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ON CAUCHY COVERING MAPS AND COMPLETE METRIC SPACES

SUDIP KUMAR PAL AND NAYAN ADHIKARY

Abstract. A function $f : (X, d_X) \to (Y, d_Y)$ is called Cauchy continuous iff for each Cauchy sequence $\{x_n\}$ in X, $\{f(x_n)\}$ is a Cauchy sequence in Y. In this article we consider the reverse implication: i.e. whenever $\{y_n\}$ is a Cauchy sequence in Y, there is a Cauchy sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$, call such a function a Cauchy covering map, and study related properties. Cauchy covering maps can be viewed also as a generalization of so called sequence covering maps. We also give an almost necessary and sufficient condition for a metric space to be complete in terms of Cauchy covering maps. We further study some generalizations of Cauchy continuous maps.

1. Introduction

In this paper we consider the class of spaces that are continuous images of metric spaces. The study of this class has been well studied over the past 50 years. In 1971, Swiec[13, 14] introduced the concept of sequence covering maps which is closely related to the question about compact covering and s-images of metric spaces. In [9], Lin discussed about sequence covering maps and its properties, also solved many open problems related to this concept.

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The notion of statistical convergence, which is an extension of the idea of usual convergence, was introduced by Fast[5] and Schoenbeg[12](see also [15]). A lot of investigations have been done on the applications of statistical convergence in topological spaces after the initial works by Fridy[6] and Šalát[11]. It seems more appropriate to follow the more general approach of [8] where the notion of $\mathcal{I}$-convergence of sequences was introduced for $\mathcal{I}$, an ideal on the set of positive integers.

In the context of real valued functions defined on metric spaces, it is known that the class of all uniformly continuous functions is a proper subset of the class of continuous functions. Between these two basic classes of continuous functions lies the class of Cauchy continuous functions, i.e. the functions that map Cauchy sequences in the domain to Cauchy sequences in the target space. Call a sequence in a metric space cofinally Cauchy if for each positive ε there exists a cofinal (rather than residual) set of indices whose corresponding terms are ε-close. A metric space is said to be cofinally complete if each cofinally Cauchy sequence has a cluster point. In [1], Aggarwal studied the class of Cauchy continuity and the class of cofinally Cauchy continuity, investigated their properties and gave some characterizations of cofinally complete metric spaces.

In section 3.1 as a natural consequence we consider the notion of Cauchy covering functions. This concept can be viewed as not only a converse implication of Cauchy continuous functions but also a generalization of sequence covering maps. We show that in general Cauchy covering and sequence covering maps are independent and impose some conditions under which these concepts can relate to each other, for example, the two notions coincide for the class of complete metric spaces. We also give an equivalent condition for a metric space to be complete in terms of the Cauchy covering maps. We further introduce some general classes of Cauchy covering maps namely, Cofinally Cauchy covering maps and $\mathcal{I}$-Cauchy covering maps and investigate their relation to Cauchy covering maps. The definition of sequence covering map is analogous to the definition of compact covering map but they are mutually independent. We also introduce the concept of pre-compact covering maps as a generalization of compact covering maps and examined its relation to Cauchy covering maps. We give various counterexamples to illustrate that many implications are proper.

In section 3.2 we consider some general classes of Cauchy continuous functions, namely, statistical-Cauchy continuous and ideal-Cauchy continuous functions. We characterize all those ideals of natural numbers for which these concepts are equivalent. In [7], Ghosal considered a similar variation of ideal-Cauchy continuity which he denoted as ideal-uniformly bounded maps. However, the author's claim about their Example 1 in

[7] is incorrect. In section 3.2 we correct this by constructing the needed counterexample. All spaces in the sequel are metric and all subsets of $\mathbb{R}$ have the usual metric. So the mention of the metric will be often omitted except when it is necessary to be clear which metric we consider.

2. Preliminaries:

We first recall the basic definitions of ideals and filters from [2, 8]. A family $\mathcal{I} \subset 2^Y$ of subsets of a non-empty set Y is said to be an ideal in Y if $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$. Further an admissible (or free) ideal $\mathcal{I}$ of Y satisfies $\{x\} \in Y$ for each $x \in Y$. An admissible ideal $\mathcal{I}$ is said to satisfy the condition (AP) (or is called a P-ideal or sometimes an AP-ideal) if, for every countable family of mutually disjoint sets $(A_1, A_2, ...)$ of $\mathcal{I}$, there exists a countable family of sets $(B_1, B_2, ...)$ such that $A_j \triangle B_j$ is finite for each $j \in \mathbb{N}$ and $\cup B_k \in \mathcal{I}$. Whereas $\mathcal{I}$ is called maximal if there does not exist any non-trivial proper ideal, properly containing $\mathcal{I}$. An ideal $\mathcal{I}$ is said to be a tall ideal if each infinite subset of $\mathbb{N}$ contains an infinite element of $\mathcal{I}$. If $\mathcal{I}$ is a proper non-trivial ideal in Y (i.e. $Y \notin \mathcal{I}$, $\mathcal{I} \neq \emptyset$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y : \exists A \in \mathcal{I}$ such that $M = Y \setminus A\}$ is called the filter associated with the ideal $\mathcal{I}$.

Definition 2.1. [8] A sequence $\{x_n\}$ in a space X is said to be $\mathcal{I}$-convergent to $x \in X$ (i.e $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = x$) if for every neighbourhood U of x, $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$.

Definition 2.2. [4] Let (X, d_X) be a metric space and $\mathcal{I}$ be an admissible ideal.
(1) A sequence $\{x_n\}$ in X is said to be $\mathcal{I}$-Cauchy if for every $\varepsilon > 0 \; \exists \; N \in \mathbb{N}$ such that $\{n : d_X(x_N, x_n) \geq \varepsilon\} \in \mathcal{I}$.
(2) A sequence $\{x_n\}$ in X is said to be $\mathcal{I}^*$-Cauchy if there exists a set $M = \{m_k : k \in \mathbb{N}\} \subset \mathbb{N},\ M \subset \mathcal{F}(\mathcal{I})$ such that the subsequence $\{x_{m_k}\}$ is an ordinary Cauchy sequence.

It is easy to check that when $\mathcal{I} = \mathcal{I}_d$ (the class of all subsets from $\mathbb{N}$ with natural density zero) we get the statistical version.

Definition 2.3. [1] Let (X, d_X) be a metric space. Then a sequence $\{x_n\}$ in X is said to be cofinally Cauchy in X if for every $\varepsilon > 0 \; \exists$ an infinite subset $\mathbb{N}_\varepsilon$ of $\mathbb{N}$ such that for each $i, j \in \mathbb{N}_\varepsilon,\ d_X(x_i, x_j) < \varepsilon$.

Definition 2.4. [1] A metric space (X, d_X) is said to be cofinally complete if every cofinally Cauchy sequence has a cluster point in X.

Definition 2.5. [1] A function $f : (X, d_X) \to (Y, d_Y)$ is said to be Cauchy continuous if for each Cauchy sequence $\{x_n\}$ in X, $\{f(x_n)\}$ is a Cauchy sequence in Y.

Definition 2.6. [9] Let $f : X \to Y$ be a map. Then f is said to be a sequence covering map if for every convergent sequence $\{y_n\}$ converges to y in Y, there is a convergent sequence $\{x_n\}$ converges to x in X with each $x_n \in f^{-1}(y_n)$ and $x \in f^{-1}(y)$.

Definition 2.7. [9] Let $f : X \to Y$ be a map. Then f is said to be a *sequentially quotient* map if for each convergent sequence $\{y_n\}$ converges to y in Y there is a convergent sequence $\{x_k\}$ converges to x in X with $f(x) = y$ and $f(x_k) = y_{n_k}$, for each k, where $\{y_{n_k}\}$ is a subsequence of $\{y_n\}$.

Definition 2.8. [9] Let $f : (X, d_X) \to (Y, d_Y)$ be a map. Then f is said to be a compact covering map if every compact subset of Y is the image of some compact subset of X.

3. Main Results

3.1. **Cauchy covering maps:** In this section we consider the notion of Cauchy covering maps and investigate its relation with the sequence covering maps. Throughout the paper (X, d_X) stands for a metric space and $\mathcal{I}$ is a non-trivial admissible ideal of $\mathbb{N}$. We first introduce the following definitions:

Definition 3.1. Let $f : (X, d_X) \to (Y, d_Y)$ be a map. Then f is said to be a Cauchy covering map if for each Cauchy sequence $\{y_n\}$ in Y, there is a Cauchy sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$.

Definition 3.2. Let $f : (X, d_X) \to (Y, d_Y)$ be a map. Then f is said to be a cofinally Cauchy covering map if whenever $\{y_n\}$ is a cofinally Cauchy sequence in Y then there is a cofinally Cauchy sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$.

Definition 3.3. [10] Let $f : (X, d_X) \to (Y, d_Y)$ be a map. Then f is said to be an $\mathcal{I}$-sequence covering map if for each $\mathcal{I}$-convergent sequence $\{y_n\}$ with $\mathcal{I}$ limit y in Y, there is an $\mathcal{I}$-convergent sequence $\{x_n\}$ with $\mathcal{I}$ limit x in X with each $x_n \in f^{-1}(y_n)$ and $x \in f^{-1}(y)$.

Definition 3.4. Let $f : (X, d_X) \to (Y, d_Y)$ be a map. Then f is said to be an $\mathcal{I}$-Cauchy covering map if for each $\mathcal{I}$-Cauchy sequence $\{y_n\}$ in Y, there is an $\mathcal{I}$-Cauchy sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$.

Definition 3.5. Let $f : (X, d_X) \to (Y, d_Y)$ be a map. Then f is said to be a precompact covering map if every totally bounded subset of Y is the image of some totally bounded subset of X.

In general the concepts of Cauchy covering maps and sequence covering maps are independent. The following examples are in this direction.

Example 3.1. Let $Y = \{y_n\} \cup \{y\}$, where $\{y_n\}$ is a real valued convergent sequence with limit y. Then Y is homeomorphic to the metric space $\{\frac{1}{n}\} \cup \{0\}$ with the usual metric of $\mathbb{R}$. Also let $X = (0, 1]$. Now define $f : X \to Y$ by $f(\frac{1}{2n}) = y_n$ for each $n \in \mathbb{N}$ and $f(x) = y$, otherwise. Any Cauchy sequence in Y is the image of some subsequence of $\{\frac{1}{n}\}$. So f is a Cauchy covering map. Furthermore every subset of X is precompact and hence f is a precomact covering map. Further $\{y_n\}$ converges to y in Y but any sequence $\{x_n\}$ in X with $x_n \in f^{-1}(y_n)$ is precisely the sequence $\{\frac{1}{2n}\}$, which is not convergent in X. Therefore f is not a sequence covering map and also not a compact covering map.

The above example also shows that the precompact covering map may not be a compact covering map.

Example 3.2. Let $f : [1, \infty) \to (0, 1]$ be defined by $f(x) = \frac{1}{x}$. Clearly f is a sequence covering map and also a compact covering map. But there is no Cauchy sequence $\{x_n\}$ in $[1, \infty)$ such that $f(x_n) = \frac{1}{n}$. So f is not a Cauchy covering map and also not precompact covering .

Definitions of Cauchy covering maps and precompact covering maps are analogous but they are mutually independent.

Example 3.3. Consider, $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{1 + \frac{1}{n} : n \in \mathbb{N}\}$ and $Y = \{y_n : n \in \mathbb{N}\}$ where $\{y_n\}$ is a real valued Cauchy sequence. Define $f : X \to Y$ as $f(\frac{1}{n}) = y_{2n}, f(1 + \frac{1}{n}) = y_{2n-1}$. Then f is a precompact covering map but not a Cauchy covering map.

Example 3.4. Let Y be any metric space and X be the class of disjoint union of all Cauchy sequences of Y. Also let f be the natural map from X onto Y. Then f is Cauchy covering but not a precompact covering map.

Note 3.1. Consider the metric space X as in Example 3.4. Then every $x(\in X)$ has a totally bounded neighbourhood. From this construction we can say that each metric space is Cauchy covering image of a metric space which has the property that each point has totally bounded neighbourhood.

Now we investigate the relation of Cauchy covering map with some other maps.

Theorem 3.6. *Suppose that $f : (X, d_X) \to (Y, d_Y)$ is a one to one Cauchy covering map. Then f is a precompact covering map.*

Proof. Let V be a totally bounded subset of Y and $\{x_n\}$ be a sequence in $f^{-1}(V)$. If $\{x_n\}$ has no Cauchy subsequence then so $\{f(x_n)\}$, which contradicts the fact that V is totally bounded. So $f^{-1}(V)$ is also totally bounded. □

Theorem 3.7. *Suppose that $f : (X, d_X) \to (Y, d_Y)$ is a one to one Cauchy covering map. Then f is a sequence covering map.*

Proof. Suppose that $\{y_n\}$ converges to y in Y. Then there exists a Cauchy sequence $\{x_n\}$ in X with $f(x_n) = y_n$. Consider the sequence $\{z_k\}$ defined by $z_{2n-1} = y_n$ and $z_{2n} = y$. Then $f(r_k) = z_k$, where $r_{2n-1} = x_n$ and $r_{2n} = x$. So $\{x_n\}$ converges to $x \in f^{-1}(y)$. Hence f is a sequence covering map. $\square$

Theorem 3.8. *Suppose that $f : (X, d_X) \to (Y, d_Y)$ is an $\mathcal{I}$-sequence covering map where $\mathcal{I}$ is an admissible ideal but $\mathcal{I}$ is not a tall ideal. Then f is a sequence covering map.*

Proof. Suppose that f is an $\mathcal{I}$-sequence covering map and $\{y_n\}$ is a convergent sequence converges to y in Y. Since $\mathcal{I}$ is not a tall ideal so there is an infinite subset $A = \{n_k : k \in \mathbb{N}\}$ of $\mathbb{N}$ with each $n_k < n_{k+1}$ such that any infinite subset of A does not belong to $\mathcal{I}$. Now define a new sequence $z_{n_k} = y_k$ and $z_n = y$, otherwise. Clearly $\{z_n\}$ converges to y. So there exists a sequence $\{x_n\}$ with each $x_n \in f^{-1}(z_n)$ and it $\mathcal{I}$-converges to a point $x \in f^{-1}(y)$. So for $\varepsilon > 0$ the set $\{n \in \mathbb{N} : d_X(x_n, x) \geq \varepsilon\} \in \mathcal{I}$, which shows that $B = \{n_k : d_X(x_{n_k}, x) \geq \varepsilon\} \in \mathcal{I}$. So B is finite being a subset of A. Hence $\{x_{n_k}\}$ converges to $x \in f^{-1}(y)$ with each $x_{n_k} \in f^{-1}(y_k)$. Thus f is a sequence covering map. $\square$

For the next Theorem we recall the following Lemma.

Lemma 3.9. *(c.f.* [8]*) If $\mathcal{I}$ is an admissible ideal with the property (AP) then in a metric space the concepts of $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence are coincide.*

Theorem 3.10. *Suppose that $f : (X, d_X) \to (Y, d_Y)$ is a Cauchy covering map and $\mathcal{I}$ is an AP ideal. Then f is an $\mathcal{I}$-Cauchy covering map.*

Proof. Let $\{y_n\}$ be an $\mathcal{I}$-Cauchy sequence. Then by Lemma 3.9 there is a subset $M = \{m_k : k \in \mathbb{N}\}$ of $\mathbb{N}$ with $M \in \mathcal{F}(\mathcal{I})$ such that $\{y_{m_k}\}$ is a Cauchy sequence. So there is a sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$ such that the subsequence $\{x_{m_k}\}$ is Cauchy. Therefore $\{x_n\}$ is an $\mathcal{I}$-Cauchy sequence in X. Hence f is an $\mathcal{I}$-Cauchy covering map. $\square$

Now we formulate an almost equivalent condition for a metric space to be complete in terms of Cauchy covering maps.

Theorem 3.11. *Let X be a metric space. If each Cauchy covering map on X is a sequence covering map then X is complete.*

Proof. Suppose X is not complete. Then there is a Cauchy sequence $\{x_n\}$ in X which is not convergent in X. Let $Y = \{y_n : n \in \mathbb{N}\} \cup \{y\}$ where $\{y_n\}$ is a convergent sequence of distinct points with its limit y.

Define a function $f : X \to Y$ as $f(x_{2n}) = y_n$ for each $n \in \mathbb{N}$ and $f(x) = y$, otherwise. Now any Cauchy sequence $\{z_n\}$ in Y is a subsequence of $\{y_n\}$. So choose $r_n \in f^{-1}(z_n)$ as a subsequence of $\{x_n\}$. Therefore f is a Cauchy covering map but not a sequence covering map, which is a contradiction. Hence X is complete. □

The converse of the theorem will be true if we impose the continuity on f.

Theorem 3.12. *Suppose that (X, d_X) is a complete metric space and $f : (X, d_X) \to (Y, d_Y)$ is a continuous Cauchy covering map. Then f is a sequence covering map.*

Proof. Suppose that $\{y_n\}$ converges to y in Y. Then there is a Cauchy sequence $\{x_n\}$ with each $x_n \in f^{-1}(y_n)$ in X. Since X is complete so $\{x_n\}$ is convergent to some point x in X. Now from the continuity of f we get $f(x_n) \to f(x)$. So $f(x) = y$. Hence f is a sequence covering map. □

The following example asserts that the continuity is necessary for the theorem.

Example 3.5. Suppose $X = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ where each point of X are distinct and $\{x_n\}$ is a real valued sequence converges to x and $Y = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0, 2\}$. Now define $f : X \to Y$ by $f(x) = 2, f(x_{2n}) = \frac{1}{n}$ and $f(x_{2n-1}) = 0$ for each $n \in \mathbb{N}$. Clearly X is complete. Any Cauchy sequence $\{z_n\}$ in Y is either eventually constant or a subsequence of $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. So in the second case $r_n \in f^{-1}(z_n)$ is a subsequence of $\{x_n\}$. Hence f is a Cauchy covering map. But f is not continuous and also not a sequence covering map.

Further, completeness of X is necessary for the above theorem. The following example is in this direction.

Example 3.6. Let $\bigwedge = \{\alpha : \alpha$ is an increasing function from $\mathbb{N}$ into $\mathbb{N}\}$ and $X_\alpha = \{\frac{1}{n} : n \in \mathbb{N}\} \times \{\alpha\}$. Also let (X_α, d_α) be a metric space where $d_\alpha((\frac{1}{m}, \alpha), (\frac{1}{n}, \alpha)) = |\frac{1}{m} - \frac{1}{n}|$. Put $X = \bigoplus_{\alpha \in \bigwedge} X_\alpha$ and the metric D on X is defined by $D(x, y) = min\{|x - y|, 1\}$ if $x, y \in X_\alpha$ for some $\alpha \in \bigwedge$ and $D(x, y) = 1$ if $x \in X_\alpha, y \in X_\beta$ for $\alpha \neq \beta$. Take a real valued convergent sequence $\{y_n\}$ converges to y and consider $Y = \{y_n : n \in \mathbb{N}\} \cup \{y\}$ with the usual metric. Here we assert that each points of Y are distinct. Clearly X is not complete. Now define $f : X \to Y$ by $f(\frac{1}{2k}, \alpha) = y_{\alpha(k)}$ and $f(\frac{1}{2k-1}, \alpha) = y$ for each $\alpha \in \bigwedge$. Then any Cauchy sequence $\{z_n\}$ in Y is a subsequence $\{y_{\alpha(k)} : k \in \mathbb{N}\} \cup \{y\}$. Choose $r_n \in f^{-1}(z_n)$ where $\{r_n\}$ is a subsequence of the sequence $\{(\frac{1}{n}, \alpha)\}$. Hence $\{r_n\}$ is Cauchy and hence f is a Cauchy covering map. Clearly f is continuous but not a sequence covering map. Hence the completeness of X is necessary for the theorem.

The following example shows that in general cofinally Cauchy sequence may not have any Cauchy subsequence.

Example 3.7. Consider the space l_∞ of all bounded real sequence $\{x_n\}$ where the norm defined as $||x_n|| = \sup\{|x_n| : n \in \mathbb{N}\}$. Let $\mathbb{N} = \cup_{k=1}^{\infty} A_k$ where each A_k is infinite and $A_i \cap A_j = \phi$. Also let $A_i = \{a_{i1}, a_{i2}, ...\}$. If $n \in A_i,\ n \neq a_{i1}$ put $x_{n,a_{i1}} = 2,\ x_{n,n} = \frac{1}{i}$ and $x_{n,k} = 0$, otherwise. If $n = a_{i1}$ put $x_{n,a_{i1}} = 10^i$ and $x_{n,k} = 0$ for other $k \in \mathbb{N}$. Let $z_n = \{x_{n,k}\}$. Now for $\varepsilon > 0$ there exists n_0 such that $\frac{1}{n_0} < \varepsilon$ and then for all $m, n \in A_{n_0},\ ||z_m - z_n|| = \frac{1}{n_0} < \varepsilon$. So $\{z_n\}$ is a cofinally Cauchy sequence in l_∞ but $\{z_n\}$ has no Cauchy subsequence.

In general the concepts of cofinally Cauchy covering maps and Cauchy covering maps are independent.

Example 3.8. Suppose that $\{z_n\}$ is a sequence described in the Example 3.7. Put $Z = \{z_n : n \in \mathbb{N}\}$ with the subspace metric induced from the sup norm of l_∞. Now define $f : \mathbb{R} \to Z$ by $f(n) = z_n$ for all $n \in \mathbb{N}$ and $f(x) = z_1$, otherwise. Clearly f is a Cauchy continuous map as well as Cauchy covering map but not a cofinally Cauchy covering map.

Example 3.9. Let $\mathcal{F}$ be the set of all one to one mapping from $\mathbb{N}$ to $\mathbb{N}$. For each $f \in \mathcal{F}$ consider $S_f = \{x_{f,n} : n \in \mathbb{N}\}$ where for all $n \in \mathbb{N}$, $x_{f,2n} = r_f,\ r_f \in \mathbb{R}$ and $x_{f,2n-1} = n$ for each $n \in \mathbb{N}$. Clearly S_f is a cofinally Cauchy sequence but not a Cauchy sequence. Put $X = \bigoplus_{f \in \mathcal{F}} S_f$ and $Y = \{\frac{1}{n} : n \in \mathbb{N}\}$. Now define f by $f(x_{f,n}) = \frac{1}{f(n)}$. Then f is a Cauchy continuous map and also cofinally Cauchy covering but not Cauchy covering.

The next result will give a new characterization of cofinally complete metric spaces.

Theorem 3.13. *Let (X, d_X) be a cofinally complete metric space. If $f : (X, d_X) \to (Y, d_Y)$ is a continuous and cofinally Cauchy covering map then f is a sequentially quotient map and hence a quotient map.*

Proof. Suppose that $y_n \to y$ in Y. Then there exists a cofinally Cauchy sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$. As X is cofinally complete, so there exists a subsequence $\{x_{n_k}\}$ converges to some point x in X. By the continuity of f, $f(x_{n_k}) \to f(x)$, which implies $x \in f^{-1}(y)$. So f is a sequentially quotient map and hence a quotient map. $\square$

Now we discuss the necessity of the condition of the theorem. The necessity of the continuity condition is proved in the Example 3.5 where f is not a sequentially quotient map (because if we take the sequence $\{\frac{1}{n}\}$

then the only pre-image $\{x_{2n}\}$ converges to x. But $x \notin f^{-1}(0)$). The following Example will prove the necessity of the cofinally completeness.

Example 3.10. Let $\{x_n\}$ be the sequence described in Example 3.7 and X and Y be same as the Example 3.6 just replaced the sequence $\{\frac{1}{n}\}$ by the sequence $\{x_n\}$ and the metric d_α on X_α is defined by $d_\alpha((x_m,\alpha),(x_n,\alpha)) = ||x_m - x_n||$. Clearly X is not cofinally complete. Now define $f : X \to Y$ by $f(x_1,\alpha) = y, f(x_k,\alpha) = y_{\alpha(k)}$ for $k \geq 1$. Then the cofinally Cauchy sequence $\{z_n\}$ in Y is a subsequence of $\{y_{\alpha_k}\}$. We choose $r_n \in f^{-1}(z_n)$ such that $\{r_n\}$ is a subsequence of $\{(x_k,\alpha)\}_{k\in\mathbb{N}}$. Hence f is a cofinally Cauchy covering map. But f is not sequentially quotient.

Note 3.2. If we consider a continuous map $f : (X,d_X) \to (Y,d_Y)$ where X, Y both are complete spaces then the concepts of sequence covering and Cauchy covering are coincide. Also the concepts of precompact map and compact covering map are equivalent. Furthermore, continuous Cauchy covering map preserves completeness.

3.2. **Cauchy continuous maps:** In this section we consider some general classes of Cauchy continuous functions and investigate their relation to Cauchy continuous maps. First we consider the following definitions:

Definition 3.14. [3] Let (X,d_X) be a metric space and $\mathcal{I}$ be an admissible ideal. Then a sequence $\{x_n\}$ in X is called $\mathcal{I}$-bounded in X if there exist an element $x \in X$ and a positive real number r such that $\{n : d_X(x_n,x) \geq r\} \in \mathcal{I}$.

Definition 3.15. Let $\mathcal{I}$ be an admissible ideal. A function $f : (X,d_X) \to (Y,d_Y)$ is said to be $\mathcal{I}$-Cauchy continuous (Statistical Cauchy continuous) if for each $\mathcal{I}$-Cauchy (Statistical Cauchy) sequence $\{x_n\}$ in X, $\{f(x_n)\}$ is an $\mathcal{I}$-Cauchy (Statistical Cauchy) sequence in Y.

Definition 3.16. A function $f : (X,d_X) \to (Y,d_Y)$ is called bounded continuous if for each bounded sequence $\{x_n\}$ in X, $\{f(x_n)\}$ is a bounded sequence in Y.

Definition 3.17. Let $\mathcal{I}$ be an admissible ideal. A function $f : (X,d_X) \to (Y,d_Y)$ is said to be $\mathcal{I}$-bounded continuous (Statistical bounded continuous) if for each $\mathcal{I}$-bounded (Statistically bounded) sequence $\{x_n\}$ in X, $\{f(x_n)\}$ is an $\mathcal{I}$-bounded (Statistically bounded) sequence in Y.

Theorem 3.18. *Let $f : (X,d_X) \to (Y,d_Y)$ be a map. Then f is statistical Cauchy continuous iff f is Cauchy continuous.*

Proof. Suppose that f is a Cauchy continuous map. Let $\{x_n\}$ be a statistical Cauchy sequence in X. Then by Lemma 3.9 there exists $M \subset \mathbb{N}$ with density, $d(M) = 1$ such that $\{x_n\}_{n\in M}$ is a Cauchy sequence. So $\{f(x_n)\}_{n\in M}$ is a Cauchy sequence and hence $\{f(x_n)\}_{n\in\mathbb{N}}$ is also a statistical Cauchy sequence.

Conversely, suppose that f is a statistical Cauchy continuous map and $\{x_n\}$ is a Cauchy sequence in X. Since every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is Cauchy so corresponding $\{f(x_{n_k})\}$ is statistical Cauchy. Thus every subsequence of $\{f(x_n)\}$ is statistical Cauchy. Now we will show that $\{f(x_n)\}$ is a Cauchy sequence.

Suppose that $\{f(x_n)\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ such that for each $k \in \mathbb{N}$ we can find m_k and n_k with $d_Y(f(x_{m_k}), f(x_{n_k})) \geq \varepsilon$. Now define a new sequence $\{y_n\}$ as $y_{2k-1} = f(x_{m_k})$ and $y_{2k} = f(x_{n_k})$ for each $k \in \mathbb{N}$. Then we have $d_Y(y_{2k-1}, y_{2k}) \geq \varepsilon$ for each $k \in \mathbb{N}$. Next we will show that $\{y_k\}$ is not statistical Cauchy. Suppose that $\{y_k\}$ is a statistical Cauchy sequence. Then there is $P \subset \mathbb{N}$ with $d(P) = 1$, $\{y_k\}_{k\in P}$ is Cauchy. Now there are infinitely many k such that P contains both $2k-1$ and $2k$, otherwise density of P, $d(P) \leq \frac{1}{2}$. Therefore for infinitely many k with $2k-1, 2k \in P$ $d_Y(y_{2k-1}, y_{2k}) \geq \varepsilon$, holds which contradicts the fact that $\{y_k\}_{k\in P}$ is a Cauchy sequence. Hence $\{y_k\}$ is not statistical Cauchy, which again contradicts that every subsequence of $\{f(x_n)\}$ is statistical Cauchy. Thus $\{f(x_n)\}$ is a Cauchy sequence. Therefore f is a Cauchy continuous map. □

Theorem 3.19. *Let $f : (X, d_X) \to (Y, d_Y)$ be a map and $\mathcal{I}$ be an admissible ideal. Then f is $\mathcal{I}$-bounded continuous iff f is bounded continuous.*

Proof. Let f be a bounded continuous function and $\{x_n\}$ be an $\mathcal{I}$-bounded sequence in X. Then there exists $M \subset \mathbb{N}$ with $M \in \mathcal{F}(\mathcal{I})$ and $\{x_n\}_{n\in M}$ is a bounded sequence. So $\{f(x_n)\}_{n\in M}$ is a bounded sequence. Hence $\{f(x_n)\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-bounded sequence.

Conversely, let f be an $\mathcal{I}$-bounded continuous map and $\{x_n\}$ be a bounded sequence in X. Since every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is bounded so corresponding $\{f(x_{n_k})\}$ is $\mathcal{I}$-bounded. Hence every subsequence of $\{f(x_n)\}$ is $\mathcal{I}$-bounded. Suppose $\{f(x_n)\}$ is not bounded. Then we can find an element x in X such that for each $k \in \mathbb{N}$ there is $f(x_{n_k})$ with $d_Y(f(x_{n_k}), x) \geq k$. Now for any positive real number a there is $k \in \mathbb{N}$ such that $k > a$. So we have $d_Y(f(x_{n_i}), x) \geq k > a$, for all $i \geq k$. Hence $\{k : d_Y(f(x_{n_k}), x) \geq a\} \notin \mathcal{I}$. Thus $\{f(x_{n_k})\}$ is not $\mathcal{I}$-bouded in Y, which contradicts that every subsequence of $\{f(x_n)\}$ is $\mathcal{I}$-bounded. Therefore $\{f(x_n)\}$ is a bounded sequence. Hence f is a bounded continuous map. □

Lemma 3.20. [3] *Let $\mathcal{I}_0$ be an admissible ideal. Then $\mathcal{I}_0$ is a maximal ideal iff $A \in \mathcal{I}_0 \vee \mathbb{N} \setminus A \in \mathcal{I}_0$ for each $A \subset \mathbb{N}$.*

Note 3.3. Let (X, d_X) be a metric space and $\{x_n\}$ be a sequence in X. If each subsequence of $\{x_n\}$ is statistical Cauchy then $\{x_n\}$ is a Cauchy sequence. But in general this is not true for any admissible ideal. Suppose that $\mathcal{I}$ is a maximal ideal. A sequence $\{x_n\}$ is defined by $x_{2k-1} = 0$ and $x_{2k} = 2$ for each $k \in \mathbb{N}$. Clearly $\{x_n\}$ is not a Cauchy sequence. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. Then there exists a subset A of $\mathbb{N}$ such that $x_{n_k} = 0$, for $k \in A$ and $x_{n_k} = 2$, otherwise. From Lemma 3.20 either $A \in \mathcal{I}$ or $A^c \in \mathcal{I}$. If $A \in \mathcal{I}$ then $\{x_{n_k}\}_{k \in A^c}$ is a constant sequence where $A^c \in \mathcal{F}(I)$. So $\{x_{n_k}\}$ is an $\mathcal{I}-$Cauchy sequence. The other case is similar. Hence every subsequence of $\{x_n\}$ is $\mathcal{I}$-Cauchy.

If we consider the case of $\mathcal{I}$-boundedness the result also holds good. Let (X, d_X) be a metric space and $\{x_n\}$ be a sequence in X. If each subsequence of $\{x_n\}$ is $\mathcal{I}$-bounded then $\{x_n\}$ is also bounded. (Suppose $\{x_n\}$ is not bounded. Then we can find an element x in X such that for each $k \in \mathbb{N}$ there is a subsequence $\{x_{n_k}\}$ such that $d_X(x_{n_k}, x) \geq k$. Now for any positive real number a there is $k \in \mathbb{N}$ such that $k > a$. So we have $d_X(x_{n_i}, x) \geq k > a$ for all $i \geq k$. Hence $\{k : d_X(x_{n_k}, x) \geq a\} \notin \mathcal{I}$. Thus $\{x_{n_k}\}$ is not an $\mathcal{I}$-bounded sequence in X, which contradicts that every subsequence of $\{x_n\}$ is $\mathcal{I}$-bounded. So $\{x_n\}$ is a bounded sequence.)

Further if we consider the case of $\mathcal{I}$-convergent then the result is also true.

Let (X, d_X) be a metric space and $\{x_n\}$ be a sequence in X. If each subsequence of $\{x_n\}$ is $\mathcal{I}$-Convergent to the same limit then $\{x_n\}$ converges to that limit. (Suppose $\{x_k\}$ is $\mathcal{I}$-convergent to x in X but not convergent to x in X. Then we can find $\varepsilon > 0$ such that there is a subsequence $\{x_{n_k}\}$ with $d_X(x_{n_k}, x) \geq \varepsilon$, for each $k \in \mathbb{N}$. Put $y_k = x_{n_k}$ for each $k \in \mathbb{N}$. So $\{y_k\}$ is a subsequence of $\{x_n\}$ that can not be $\mathcal{I}$-convergent to x. Therefore $\{x_n\}$ converges to x.)

In general $\mathcal{I}$-Cauchy continuity does not imply the Cauchy continuity.

Example 3.11. Suppose $\mathcal{I}$ is an admissible maximal ideal. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and $Y = \{0, 2\}$ with the discrete metric. Now define $f : (X, d_X) \to (Y, d_Y)$ by $f(\frac{1}{2n}) = 2$ and $f(x) = 0$, otherwise. Then f is an $\mathcal{I}$-Cauchy continuous map as every sequence in Y is $\mathcal{I}$-Cauchy (from Note 3.3). But f is not a continuous map and so not a Cauchy continuous map.

Theorem 3.21. *Let $f : (X, d_X) \to (Y, d_Y)$ be a map and $\mathcal{I}$ be an admissible ideal, satisfies AP condition. Then f is Cauchy continuous implies f is $\mathcal{I}$-Cauchy continuous.*

Proof. Let f be a Cauchy continuous map and $\{x_n\}$ be an $\mathcal{I}$-Cauchy sequence. Then by Lemma 3.9 there is $M = \{n_k : k \in \mathbb{N}\} \subset \mathbb{N}$ with $M \in \mathcal{F}(\mathcal{I})$ such that $\{x_{n_k}\}$ is a Cauchy sequence. Therefore $\{f(x_{n_k})\}$ is a Cauchy sequence which shows that $\{f(x_n)\}$ is an $\mathcal{I}$-Cauchy sequence. This completes the proof. □

The following theorem has classified all those ideals for which $\mathcal{I}$-Cauchy continuity of a function is equivalent to the continuity of f.

Theorem 3.22. *Let $f : (X, d_X) \to (Y, d_Y)$ be a map. Also let $\mathcal{I}$ be an admissible ideal with the property that there exist disjoint subsets A_1, A_2 of $\mathbb{N}$ where $A_1, A_2 \notin \mathcal{I}$ and $\mathbb{N} = A_1 \cup A_2$. If f is an $\mathcal{I}$-Cauchy continuous map then f is also a continuous map.*

Proof. Suppose f is $\mathcal{I}$-Cauchy continuous but f is not continuous. Then there is a convergent sequence $\{x_n\}$ converges to x such that $\{f(x_n)\}$ is not convergent to $f(x)$. So there exist $\varepsilon > 0$ and a subsequence $\{f(x_{n_k})\}$ with $d_Y(f(x_{n_k}), f(x)) > \varepsilon$ for each $k \in \mathbb{N}$. Define a new sequence $z_k = x_{n_k}$ if $k \in A_1$ and $z_k = x$ if $k \in A_2$. Clearly $\{z_k\}$ is an $\mathcal{I}$-Cauchy sequence in X. Now for each $k \in \mathbb{N}$ we have two cases either $k \in A_1$ or $k \in A_2$. If $k \in A_1$ then $f(z_k) = f(x_{n_k})$ and so $A_2 \subset \{n : d_Y(f(z_n), f(z_k)) > \varepsilon\} \notin \mathcal{I}$. On the other hand if $k \in A_2$ then $f(z_k) = f(x)$ and so $A_1 \subset \{n : d_Y(f(z_n), f(z_k)) > \varepsilon\} \notin \mathcal{I}$. So $\{f(z_k)\}$ cannot be $\mathcal{I}$-Cauchy, which contradicts that f is $\mathcal{I}$-Cauchy continuous. Hence f is continuous. □

Note 3.4. For an AP ideal satisfy the property mention in the theorem we have Cauchy continuity $\Rightarrow$ $\mathcal{I}$-Cauchy continuity $\Rightarrow$ continuity. But continuity may not imply $\mathcal{I}$-Cauchy continuity for an admissible ideal.

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