^*$ from A and assume that for every $\beta < \alpha$, there are $\delta \in S$ and B a pseudo-intersection of $\{A_\xi : \xi < \delta\}$ so that $A \cap (\omega \times \beta) \subseteq B \times \delta$.

By removing $\omega \times \alpha^*$, where α^* is the supremum of the ordinals $\sup\{\xi : (n, \xi) \in A\}$ which are less than α, we may assume that for every $n \in \pi_0[A]$, $\{\xi : (n, \xi) \in A\}$ is unbounded in α. By the work of the previous paragraph, $\pi_0[A]$ is contained in a pseudo-intersection B^* of $\{A_\xi : \xi < \alpha\}$. Therefore, A is contained in the closed set $B^* \times \alpha$, a contradiction. Ⓒ

□

The space constructed above is of cardinality and character $\mathfrak{p}$. We can also construct an example of size $\aleph_1$, even if $\mathfrak{p} > \omega_1$.

Theorem 9. *There is a Fréchet* CFC *space which is not* $(\omega+1)$*-selective with cardinality* $\aleph_1$*.*

Proof. Let T be an Aronszajn tree. Define an order-preserving map σ from $(T, <)$ into $([\omega]^\omega, \supseteq^*)$ by induction on the level, so that on each level of T the image of σ consists of pairwise almost-disjoint sets.

More explicitly, let $\sigma(\emptyset) = \omega$ and assume for $\alpha < \omega_1$ that σ is defined on all levels $< \alpha$. If $\alpha = \beta + 1$ is successor, then for each $s \in T_\beta$, partition $\sigma(s)$ into pairwise disjoint infinite sets $\{a_{s,i} : i < \omega\}$ and enumerate the set of $t > s$ on level α as $\{t_i : i < i^* \leq \omega\}$. Then put $\sigma(t_i) = a_{s,i}$.

If α is limit, then for each $t \in T_\alpha$, let $\sigma(t)$ be a pseudo-intersection of $\{\sigma(s) : s < t\}$, so that $\sigma(t) \subseteq^* \sigma(s)$ for all $s < t$.

Let I_α be the ideal on ω generated by $\{\sigma(t) : t \in T_\alpha\}$, i.e., $a \in I_\alpha$ if and only if a is contained in a finite union of sets of the form $\sigma(t), t \in T_\alpha$.

Notice that the sequence $\{I_\alpha : \alpha < \omega_1\}$ is decreasing. In addition, we have:

Claim 5.3. For every infinite set B, there is some $\alpha < \omega_1$ so that $B \notin I_\alpha$.

Proof of Claim. Assume towards a contradiction that there is a set B so that $B \in I_\alpha$ for every $\alpha < \omega_1$. Let $T_B = \{t \in T : \sigma(t) \cap B \text{ is infinite}\}$. T_B is a subtree of T, since if $t \in T_B$ and $s < t$, then $\sigma(s) \supseteq^* \sigma(t)$ and $\sigma(s) \cap B$ is infinite, so $s \in T_B$. The levels of T_B are finite, since $B \in I_\alpha$ and hence B is covered by a finite union of sets from the pairwise almost-disjoint family $\{\sigma(t) : t \in T_\alpha\}$. T_B has height ω_1.

Since T_B is a subset of the Aronszajn tree T, T_B has no cofinal branch. But any tree with finite levels and height ω_1 must have a cofinal branch, a contradiction (this fact can be seen by taking a uniform ultrafilter U on T_B, so $\{s \in T_B : \{t \in T_B : s < t\} \in U\}$ determines a cofinal branch). ©

Now we can define a space X_T whose underlying set is $\{\infty\} \cup (\omega \times \omega_1)$ so that points of $\omega \times \omega_1$ are isolated and the neighborhood filter for ∞ is generated by sets of the form:

(1) $X_T \setminus (n \times \omega_1)$, where $n < \omega$,
(2) $X_T \setminus (B \times \alpha)$, where $\alpha < \omega_1$ and $B \in I_\alpha$.

X_T is not $(\omega + 1)$-selective, as witnessed by the lower semicontinuous function $\varphi : \omega + 1 \to \mathcal{F}(X_T)$ where $\varphi(\omega) = \infty$ and $\varphi(n) = \{n\} \times \omega_1$. Any selection has countable range bounded by some $\alpha < \omega_1$, and I_α contains an infinite set, so there are no continuous selections.

It is clear that X_T is CFC. It remains to check that it is Fréchet, and since X_T is CFC, it suffices to show that X_T has countable tightness.

Suppose $\infty \in \overline{A}$. We will show that $\infty \in \overline{A'}$ for some countable $A' \subseteq A$. Let $\pi_0 : X_T \to \omega$ and $\pi_1 : X_T \to \omega_1$ denote the first and second projection, respectively. We may assume that the set $A_u = \{n : A \cap (\{n\} \times \omega_1) \text{ is uncountable}\}$ is equal to $\pi_0[A]$, since either

$$\infty \in \overline{A \cap (A_u \times \omega_1)}$$

or

$$\infty \in \overline{A \cap ((\omega \setminus A_u) \times \omega_1)},$$

and in the second case we are already done.

The set $\pi_0[A]$ must be infinite, so by the earlier claim there is some $\alpha < \omega_1$ so that $\pi_0[A] \notin I_\alpha$. By increasing α, we can take α to be a limit point of $\pi_1[A \cap (\{n\} \times \omega_1)]$ for all $n \in \pi_0[A]$.

Any basic open neighborhood of ∞ either contains

- $(n, \omega) \times (\beta, \alpha)$ for some $n < \omega$ and $\beta < \alpha$
- or $B \times \alpha$ for some $B \subseteq \pi_0[A]$ whose complement is in I_α.

Therefore, any neighborhood of ∞ intersects $A \cap (\omega \times \alpha)$, so $\infty \in \overline{A \cap (\omega \times \alpha)}$. □

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