http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 57, 2021

Pages 77–86

http://topology.nipissingu.ca/tp/

A short account of why Thompson's group F is of type F_∞

by

Matthew C. B. Zaremsky

Electronically published on June 13, 2020

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 57 (2021)
Pages 77-86

E-Published on June 13, 2020

A SHORT ACCOUNT OF WHY THOMPSON'S GROUP F IS OF TYPE F_∞

MATTHEW C. B. ZAREMSKY

ABSTRACT. In 1984 Brown and Geoghegan proved that Thompson's group F is of type F_∞, making it the first example of an infinite dimensional torsion-free group of type F_∞. Over the decades a different, shorter proof has emerged, which is more streamlined and generalizable to other groups. It is difficult, however, to isolate this proof in the literature just for F itself, with no complicated generalizations considered and no additional properties proved. The goal of this expository note then is to present the "modern" proof that F is of type F_∞, and nothing else.

Introduction and History. A *classifying space* for a group G is a CW complex Y with $\pi_1(Y) \cong G$ and $\pi_k(Y) = 0$ for all $k \neq 1$. If G admits a classifying space with finite n-skeleton, we say G is of *type F_n*. Equivalently, G is of type F_n if it admits a free, cocompact, cellular action on an $(n-1)$-connected CW complex. Being of type F_1 is equivalent to being finitely generated, and being of type F_2 is equivalent to being finitely presented. We say G is of *type F_∞* if it is of type F_n for all n. Thompson's group F was the first example of a torsion-free group of type F_∞ with no finite dimensional classifying space. Indeed, F cannot have a finite dimensional classifying space since it turns out to contain infinite rank free abelian subgroups, and so in some sense is an "infinite dimensional" group.

The fastest definition of F is via the infinite presentation

$$F = \langle x_0, x_1, x_2, \cdots \mid x_j x_i = x_i x_{j+1} \text{ for all } i < j\rangle.$$

2020 *Mathematics Subject Classification.* Primary 20F65; Secondary 57M07.
Key words and phrases. Thompson's group, finiteness properties, classifying space.

The endomorphism $F \to F$ sending each x_i to x_{i+1} is clearly idempotent up to conjugation, and so encodes a "free homotopy idempotent". This is another source of the group's fame, as it turns out F is the universal group encoding a free homotopy idempotent. This was proved by Freyd–Heller [10] (written in the 1970s, published in the 1990s) and, independently, Dydak [8]. The notation "F" originated from Freyd and Heller's work, standing for "free homotopy idempotent" (and later sometimes also standing for "Freyd–Heller group"). The group now called F actually predated Freyd and Heller's work, as it arose in Richard Thompson's work on logic in the 1960s (see, e.g., [13]), and so over the years has come to be universally called "Thompson's group F".

The original proof that F is of type F_∞ was given by Brown and Geoghegan in [4], making heavy use of the map $F \to F$ that is idempotent up to conjugation. This proof is very specific to F itself, but Brown subsequently found a new proof in [5] that generalized more easily to variations of F. This proof approach was then simplified and further generalized over the years by Stein [15], Farley [9], and others, in a variety of applications to families of "Thompson-like" groups. There are too many examples of this to list here, but lists of such examples can be found in, e.g., [16, 17].

By now a comparatively short, easy proof that F is of type F_∞ exists, thanks to all this work over the years, but isolating it in the literature is difficult. Many (but not all) of the most important steps can be found in [11, Section 9.3] or [6]. Also, one can sort out the full "modern" F_∞ proof for F from the (long) F_∞ proof for the braided Thompson groups in [3], but this requires quite a bit of effort.

The purpose of this note then is to present the most modern form of the F_∞ proof for Thompson's group F, and only for F, with no other groups considered and no other properties proved. The target audience is people interested in understanding the most basic situation, just for F, before venturing into more complicated generalizations.

Acknowledgments. Thanks are due to a number of people for encouraging me to write this up, including Brendan Mallery, David Rosenthal, Rachel Skipper, Rob Spahn, and Marco Varisco. I am also grateful to the anonymous referee for many excellent suggestions, which greatly improved the discussion of the history of F. This work is supported by grant #635763 from the Simons Foundation.

A: Trees and forests. Throughout this note, a *tree* will mean a finite rooted binary tree. A *forest* is a disjoint union of finitely many trees. The roots and leaves of a tree or forest are always ordered left to right.

The *trivial tree* is the tree with 1 leaf (which is also its root). A *trivial forest* is a forest each of whose trees is trivial. We denote the trivial forest with n roots (and hence n leaves) by 1_n. If we want to avoid specifying n, we will just write $1 = 1_n$.

A *caret* is a tree with 2 leaves. Given a forest $\mathfrak{f}$, a *simple expansion* of $\mathfrak{f}$ is a forest obtained by adding one new caret to $\mathfrak{f}$, with the root of the caret identified with a leaf of $\mathfrak{f}$. If it is the kth leaf, this is the *kth simple expansion* of $\mathfrak{f}$ (see Figure 1). An *expansion* of $\mathfrak{f}$ is recursively defined to be $\mathfrak{f}$ or a simple expansion of an expansion of $\mathfrak{f}$. Note that if $\mathfrak{f}$ and $\mathfrak{f}'$ have the same number of roots then (and only then do) they have a common expansion. For example any two trees have a common expansion.

FIGURE 1. A tree, and a simple expansion of the tree (namely the 2nd simple expansion).

B: The group. A *tree pair* is a pair $(\mathfrak{t}_-, \mathfrak{t}_+)$ where $\mathfrak{t}_\pm$ are trees with the same number of leaves. A *simple expansion* of a tree pair is a tree pair $(\mathfrak{t}'_-, \mathfrak{t}'_+)$ such that there exists k where $\mathfrak{t}'_\pm$ is the kth simple expansion of $\mathfrak{t}_\pm$. An *expansion* of $(\mathfrak{t}_-, \mathfrak{t}_+)$ is recursively defined to be $(\mathfrak{t}_-, \mathfrak{t}_+)$ or a simple expansion of an expansion of $(\mathfrak{t}_-, \mathfrak{t}_+)$. Thompson's group F is the set of equivalence classes $[\mathfrak{t}_-, \mathfrak{t}_+]$ of tree pairs $(\mathfrak{t}_-, \mathfrak{t}_+)$, with the equivalence relation generated by $(\mathfrak{t}_-, \mathfrak{t}_+) \sim (\mathfrak{t}'_-, \mathfrak{t}'_+)$ whenever $(\mathfrak{t}'_-, \mathfrak{t}'_+)$ is an expansion of $(\mathfrak{t}_-, \mathfrak{t}_+)$ (for more details on this and some equivalent definitions of F, including relating this to the infinite presentation given above, see, e.g., [7, 2]).

The point of expansions is that one can multiply equivalence classes $[\mathfrak{t}_-, \mathfrak{t}_+]$ and $[\mathfrak{u}_-, \mathfrak{u}_+]$ by expanding until without loss of generality $\mathfrak{t}_+ = \mathfrak{u}_-$, and then $[\mathfrak{t}_-, \mathfrak{t}_+][\mathfrak{u}_-, \mathfrak{u}_+] := [\mathfrak{t}_-, \mathfrak{u}_+]$. In this way, F is a group. The identity is $[1_1, 1_1]$ and the inverse of an element $[\mathfrak{t}_-, \mathfrak{t}_+]$ is $[\mathfrak{t}_+, \mathfrak{t}_-]$.

C: The groupoid. A *groupoid* is a set with all the axioms of a group except the product gh need not necessarily be defined for every pair of elements g, h. A standard example is the set of all square matrices, where two elements can be multiplied if and only if they have the same dimension. Thompson's group F naturally lives in the groupoid where we generalize trees to forests, which we describe now.

A *forest pair* is a pair $(\mathfrak{f}_-, \mathfrak{f}_+)$ where $\mathfrak{f}_\pm$ are forests with the same number of leaves. An *expansion* of a forest pair is defined analogously to an expansion of a tree pair, and we define equivalence of forest pairs similarly to equivalence of tree pairs. Let $\mathcal{F}$ be the set of all equivalence classes $[\mathfrak{f}_-, \mathfrak{f}_+]$ of forest pairs $(\mathfrak{f}_-, \mathfrak{f}_+)$. Since any two forests with the same number of roots have a common expansion, we can multiply two elements $[\mathfrak{f}_-, \mathfrak{f}_+]$ and $[\mathfrak{e}_-, \mathfrak{e}_+]$ of $\mathcal{F}$ provided the number of roots of $\mathfrak{f}_+$ and $\mathfrak{e}_-$ are the same. In this case we expand until $\mathfrak{f}_+ = \mathfrak{e}_-$ and then $[\mathfrak{f}_-, \mathfrak{f}_+][\mathfrak{e}_-, \mathfrak{e}_+] := [\mathfrak{f}_-, \mathfrak{e}_+]$. In this way, $\mathcal{F}$ is a groupoid. Note that the group F is a subgroupoid of $\mathcal{F}$.

D: The poset. Define a *split* to be an element of $\mathcal{F}$ of the form $[\mathfrak{f}, 1]$. For $[\mathfrak{f}_-, \mathfrak{f}_+] \in \mathcal{F}$, declare that $[\mathfrak{f}_-, \mathfrak{f}_+] \leq [\mathfrak{f}_-, \mathfrak{f}_+][\mathfrak{f}, 1]$ for any split $[\mathfrak{f}, 1]$ such that this product is defined.

Lemma 1. *The relation $\leq$ is a partial order.*

Proof. Clearly $\leq$ is reflexive, since any $[1_n, 1_n]$ is a split. A product of splits is itself a split, because any forest with n roots is an expansion of 1_n, so $\leq$ is transitive. Finally, a product of non-trivial splits is non-trivial since any expansion of a non-trivial forest is non-trivial, so $\leq$ is antisymmetric. $\square$

Let $\mathcal{F}_1$ be the subset of $\mathcal{F}$ consisting of all $[\mathfrak{t}, \mathfrak{f}]$ for $\mathfrak{t}$ a tree (and $\mathfrak{f}$ a forest with the same number of leaves as $\mathfrak{t}$). The groupoid product on $\mathcal{F}$ restricts to a left action of F on $\mathcal{F}_1$. It is clear that $\leq$ restricts to $\mathcal{F}_1$, and that this partial order on $\mathcal{F}_1$ is F-invariant, since left multiplication by an element of F commutes with right multiplication by a split. In this way, $\mathcal{F}_1$ is an F-poset.

The *geometric realization* $|P|$ of a poset P is the simplicial complex with a simplex for every chain $x_0 < \cdots < x_k$ of elements $x_i \in P$, with face relation given by taking subchains. A poset is *directed* if any two elements have a common upper bound. It is a standard fact that the geometric realization of a directed poset is contractible.

Lemma 2. *The poset $\mathcal{F}_1$ is directed, and so the geometric realization $|\mathcal{F}_1|$ is contractible.*

Proof. Note that $[\mathfrak{t},\mathfrak{f}][\mathfrak{f},1] = [\mathfrak{t},1]$, so any element of $\mathcal{F}_1$ has an upper bound of the form $[\mathfrak{t},1]$ for $\mathfrak{t}$ a tree. Given two such elements $[\mathfrak{t},1]$ and $[\mathfrak{u},1]$, let $\mathfrak{v}$ be a common expansion of $\mathfrak{t}$ and $\mathfrak{u}$, and now $[\mathfrak{v},1]$ is a common upper bound of $[\mathfrak{t},1]$ and $[\mathfrak{u},1]$. $\square$

Since the action of F on $\mathcal{F}_1$ is order preserving, it induces a simplicial action of F on the contractible complex $|\mathcal{F}_1|$.

Lemma 3. *The action of F on $|\mathcal{F}_1|$ is free.*

Proof. The action of F on $|\mathcal{F}_1|^{(0)} = \mathcal{F}_1$ is free, since it is an action of a subgroup of a groupoid on the groupoid by left translation. Since the action of F on $\mathcal{F}_1$ is order preserving, the stabilizer of the simplex $x_0 < \cdots < x_k$ lies in the stabilizer of x_0, hence is trivial. $\square$

E: The Stein complex. In [5] Brown used the action of F on $|\mathcal{F}_1|$ to give a new proof that F is of type F_∞, which generalized to many additional groups. The topological analysis in [5] was still quite complicated though. The complex $|\mathcal{F}_1|$ deformation retracts to a smaller, more manageable subcomplex X now called the *Stein complex.* This complex was first constructed by Stein in [15] (also see [6]), and simplified the F_∞ proof for F in [5] quite a bit.

To define X we need the notion of "elementary" forests, splits, and simplices. First, call a forest $\mathfrak{f}$ *elementary* if every tree in $\mathfrak{f}$ is either trivial or a single caret (see Figure 2). Call a split $[\mathfrak{f},1]$ *elementary* if $\mathfrak{f}$ is an elementary forest. If $x \in \mathcal{F}_1$ and s is a split, so $x \le xs$, then write $x \preceq xs$ if s is an elementary split. (Note that $\preceq$ is reflexive and antisymmetric, but not transitive.) Call a simplex $x_0 < \cdots < x_k$ in $|\mathcal{F}_1|$ *elementary* if $x_i \preceq x_j$ for all $i < j$. The elementary simplices form a subcomplex X, called the *Stein complex.* Note that X is invariant under the action of F.

FIGURE 2. An example of an elementary forest and a non-elementary forest.

Proposition 4. *The Stein complex X is homotopy equivalent to $|\mathcal{F}_1|$, hence is contractible.*

Proof. Given a forest $\mathfrak{f}$, there is a unique maximal elementary forest with $\mathfrak{f}$ as an expansion, namely the elementary forest whose kth tree is non-trivial (hence a caret) if and only if the kth tree of $\mathfrak{f}$ is non-trivial, for each k. Call this the *elementary core* of $\mathfrak{f}$, denoted $\mathrm{core}(\mathfrak{f})$. Note that if $\mathfrak{f}$ is non-trivial then so is $\mathrm{core}(\mathfrak{f})$. If $\mathfrak{e} = \mathrm{core}(\mathfrak{f})$, call $[\mathfrak{e}, 1]$ the *elementary core* of $[\mathfrak{f}, 1]$, and write $\mathrm{core}([\mathfrak{f}, 1]) := [\mathfrak{e}, 1]$. Now let $x \leq z$ with $x \not\preceq z$, and consider $(x, z) := \{y \mid x < y < z\}$. Since any $y \in (x, z)$ is of the form xs for s a non-trivial split, we can define a map $\phi\colon (x, z) \to (x, z)$ via $\phi(xs) := x\,\mathrm{core}(s)$. This is clearly a poset map that restricts to the identity on its image, and satisfies $\phi(y) \leq y$ for all y. Finally, note that $\phi(y) \leq \phi(z) \in (x, z)$ for all y. Standard poset theory (see, e.g., [14, Section 1.5]) now tells us that $|(x, z)|$ is contractible (intuitively, ϕ "retracts" it to a cone on the point $\phi(z)$).

Now our goal is to build up from X to $|\mathcal{F}_1|$ by gluing in the missing simplices, in such a way that whenever we add a new simplex it is along a contractible relative link, which will imply that $X \simeq |\mathcal{F}_1|$. The missing simplices are precisely the non-elementary ones. Let us actually glue in all the non-elementary simplices in chunks, by gluing in (contractible) subcomplexes of the form $|\{y \mid x \leq y \leq z\}|$ for $x < z$ non-elementary. We glue these in, in order of increasing $f(z) - f(z)$ value, where $f\colon \mathcal{F}_1 \to \mathbb{N}$ sends $[\mathfrak{t}, \mathfrak{f}]$ to the number of roots of $\mathfrak{f}$. (Think of $f(z) - f(x)$ as the number of carets in the split taking x to z.) When we glue in $|\{y \mid x \leq y \leq z\}|$, the relative link is $|\{y \mid x \leq y < z\}| \cup |\{y \mid x < y \leq z\}|$. This is the suspension of $|\{y \mid x < y < z\}|$, which is contractible by the first paragraph of the proof. □

Note that the action of F on $|\mathcal{F}_1|$ restricts to an action of F on X.

F: The Stein–Farley cube complex. The Stein complex X is easier to use than $|\mathcal{F}_1|$, but there is one further simplification that makes it still easier, namely, the simplices of X can be glommed together into cubes, making X a cube complex. This was observed by Stein in [15] and further developed in [9], where X was shown to even be a CAT(0) cube complex.

Given $x \preceq z$, say $z = xs$ for $s = [\mathfrak{e}, 1]$ an elementary split, the set $\{y \mid x \leq y \leq z\}$ is a boolean lattice. This is because the forests $\mathfrak{e}'$ with $x[\mathfrak{e}', 1] \leq x[\mathfrak{e}, 1]$ are all obtained by assigning a 0 or a 1 to each caret in $\mathfrak{e}$ and including said caret in $\mathfrak{e}'$ if and only if it was assigned a 1. The geometric realizations of these boolean lattices, which are metric cubes, cover X, and any non-empty intersection of such cubes is itself such a cube, so in this way X has the structure of a cube complex. When we view X as a cube complex instead of a simplicial complex, we will call it the *Stein-Farley complex.*

The action of F on X takes cubes to cubes, so F acts cellularly on the Stein-Farley complex.

G: Sublevel complexes. At this point we have a free cellular action of F on the contractible cube complex X. If the orbit space were of finite type, meaning if it had finitely many cells per dimension, then we would be done proving F is of type F_∞. In fact this is not the case, but X does admit a natural filtration into cocompact subcomplexes that are increasingly highly connected, as we now explain.

Let $f\colon \mathcal{F}_1 \to \mathbb{N}$ be the function from the proof of Proposition 4, so $f([\mathsf{t},\mathfrak{f}])$ equals the number of roots of $\mathfrak{f}$. Note that f is F-invariant. For each $m \in \mathbb{N}$ let $X^{f\le m}$ be the full subcomplex of X spanned by vertices $x \in X^{(0)} = \mathcal{F}_1$ with $f(x) \le m$. The $X^{f\le m}$ are called *sublevel complexes*. Note that the $X^{f\le m}$ are nested, and their union is all of X, so they form a filtration of X. Each $X^{f\le m}$ is F-invariant.

Lemma 5. *Each $X^{f\le m}$ is cocompact under the action of F.*

Proof. Since there are only finitely many elementary forests with a given number of roots or a given number of leaves, X is locally finite. Hence it suffices to show $X^{f\le m}$ has finitely many F-orbits of vertices, and for this we claim that if $x, x' \in X^{(0)}$ with $f(x) = f(x')$ then $F.x = F.x'$. Indeed, $f(x) = f(x')$ ensures that $x'x^{-1}$ is an allowable product in $\mathcal{F}$, and clearly $x'x^{-1} \in F$ with $(x'x^{-1})x = x'$. $\square$

To summarize, for each $m \in \mathbb{N}$, F acts freely, cocompactly, and cellularly on $X^{f\le m}$. To show that F is of type F_∞, i.e., of type F_n for all n, it just remains to show that for each n there exists m such that $X^{f\le m}$ is $(n-1)$-connected.

Let $\nu(m) := \left\lfloor \dfrac{m-2}{3} \right\rfloor$.

Proposition 6. *The complex $X^{f\le m}$ is $(\nu(m+1)-1)$-connected.*

We will prove Proposition 6 shortly. First let us see why we will be done after this.

Theorem 7. *F is of type F_∞.*

Proof. For each $m \in \mathbb{N}$, F acts freely, cocompactly, and cellularly on the $(\nu(m+1)-1)$-connected complex $X^{f\le m}$. Hence F is of type $\mathrm{F}_{\nu(m+1)}$ for all m. Since $\nu(m+1)$ goes to ∞ as m goes to ∞, F is of type F_∞. $\square$

H: Descending links. To prove Proposition 6 we will use Bestvina–Brady Morse theory (see [1]). Given an affine cell complex Y, e.g., a simplicial or cube complex, a map $h\colon Y \to \mathbb{R}$ is a *Morse function* if h is affine on cells, non-constant on edges, and discrete on vertices. Given a Morse function $h\colon Y \to \mathbb{R}$ and a cell C in Y, h achieves its maximum value on C at a unique vertex, called the *top* of C. The *descending link* $\mathrm{lk}^{\downarrow}y$ of a vertex $y \in Y^{(0)}$ is the link of y in all the cells with y as their top. The point of Morse theory is that a sufficient understanding of descending links can translate into knowledge about sublevel complexes (see [1, Corollary 2.6]).

Proof of Proposition 6. We can extend $f\colon X^{(0)} \to \mathbb{N}$ to a map $f\colon X \to \mathbb{R}$ by extending affinely to each cube, and this is a Morse function. Since X is contractible, it now suffices by [1, Corollary 2.6] to prove that for every $x \in X^{(0)}$ with $f(x) > m$, the descending link $\mathrm{lk}^{\downarrow}x$ is $(\nu(m+1)-1)$-connected. The descending link of x is the simplicial complex with a k-simplex for each $x' = x[1, \mathfrak{e}]$, where $\mathfrak{e}$ is an elementary forest with $k+1$ carets and $f(x)$ leaves, with face relation given by removing carets. (For this, it is important that we are using the cubical structure on X, not the simplicial structure.) If $f(x) = n$ then this is isomorphic to the matching complex on the graph L_n. Here L_n is the graph with vertex set $\{1, \dots, n\}$ and an edge $\{i, i+1\}$ for each $1 \le i \le n-1$, and the *matching complex* $\mathcal{M}(\Gamma)$ of a graph Γ is the simplicial complex with a simplex for each non-empty finite collection of pairwise disjoint edges of Γ with face relation given by inclusion (see Figure 3).

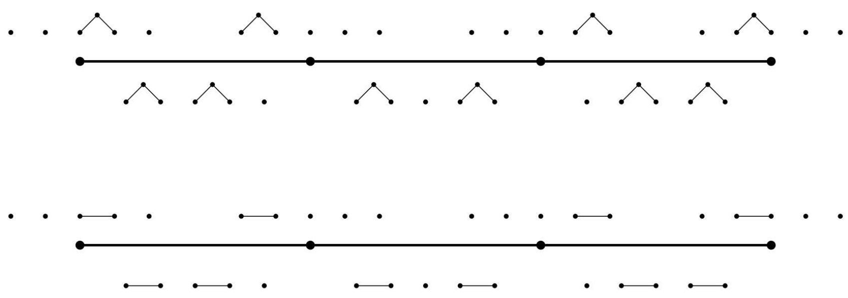

FIGURE 3. The correspondence between $\mathrm{lk}^{\downarrow}x$ with $f(x) = 5$ (the top picture, with the forests $\mathfrak{e}$ representing the simplices) and $\mathcal{M}(L_5)$ (the bottom picture).

Since $n > m$, it now suffices to show that $\mathcal{M}(L_n)$ is $(\nu(n)-1)$-connected.

This is well known (see, e.g., [12, Proposition 11.16]), but it is easy to prove so we present a proof here. We will induct on n to prove that this holds, and that moreover $\mathcal{M}(L_n)$ is contractible whenever $n \equiv 2 \mod 3$, and that the inclusion $\mathcal{M}(L_{n-1}) \to \mathcal{M}(L_n)$ is a homotopy equivalence whenever $n \equiv 1 \mod 3$. As a base case we can check "by hand" that $\mathcal{M}(L_n)$ is non-empty (i.e., (-1)-connected) for $n \geq 2$, $\mathcal{M}(L_2)$ is contractible, and $\mathcal{M}(L_3) \to \mathcal{M}(L_4)$ is a homotopy equivalence. Now assume $n \geq 5$. Clearly $\mathcal{M}(L_n)$ is isomorphic to $\mathcal{M}(L_{n-1})$ union the star of $\{n-1, n\}$, and the intersection of $\mathcal{M}(L_{n-1})$ with this star is $\mathcal{M}(L_{n-2})$. Hence $\mathcal{M}(L_n)$ is homotopy equivalent to the mapping cone of the inclusion $\mathcal{M}(L_{n-2}) \to \mathcal{M}(L_{n-1})$. If $n \equiv 0, 1 \mod 3$ then $\nu(n-1) = \nu(n)$, so $\mathcal{M}(L_{n-1})$ is $(\nu(n)-1)$-connected, and moreover $\mathcal{M}(L_{n-2})$ is $(\nu(n)-2)$-connected, so $\mathcal{M}(L_n)$ is $(\nu(n)-1)$-connected (this follows for example from Van Kampen, Mayer–Vietoris, and Hurewicz). If $n \equiv 2 \mod 3$ then the inclusion $\mathcal{M}(L_{n-2}) \to \mathcal{M}(L_{n-1})$ is a homotopy equivalence, so $\mathcal{M}(L_n)$ is contractible. Lastly, if $n \equiv 1 \mod 3$ then $\mathcal{M}(L_{n-2})$ is contractible, so the inclusion $\mathcal{M}(L_{n-1}) \to \mathcal{M}(L_n)$ is a homotopy equivalence. □

References

[1] Mladen Bestvina and Noel Brady, *Morse theory and finiteness properties of groups*, Invent. Math., **129**(3) (1997), 445–470.

[2] James Michael Belk, *Thompson's group F*. ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–Cornell University.

[3] Kai-Uwe Bux, Martin G. Fluch, Marco Marschler, Stefan Witzel, and Matthew C. B. Zaremsky, *The braided Thompson's groups are of type* F_∞, J. Reine Angew. Math., **718** (2016), 59–101. With an appendix by Zaremsky.

[4] Kenneth S. Brown and Ross Geoghegan, *An infinite-dimensional torsion-free* FP_∞ *group*, nvent. Math., **77**(2) (1984), 367–381.

[5] Kenneth S. Brown, *Finiteness properties of groups*, In Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985), volume 44, pages 45–75, 1987.

[6] ———, *The geometry of finitely presented infinite simple groups*, In *Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989)*, volume 23 of Math. Sci. Res. Inst. Publ., pages 121–136. Springer, New York, 1992.

[7] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Introductory notes on Richard Thompson's groups*, Enseign. Math. (2), **42**(3-4) (1996), 215–256.

[8] Jerzy Dydak, *A simple proof that pointed FANR-spaces are regular fundamental retracts of ANR's*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **25**(1) (1977), 55–62.

[9] Daniel S. Farley, *Finiteness and* CAT(0) *properties of diagram groups*, Topology, **42**(5) (2003), 1065–1082.

[10] Peter Freyd and Alex Heller, *Splitting homotopy idempotents. II*, J. Pure Appl. Algebra, **89** (1-2) (1993), 93–106.

[11] Ross Geoghegan, *Topological methods in group theory*, volume 243 of Graduate Texts in Mathematics. Springer, New York, 2008.

[12] Dmitry Kozlov, *Combinatorial algebraic topology*, volume 21 of Algorithms and Computation in Mathematics. Springer, Berlin, 2008.

[13] Ralph McKenzie and Richard J. Thompson, *An elementary construction of unsolvable word problems in group theory*, In *Word problems: decision problems and the Burnside problem in group theory (Conf., Univ. California, Irvine, Calif. 1969; dedicated to Hanna Neumann)*, volume 71 of Studies in Logic and the Foundations of Math., pages 457–478, 1973.

[14] Daniel Quillen, *Homotopy properties of the poset of nontrivial p-subgroups of a group*, Adv. in Math., **28**(2) (1978), 101–128.

[15] Melanie Stein, *Groups of piecewise linear homeomorphisms*, Trans. Amer. Math. Soc., **332**(2) (1992), 477–514.

[16] Rachel Skipper, Stefan Witzel, and Matthew C. B. Zaremsky, *Simple groups separated by finiteness properties*, Invent. Math., **215**(2) (2019), 713–740.

[17] Stefan Witzel, *Classifying spaces from Ore categories with Garside families*, Algebr. Geom. Topol., **19**(3) (2019), 1477–1524.

Department of Mathematics and Statistics, University at Albany (SUNY), Albany, NY 12222

Email address: `mzaremsky@albany.edu`