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by

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MULTIPLE FACETS OF INVERSE CONTINUITY

SZYMON DOLECKI

ABSTRACT. Inversion of various inclusions that characterize continuity in topological spaces results in numerous variants of quotient and perfect maps. In the framework of convergences, the said inclusions are no longer equivalent, and each of them characterizes continuity in a different concretely reflective subcategory of convergences. On the other hand, it turns out that the mentioned variants of quotient and perfect maps are quotient and perfect maps with respect to these subcategories. This perspective enables use of convergence-theoretic tools in quests related to quotient and perfect maps, considerably simplifying the traditional approach. Similar techniques would be unconceivable in the framework of topologies.

INTRODUCTION

This paper is designed for a broad mathematical audience. Its purpose is to show the utility of the convergence theory approach to classical themes of general topology. Therefore, I focus on convergence-theoretic methods rather than on detailed applications. For this reason, I shall provide only a small number of examples, referring for more details to classical books on general topology, for example, to [16] of R. ENGELKING, and to research papers cited below.

It will be shown that the arguments deployed would be impossible without a broader framework transcending that of topologies.

Continuity of maps between topological spaces can be characterized in many ways, in particular, in terms of adherences of filters from various classes, their images, and preimages.

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It appears that inversion of the inclusions occurring in these characterizations results in numerous variants of quotient and perfect maps. Extending these definitions to general convergences leads, on one hand, to quotients with respect to various reflective subcategories, like topologies, pretopologies, paratopologies, and pseudotopologies, and, on the other, to fiber relations having compactness type related to the listed convergence categories. This vision enables to see the quintuple quotient quest of E. Michael [21] as a single unifying quotient quest presented by the author in [5].

It turns out that continuity is not pertinent to many considerations involving quotient-like and perfect-like maps. In fact, the latter properties correspond to certain stability features of fiber relations, that is, of the inverse relations of mappings, rather than to mappings themselves, and can be studied advantageously independently from continuity.

Accordingly, no continuity assumption will be implicitly made. Let me also say that no separation axiom will be assumed in the forthcoming definitions, unless explicitly stated.

Convergences (in particular, pseudotopologies) were introduced and investigated by G. Choquet in [4], because topologies turned out to be inadequate in the study of hyperspaces ([1]). The category of convergences is *exponential* ([2]), contrary to the category of topologies. This feature of convergences made them suitable for the just mentioned study of hyperspaces. But their usefulness proved to be much broader, as I am going to show in this paper.

Throughout this paper, f^- stands for the *inverse relation* of a *map* f, hence we use the convention that

$$f^-(B) := f^{-1}(B)$$

denotes the *preimage* of B under f.

To deal with many cross-references more easily, Table of Contents has been placed at the end of the paper, not at its beginning, because it is very detailed, and thus lengthy.

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[1]More precisely, of upper and lower limits, said of *Kuratowski*; in fact, they were introduced by G. Peano in [23], [24].

[2]The category of convergences with continuous maps as morphisms is *exponential* (or *Cartesian closed*), which means that for two given convergences ξ, σ, there exists a *coarsest convergence* (denoted by $[\xi, \sigma]$) on the set of continuous functions $C(\xi, \sigma)$, for which the canonical coupling $\langle x, f\rangle := f(x)$ is continuous: $\langle,\rangle \in C(\xi \times [\xi, \sigma], \sigma)$. In particular, hyperconvergences are spaces of continuous maps with respect to $\sigma = \$$, the *Sierpiński topology* on a two elements set, say $\{0, 1\}$, the open sets of which are $\varnothing, \{1\}, \{0, 1\}$. The category of topologies with continuous maps as morphisms is not exponential, as for two topologies ξ, σ, the convergence $[\xi, \sigma]$ need not be topological.

1. Continuity in topological spaces

Consider non-empty sets X and Y ([3]), topologies ξ on X and τ on Y, a map $f : X \to Y$, and $A \subset X$. By definition, f is *continuous* (from ξ to τ) if the preimage under f of each τ-open set is ξ-open. We denote by $C(\xi, \tau)$ the set of all functions that are continuous from ξ to τ, by $\mathrm{cl}_\xi A$ the *closure*, and by $\mathrm{int}_\xi A$ the *interior* of A with respect to ξ, and so on.

The following equivalences can be found in [16, Theorem 1.4.1], but constitute also a standard exercise in topology courses.

Proposition 1. *Let ξ and τ be topologies. Then the following statements are equivalent:*

$$f \in C(\xi, \tau), \tag{1.1}$$

$$\mathrm{cl}_\xi f^-(B) \subset f^-(\mathrm{cl}_\tau B) \textit{ for each } B \subset Y, \tag{1.2}$$

$$f(\mathrm{cl}_\xi A) \subset \mathrm{cl}_\tau f(A) \textit{ for each } A \subset X. \tag{1.3}$$

If we invert the two inclusions above ([4]), then we get

$$f^-(\mathrm{cl}_\tau B) \subset \mathrm{cl}_\xi f^-(B) \text{ for each } B \subset Y, \tag{1.4}$$

$$\mathrm{cl}_\tau f(A) \subset f(\mathrm{cl}_\xi A) \text{ for each } A \subset X. \tag{1.5}$$

The properties so obtained can be described as inverse to (1.2) and (1.3), respectively. It turns out that they are not equivalent! The first characterizes *quotient maps*, and the second, *closed maps*. Recall that a map $f \in C(\xi, \tau)$ is said to be

(1) *quotient* if B is τ-closed whenever $f^-(B)$ is ξ-closed,
(2) *closed* if $f(A)$ is τ-closed for every ξ-closed set A.

In the sequel, we will consider only surjective maps, a non-essential restriction that simplifies the presentation. The definitions and conditions above make sense without the continuity assumption and, moreover, the continuity assumption is superfluous in the following characterizations.

We denote by $|\xi|$ the underlying set of a topology ξ. Hence, here $|\xi| = X$ and $|\tau| = Y$.

Proposition 2. *If a map $f : |\xi| \longrightarrow |\tau|$ is surjective, then (1.4) holds if and only if f is quotient.*

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[3]We shall assume that the sets considered are non-empty unless stated otherwise explicitly.

[4]These conditions are also equivalent to $f^-(\mathrm{int}_\tau B) \subset \mathrm{int}_\xi f^-(B)$ for each $B \subset Y$ (see [16, Theorem 1.4.1]); the inversion of this inclusion $\mathrm{int}_\xi f^-(B) \subset f^-(\mathrm{int}_\tau B)$ turns out to be equivalent to (1.5). However, a formally similar condition $\mathrm{int}_\tau f(A) \subset f(\mathrm{int}_\xi A)$ is unrelated to continuity.

Proof. If $f^{-}(B)$ is ξ-closed, that is, $\mathrm{cl}_\xi f^{-}(B) \subset f^{-}(B)$, which together with (1.4) yields $f^{-}(\mathrm{cl}_\tau B) \subset f^{-}(B)$, hence $\mathrm{cl}_\tau B \subset B$, because f is surjective. Conversely, if f is quotient and $\mathrm{cl}_\xi f^{-}(B) = f^{-}(B)$, then $\mathrm{cl}_\tau B = B$, hence $f^{-}(\mathrm{cl}_\tau B) = f^{-}(B)$, and thus (1.4). □

Proposition 3. *If a map $f : |\xi| \longrightarrow |\tau|$ is surjective, then (1.5) holds if and only if f is closed.*

Proof. If $\mathrm{cl}_\xi A \subset A$, by (1.5), $\mathrm{cl}_\tau f(A) \subset f(A)$, that is, $f(A)$ is τ-closed. Conversely, if $\mathrm{cl}_\xi A = A$ implies that $\mathrm{cl}_\tau f(A) \subset f(A)$, hence (1.5). □

Let us now enlarge our perspective by making use of the concept of filter, which will enable us to formulate new characterizations of continuity.

1.1. **Convergence of filters in topological spaces.** One of the ways of defining a topology is to specify its converging *filters*. A non-empty family $\mathcal{F}$ of subsets of a set X is called a *filter* whenever

$$(F_0 \in \mathcal{F}) \wedge (F_1 \in \mathcal{F}) \Longleftrightarrow F_0 \cap F_1 \in \mathcal{F}.$$

A filter $\mathcal{F}$ is called *non-degenerate* if

$$\varnothing \notin \mathcal{F}.$$

Basic facts about filters are gathered in Appendix 11.2. Let us only recall here that the family $\mathcal{N}_\theta(x)$ (of *neighborhoods* of x for a topology θ) is a non-degenerate filter.

Convergence of filters in topological spaces generalizes convergence of sequences. Indeed, the family $\mathcal{O}_\theta(x)$ of all θ-open sets that contain x, is a *base* of the neighborhood filter $\mathcal{N}_\theta(x)$ (see Appendix 11.2). On the other hand, a sequence $(x_n)_n$ converges to x in θ whenever, for each θ-open set O containing x, there is $m \in \mathbb{N}$ such that $\{x_n : n \geq m\} \subset O$, that is, whenever O belongs to the filter *generated* (Appendix 11.3) by

$$\{\{x_n : n \geq m\} : m \in \mathbb{N}\}.$$

Definition 4. If θ is a topology on a set X, then we say that a *filter* $\mathcal{F}$ *converges* to a point x,

$$x \in \lim\nolimits_\theta \mathcal{F},$$

whenever $\mathcal{O}_\theta(x) \subset \mathcal{F}$. As $\mathcal{O}_\theta(x)$ is a base of the *neighborhood filter* $\mathcal{N}_\theta(x)$ of x for θ, we get

$$x \in \lim\nolimits_\theta \mathcal{F} \Longleftrightarrow \mathcal{N}_\theta(x) \subset \mathcal{F}. \tag{1.6}$$

More generally, if $\mathcal{B}$ is a base of $\mathcal{F}$, then by definition $\lim_\theta \mathcal{B} = \lim_\theta \mathcal{F}$.

1.2. **Some operations on families of sets.** The symbol $\mathcal{A}^{\uparrow}$ for a given family $\mathcal{A}$ ([5]) of subsets of a set X stands for

(isotonization) $$\mathcal{A}^{\uparrow} := \bigcup_{A\in\mathcal{A}}\{H \subset X : H \supset A\}.$$

If $\mathcal{A}$ and $\mathcal{D}$ are families, then $\mathcal{A}$ is said to be *coarser* than $\mathcal{D}$ ($\mathcal{D}$ *finer* than $\mathcal{A}$)

$$\mathcal{A} \leq \mathcal{D}$$

if for each $A \in \mathcal{A}$ there is $D \in \mathcal{D}$ such that $D \subset A$. The *grill* $\mathcal{A}^{\#}$ of a family $\mathcal{A}$ on X is defined by G. Choquet in [3] by

$$\mathcal{A}^{\#} := \{H \subset X : \underset{A\in\mathcal{A}}{\forall}\ H \cap A \neq \varnothing\}.$$

If $\mathcal{A}$ and $\mathcal{D}$ are families, then we write $\mathcal{A}\#\mathcal{D}$ ($\mathcal{H}$ *meshes* $\mathcal{A}$) whenever $\mathcal{D} \subset \mathcal{A}^{\#}$, equivalently, $\mathcal{A} \subset \mathcal{D}^{\#}$. This means that $H \cap A \neq \varnothing$ for each $H \in \mathcal{H}$ and each $A \in \mathcal{A}$.

Let $\mathcal{A}$ be a family of subsets of X, $\mathcal{G}$ a family of subsets of Y, and $f : X \to Y$. Then ([6])

$$f[\mathcal{A}] := \{f(A) : A \in \mathcal{A}\}, \qquad f^{-}[\mathcal{B}] := \{f^{-}(B) : B \in \mathcal{B}\}.$$

It is straightforward that if $f : X \to Y$ is surjective ([7]), $\mathcal{G}$ and $\mathcal{H}$ are filters, then

$$\mathcal{G} \geq f^{-}[f[\mathcal{G}]], \qquad f[f^{-}[\mathcal{H}]] = \mathcal{H}. \tag{1.7}$$

In the sequel, we make continually use of the formula (see Appendix 11.1)

$$f[\mathcal{A}]\#\mathcal{B} \Longleftrightarrow \mathcal{A}\#f^{-}[\mathcal{B}]. \tag{1.8}$$

1.3. **Adherences.** Let ξ be a topology on a set X. The *adherence* of a family $\mathcal{A}$ of subsets of X is defined by ([8])

$$\mathrm{adh}_{\xi}\,\mathcal{A} := \bigcup_{\mathcal{H}\#\mathcal{A}} \lim_{\xi}\mathcal{H}. \tag{1.9}$$

If $\mathcal{A}$ is a filter, then $\mathrm{adh}_{\xi}\,\mathcal{A} = \bigcup_{\mathcal{H}\supset\mathcal{A}} \lim_{\xi}\mathcal{H}$ is an alternative formula for (1.9). In particular, we abridge $\mathrm{adh}_{\xi}\,A := \mathrm{adh}_{\xi}\{A\}^{\uparrow}$ (see (isotonization) above). Accordingly

$$\mathcal{A} \leq \mathcal{D} \Longrightarrow \mathrm{adh}_{\xi}\,\mathcal{D} \subset \mathrm{adh}_{\xi}\,\mathcal{A}. \tag{1.10}$$

[5] If $\mathcal{A} = \varnothing$ then $\varnothing^{\uparrow} = \varnothing$.

[6] If $\mathcal{A}$ and $\mathcal{B}$ are filters, then $f[\mathcal{A}]$ and $f^{-}[\mathcal{B}]$ are filter-bases.

[7] Actually, the first inequality is valid for any map.

[8] It is implicit that $\mathcal{H}$ is either a filter or a filter-base on X, because the operation $\lim_{\xi}\mathcal{H}$ is defined only for such $\mathcal{H}$.

Lemma 5. *If ξ is a topology on a set X, then A is ξ-open if and only if $A \in \mathcal{F}$ for every filter $\mathcal{F}$ such that $A \cap \lim_\xi \mathcal{F} \neq \varnothing$ ([9]).*

1.4. **Back to continuity.** In terms of filters, $f \in C(\xi, \tau)$ whenever

$$f(\lim_\xi \mathcal{F}) \subset \lim_\tau f[\mathcal{F}] \tag{1.11}$$

for each filter $\mathcal{F}$ ([10]).

We shall now characterize continuity in terms of adherences of filters. Let us remind that a filter on X is called principal if it is of the form $\{A\}^\uparrow := \{F \subset X : A \subset F\}$, where $A \subset X$.

We write $\mathcal{F} \in \mathbb{F}$ whenever $\mathcal{F}$ is a filter, and $\mathcal{F} \in \mathbb{F}X$ if $\mathcal{F}$ is a filter on X. In other terms, $\mathbb{F}$ is the class of all filters, and $\mathbb{F}X$ the set of all filters on a set X. It follows from *Set Theory* that $\mathbb{F}$ is not a set. A class $\mathbb{J}$ is a *subclass* of $\mathbb{F}$ whenever $\mathcal{J} \in \mathbb{J}$ implies that there is a set X such that $\mathcal{J} \in \mathbb{F}X$. Then we write $\mathbb{J} \subset \mathbb{F}$, and we understand that $\mathbb{J}X = \mathbb{J} \cap \mathbb{F}X$.

Proposition 6. *Let ξ and τ be topologies on X and Y, respectively. Let $\mathbb{H}$ and $\mathbb{G}$ be classes of filters that contain all principal filters. Then the following statements are equivalent:*

(1) *$f \in C(\xi, \tau)$,*
(2) *for each $\mathcal{H} \in \mathbb{H}Y$,*
$$\operatorname{adh}_\xi f^-[\mathcal{H}] \subset f^-(\operatorname{adh}_\tau \mathcal{H}),$$
(3) *for each $\mathcal{G} \in \mathbb{G}X$,*
$$f(\operatorname{adh}_\xi \mathcal{G}) \subset \operatorname{adh}_\tau f[\mathcal{G}].$$

Proof. It is enough to prove these equivalences for the class $\mathbb{F}$ of all filters. Indeed, Proposition 1 establishes the analogous equivalence for principal filters. These two arguments entail Proposition 6, because $\mathbb{H}$ and $\mathbb{G}$ are classes of filters and both contain all principal filters.

Let (1) and $x \in \operatorname{adh}_\xi f^-[\mathcal{H}]$. Then there exists a filter $\mathcal{F}$ such that $x \in \lim_\xi \mathcal{F}$ and $\mathcal{F} \# f^-[\mathcal{H}]$, equivalently $f[\mathcal{F}] \# \mathcal{H}$. By continuity, $f(x) \in \lim_\tau f[\mathcal{F}]$, hence $f(x) \in \operatorname{adh}_\tau \mathcal{H}$ because of $f[\mathcal{F}] \# \mathcal{H}$, that is, $x \in f^-(\operatorname{adh}_\tau \mathcal{H})$, which is (2).

[9] If A is open, then the condition holds by definition. If A is not ξ-open, then there exists $x \in A$ such that $O \setminus A \neq \varnothing$ for each $O \in \mathcal{O}_\xi(x)$. In other words, $A^c := (X \setminus A) \in \mathcal{O}_\xi(x)^\#$, which means that $x \in \operatorname{adh}_\xi A^c$. Therefore, there exists a filter $\mathcal{F}$ such that $A^c \in \mathcal{F}$ and $x \in \lim_\xi \mathcal{F}$. Obviously, $A \notin \mathcal{F}$.

[10] Let $f \in C(\xi, \tau)$, and $x \in \lim_\xi \mathcal{F}$. To prove that $f(x) \in \lim_\tau f[\mathcal{F}]$, let $O \in \mathcal{O}_\tau(f(x))$. By continuity, $f^-(O) \in \mathcal{O}_\xi(x)$ and thus $f^-(O) \in \mathcal{F}$. Consequently, $O \supset f(f^-(O)) \in f[\mathcal{F}]$, that is, $f(x) \in \lim f[\mathcal{F}]$.
Conversely, assume (1.11) for each filter $\mathcal{F}$ and let O be a τ-open set. By Lemma 5, to show that $f^-(O)$ is ξ-open, let $x \in f^-(O)$ and take a filter $\mathcal{F}$ such that $x \in \lim_\xi \mathcal{F}$. By assumption, $f(x) \in \lim_\tau f[\mathcal{F}]$ and $f(x) \in O$, hence $O \in f[\mathcal{F}]$. It follows that $x \in f^-(f(x)) \subset f^-(O) \in f^-[f[\mathcal{F}]] \leq \mathcal{F}$, hence $f^-(O) \in \mathcal{F}$.

Let $\mathcal{H} := f[\mathcal{G}]$ and apply f to the inclusion in (2), getting

$$f(\mathrm{adh}_\xi f^-[f[\mathcal{G}]]) \subset f(f^-(\mathrm{adh}_\tau f[\mathcal{G}])),$$

and by (1.7) and (1.10), $\mathrm{adh}_\xi \mathcal{G} \subset \mathrm{adh}_\xi f^-[f[\mathcal{G}]]$, and thus

$$f(\mathrm{adh}_\xi \mathcal{G}) \subset f(\mathrm{adh}_\xi f^-[f[\mathcal{G}]]) \subset f(f^-(\mathrm{adh}_\tau f[\mathcal{G}])) \subset \mathrm{adh}_\tau f[\mathcal{G}].$$

Let (3) and let $A \subset X$. Set $\mathcal{G} := \{A\}^\uparrow$. Then we get $f(\mathrm{cl}_\xi A) \subset \mathrm{cl}_\tau f(A)$, that is, (3) of Proposition 1. It follows that $f \in C(\xi, \tau)$. □

Let us reconsider Proposition 6 in a broader context of convergence spaces, which will give us a new perspective, unavailable within the framework of topological spaces.

2. Framework of convergences

Topological spaces constitute a subclass of convergence spaces (see the foundational paper [4] of G. Choquet, and references [10],[15]).

A relation ξ between (non-degenerate) filters $\mathcal{F}$ on X and points of X ([11])

$$x \in \lim_\xi \mathcal{F} \tag{2.1}$$

is called a *convergence* provided that, for each $x, \mathcal{F}_0$ and $\mathcal{F}_1$,

$$\mathcal{F}_0 \subset \mathcal{F}_1 \Longrightarrow \lim_\xi \mathcal{F}_0 \subset \lim_\xi \mathcal{F}_1, \tag{2.2}$$

$$x \in \lim_\xi \{x\}^\uparrow. \tag{2.3}$$

If (2.1) holds, then we say that x *is a limit (point) of* $\mathcal{F}$ or $\mathcal{F}$ *converges to* x. We denote by $|\xi|$ the *underlying set* of ξ, that is, $|\xi| = X$ ([12]).

A convergence ζ is *finer* than a convergence ξ (equivalently, ξ is coarser than ζ)

$$\zeta \geq \xi,$$

if $\lim_\zeta \mathcal{F} \subset \lim_\xi \mathcal{F}$ for every filter $\mathcal{F}$. The ordered set of convergences on a given set is a complete lattice, that is, each non-empty set of convergences (on a given set) admits infima and suprema,

$$\lim_{\bigvee \Xi} \mathcal{F} = \bigcap_{\xi \in \Xi} \lim_\xi \mathcal{F}, \qquad \lim_{\bigwedge \Xi} \mathcal{F} = \bigcup_{\xi \in \Xi} \lim_\xi \mathcal{F}. \tag{2.4}$$

[11] As $\xi \subset \mathbb{F}X \times X$, a usual notation for the image of an element $\mathcal{F}$ of $\mathbb{F}X$ under ξ is $\xi(\mathcal{F})$. A purpose of our notation, $\lim_\xi \mathcal{F}$ instead of $\xi(\mathcal{F})$, is to emphasize the particular role of this relation, which should enhance readability.

[12] An underlying set can be restored from a convergence. In fact, $|\xi| = \bigcup_{\mathcal{F} \in \mathbb{F}} \lim_\xi \mathcal{F}$, beacuse ξ is a convergence on X, then $X \supset \bigcup_{\mathcal{F} \in \mathbb{F}} \lim_\xi \mathcal{F} \supset \bigcup_{x \in X} \lim_\xi \{x\}^\uparrow \supset X$.

2.1. **Open and closed sets for arbitrary convergences.** Convergence of filters in a topological space was defined in Definition 4 in terms of open sets, which were then characterized in Lemma 5 in terms of filters. Now, open sets will be defined for an arbitrary convergence!

Definition 7. Let ξ be a convergence. A subset O of $|\xi|$ is said to be ξ-*open* (with respect to ξ) if

$$O \cap \lim_\xi \mathcal{F} \neq \varnothing \Longrightarrow O \in \mathcal{F}.$$

A subset of $|\xi|$ is said to be ξ-*closed* if its complement is ξ-open. For each subset A of $|\xi|$, there exists the least ξ-closed set $\mathrm{cl}_\xi A$ including A, called the ξ-closure of A.

The family of all ξ-open sets is denoted by $\mathcal{O}_\xi$, and that of ξ-open sets containing x by $\mathcal{O}_\xi(x)$. Notice that $\mathcal{O}_\xi(x)$ is a filter base. The filter that it generates is called the *neighborhood filter* and is denoted by $\mathcal{N}_\xi(x)$.

We notice that the family $\mathcal{O}_\xi$ of all ξ-open sets fulfills the axioms of open sets of a topology ([13]). Denote this topology by $\mathrm{T}\xi$. It is straightforward that $\zeta \geq \xi$ implies $\mathrm{T}\zeta \geq \mathrm{T}\xi$; moreover, $\mathrm{T}\xi \leq \xi$, and $\mathrm{T}\xi \leq \mathrm{T}(\mathrm{T}\xi)$ for every convergence ξ.

A convergence θ is called a *topology* if

$$\mathcal{O}_\theta(x) \subset \mathcal{F} \Longrightarrow x \in \lim_\theta \mathcal{F}$$

for each x and $\mathcal{F}$, the converse implication being true for an arbitrary convergence.

It is obvious that a convergence θ is a topology if and only if $\theta \leq \mathrm{T}\theta$.

2.2. **Continuity in convergence spaces.** Let ξ and τ be convergences on X and Y, respectively. Let $f : X \to Y$ be a map. We say that f is *continuous* from ξ to τ if (1.11)

$$f(\lim_\xi \mathcal{F}) \subset \lim_\tau f[\mathcal{F}]$$

holds for each filter $\mathcal{F}$ on X ([14]).

For each convergence ξ on X, there exists the finest among the convergences τ on Y for which $f \in C(\xi, \tau)$. It is denoted by $f\xi$ and called the *final* convergence. Therefore, $f \in C(\xi, \tau)$ whenever $f\xi \geq \tau$.

[13]That is, $\varnothing$ and X are open, each union and each finite intersection of open sets is open.

[14]Continuity determines the notions of *subconvergence*, *quotient* convergence, *product* convergence, and so on. More precisely, a convergence ζ is called a subconvergence ξ if the *identity map* $i : |\zeta| \to |\xi|$ is an *embedding*, that is, a *homeomorphism* from $|\zeta|$ to $i(|\zeta|)$. The quotient convergence has already been defined. If Ξ is a set of convergences, then the product convergence is the initial convergence on $\prod_{\xi\in\Xi} |\xi|$ with respect to the projections $p_\theta : \prod_{\xi\in\Xi} |\xi| \to |\theta|$, that is, the coarsest convergence, for which p_θ is continuous to θ for each $\theta \in \Xi$.

On the other hand, for each convergence τ on Y, there exists the coarsest among the convergences ξ on X, for which $f \in C(\xi, \tau)$. It is denoted by $f^-\tau$ and called the *initial* convergence for (f, τ). Therefore, $f \in C(\xi, \tau)$ whenever $\xi \geq f^-\tau$.

Summarizing

$$f\xi \geq \tau \Longleftrightarrow f \in C\,(\xi, \tau) \Longleftrightarrow \xi \geq f^-\tau. \tag{2.5}$$

Let me point out here that $f^-\tau$ is a topology provided that τ is a topology, while the topologicity of ξ does not entail that of $f\xi$ (!).

Let us write down formulas for adherences in initial and final convergences for surjective maps ([15]), which will be used in the sequel (see [15, p. 215]):

$$\mathrm{adh}_{f\xi}\, \mathcal{H} = f(\mathrm{adh}_\xi\, f^-[\mathcal{H}]), \qquad \mathrm{adh}_{f^-\tau}\, \mathcal{G} = f^-(\mathrm{adh}_\tau\, f[\mathcal{G}]). \tag{2.6}$$

2.3. **Functors, reflectors, coreflectors.** From the category theory viewpoint, convergences are objects and continuous maps are morphisms of a category, which is concrete over the category of sets. Therefore, it is possible, and handy, to apply category theory objectwise. Functors are certain maps defined on classes of morphisms, and then specialized to the classes of objects viewed as identity morphisms. Here it is enough to define functors directly on objects.

We say that that an operator H associating to each convergence ξ, a convergence $H\xi$ on $|\xi|$, is called a *concrete functor* ([16]) if, for any convergences ζ, ξ

$$\zeta \geq \xi \Longrightarrow H\zeta \geq H\xi, \tag{isotone}$$

and

$$C(\xi, \tau) \subset C(H\xi, H\tau), \tag{functorial}$$

that is, each map continuous from ξ to τ, remains continuous from $H\xi$ to $H\tau$. It is straightforward that if H fulfills (isotone), then (functorial) holds if and only if, for each convergence τ and each map f,

$$H(f^-\tau) \geq f^-(H\tau), \tag{functorial 1}$$

equivalently, for each convergence ξ and each map f,

$$f(H\xi) \geq f(H\xi). \tag{functorial 2}$$

A concrete functor is called a *reflector* whenever

$$H(H\xi) = H\xi, \tag{idempotent}$$

$$H\xi \leq \xi. \tag{contractive}$$

---

[15]The formula for the adherence of the initial map also holds if f is not surjective.

[16]The term *concrete* means that $|H\xi| = |\xi|$.

The topologizer T, the pretopologizer S_0 and the pseudotopologizer S are reflectors. A concrete functor is called a *coreflector* if (idempotent) and

$$\text{(expansive)} \qquad \xi \leq H\xi.$$

Other examples of reflectors and coreflectors are given in Appendix 11.4 and 11.7.

2.4. **Adherence-determined reflectors.** The adherence $\mathrm{adh}_\xi \mathcal{A}$ of a family $\mathcal{A}$ with respect to a convergence ξ is defined, as in the topological case, by (1.9). In general, $\mathrm{adh}_\xi \mathcal{A} \subset \bigcap_{A\in\mathcal{A}} \mathrm{cl}_\xi A$, the equality holding if ξ is a topology. Hence, for a subset A of a topological space, $\mathrm{adh}_\xi A = \mathrm{cl}_\xi A$, which means that, in topological spaces, adherence coincides with the usual closure.

We say that a class of filters $\mathbb{H}$ is *initial* if $f^-[\mathcal{H}] \in \mathbb{H}$ for each $\mathcal{H} \in \mathbb{H}$. An operator H is called *adherence-determined* if there exists an initial class of filters $\mathbb{H}$ containing all principal filters and such that

$$\lim_{H\theta} \mathcal{F} = \bigcap_{\mathbb{H}\ni\mathcal{H}\#\mathcal{F}} \mathrm{adh}_\theta \mathcal{H}. \tag{2.7}$$

It turns out that each adherence-determined operator is a concrete reflector. Moreover, for each filter $\mathcal{H}$,

$$\mathcal{H} \in \mathbb{H} \Longrightarrow \mathrm{adh}_{H\theta} \mathcal{H} = \mathrm{adh}_\theta \mathcal{H} \tag{2.8}$$

We shall consider here the cases of

(1) *pretopologies* ($\mathbb{H} = \mathbb{F}_0$ is the class of *principal filters*, and $H = \mathrm{S}_0$ is the *pretopologizer*),
(2) *paratopologies* ($\mathbb{H} = \mathbb{F}_1$ is the class of *countably based filters*, and $H = \mathrm{S}_1$ is the *paratopologizer*),
(3) *pseudotopologies* ($\mathbb{H} = \mathbb{F}$ is the class of all filters, and $H = \mathrm{S}$ is the *pseudotopologizer*).

Topologies are adherence-determined in a generalized sense (see Appendix 11.4).

3. Characterizations of continuity

We are now in a position to revisit Proposition 6 in the context of convergences. It turns out that now the characterizations in terms of adherences of filters $\mathcal{G}$ and $\mathcal{H}$ depend on the classes over which they range.

A class $\mathbb{J}$ of filters is said to be *transferable* ([17]) if $\mathcal{J} \in \mathbb{J}$ implies that $R[\mathcal{J}] \in \mathbb{J}$ (see Appendix 11.1) for each relation R. In particular, maps

[17] In a broader context (see [15]), they are called $\mathbb{F}_0$-composable.

and their inverse relations are relations. Therefore, transferable classes are initial. The classes of all filters, countably based filters, and principal filters are transferable.

Theorem 8. *Let $\mathbb{J}$ be a transferable class of filters. If J is a $\mathbb{J}$-adherence-determined reflector (2.7), then the following statements are equivalent:*

(1) $f \in C(J\xi, J\tau)$,

(2) *for each* $\mathcal{H} \in \mathbb{J}$,

$$\operatorname{adh}_\xi f^-[\mathcal{H}] \subset f^-(\operatorname{adh}_\tau \mathcal{H}), \tag{3.1}$$

(3) *for each* $\mathcal{G} \in \mathbb{J}$,

$$f(\operatorname{adh}_\xi \mathcal{G}) \subset \operatorname{adh}_\tau f[\mathcal{G}]. \tag{3.2}$$

Proof. If $f \in C(J\xi, J\tau)$, that is, $f\xi \geq f(J\xi) \geq J\tau$, hence, for each filter $\mathcal{H}$,

$$\operatorname{adh}_{f\xi} \mathcal{H} \subset \operatorname{adh}_{J\tau} \mathcal{H}, \tag{3.3}$$

and if $\mathcal{H} \in \mathbb{J}$, by (2.8), $f(\operatorname{adh}_\xi f^-[\mathcal{H}]) = \operatorname{adh}_{f\xi} \mathcal{H} \subset \operatorname{adh}_\tau \mathcal{H}$. On applying f^- to this inequality, we get (3.1).

If $\mathcal{G} \in \mathbb{J}$, then $\mathcal{H} := f[\mathcal{G}] \in \mathbb{J}$ and $f^-[\mathcal{H}] = f^-[f[\mathcal{G}]] \leq \mathcal{G}$. Thus (3.1) implies $\operatorname{adh}_\xi \mathcal{G} \subset f^-(\operatorname{adh}_\tau f[\mathcal{G}])$, thus on applying f, we recover (3.2).

On applying f^- to (3.2), we assure that, for each filter $\mathcal{G} \in \mathbb{J}$,

$$\operatorname{adh}_\xi \mathcal{G} \subset \operatorname{adh}_{f^-\tau} \mathcal{G}. \tag{3.4}$$

Hence, for each filter $\mathcal{F}$,

$$\lim_\xi \mathcal{F} \subset \bigcap_{\mathbb{J} \ni \mathcal{G} \# \mathcal{F}} \operatorname{adh}_\xi \mathcal{G} \subset \bigcap_{\mathbb{J} \ni \mathcal{G} \# \mathcal{F}} \operatorname{adh}_{f^-\tau} \mathcal{G} = \lim_{J(f^-\tau)} \mathcal{F},$$

that is, $\xi \geq J(f^-\tau)$, hence $J\xi \geq J(f^-\tau) \geq f^-(J\tau)$, because J is a reflector. □

Remark 9. In the case of topologies, continuity is equivalent to the formulas above for every class $\mathbb{J}$ of filters that contains all principal filters (see Propositions 1 and 6, and the comments at the end of the section). This is because $\mathrm{T} \leq J$ for every $\mathbb{J}$-adherence-determined reflector corresponding to such a class $\mathbb{J}$.

4. Inverse continuity

Definition 10. A surjective map $f : |\xi| \to |\tau|$ is called $\mathbb{H}$-*quotient* if

$$f^-(\operatorname{adh}_\tau \mathcal{H}) \subset \operatorname{adh}_\xi f^-[\mathcal{H}] \tag{4.1}$$

holds for each $\mathcal{H} \in \mathbb{H}$, and it is called $\mathbb{G}$-*perfect* if

$$\operatorname{adh}_\tau f[\mathcal{G}] \subset f(\operatorname{adh}_\xi \mathcal{G}) \tag{4.2}$$

holds for each $\mathcal{G} \in \mathbb{G}$.

Observe that (4.1) and (4.2) are obtained from (3.1) and (3.2), respectively, by reversing the inclusion.

Proposition 11. *Let $\mathbb{J}$ be a transferable class of filters. If a surjective map is $\mathbb{J}$-perfect then it is $\mathbb{J}$-quotient.*

Proof. Assume (4.2), $\mathcal{H} \in \mathbb{J}$ and $x \in f^-(\mathrm{adh}_\tau \mathcal{H})$. Let $\mathcal{G} := f^-[\mathcal{H}] \in \mathbb{J}$ by transferability, and $f[f^-[\mathcal{H}]] = \mathcal{H}$, because f is surjective. Then, by (4.2), $x \in f^-(\mathrm{adh}_\tau \mathcal{H}) \subset f^-(f(\mathrm{adh}_\xi f^-[\mathcal{H}])) \subset \mathrm{adh}_\xi f^-[\mathcal{H}]$, which is (4.3). $\square$

The converse is not true, as shows Example 32.

Remark 12. If $\mathbb{J}$ is transferable, then a bijective map is $\mathbb{J}$-quotient if and only if it is $\mathbb{J}$-perfect.

In order to better appreciate the sense of the defined notions, let us express them in terms of adherences with respect to final and initial convergences. By using (2.6), we conclude that (4.1) is equivalent to

$$\mathrm{adh}_\tau \mathcal{H} \subset \mathrm{adh}_{f\xi} \mathcal{H}, \tag{4.3}$$

which is the inversion of (3.3), whereas (4.2) is equivalent to

$$\mathrm{adh}_{f^-\tau} \mathcal{G} \subset f^-(f(\mathrm{adh}_\xi \mathcal{G})), \tag{4.4}$$

which is not the inversion of (3.4)!

Remark 13. Contemplating this asymmetry, remember that the initial convergence can be seen as a convergence of fibers rather than that of individual points. I mean by *convergence of fibers of f* the property that $x \in \lim_{f^-\tau} \mathcal{F}$ implies that

$$f^-(f(x)) \subset \lim_{f^-\tau} f^-[f[\mathcal{F}]].$$

Indeed, $x \in \lim_{f^-\tau} \mathcal{F}$ amounts to $f(x) \in \lim_\tau f[\mathcal{F}]$, and thus $f^-(f(x)) \subset f^-(\lim_\tau f[\mathcal{F}]) = \lim_{f^-\tau} \mathcal{F}$. On the other hand, $f(f^-(f(A))) = f(A)$ for each (not necessarily surjective) map f and each A, and thus

$$\lim_{f^-\tau} f^-[f[\mathcal{F}]] = f^-(\lim_\tau f[f^-[f[\mathcal{F}]]]) = f^-(\lim_\tau f[\mathcal{F}]) = \lim_{f^-\tau} \mathcal{F}.$$

Theorem 14. [5] *If H is a reflector adherence-determined by $\mathbb{H}$, then a map $f : |\xi| \to |\tau|$ is* $\mathbb{H}$-quotient *if and only if*

$$\tau \geq H(f\xi). \tag{4.5}$$

Proof. By (2.7), (4.3) holds for each $\mathcal{H} \in \mathbb{H}$, if and only if, for each filter $\mathcal{F}$,

$$\lim_\tau \mathcal{F} \subset \lim_{H\tau} \mathcal{F} = \bigcap_{\mathbb{H} \ni \mathcal{H} \# \mathcal{F}} \mathrm{adh}_\tau \mathcal{H} \subset \bigcap_{\mathbb{H} \ni \mathcal{H} \# \mathcal{F}} \mathrm{adh}_{f\xi} \mathcal{H} = \lim_{H(f\xi)} \mathcal{F}.$$

$\square$

Formula (4.5) reveals that $\mathbb{H}$-quotient maps are in fact quotient maps with respect to a reflector H that is *adherence-determined* by $\mathbb{H}$. This way, (4.5) extends the definition to arbitrary concrete reflectors: f is *H-quotient* whenever (4.5) holds. It is practical to use either the term *H-quotient* or *$\mathbb{H}$-quotient* when H is adherence-determined by $\mathbb{H}$.

An important example of a non-adherence-determined reflector is the identity reflector I, the corresponding reflective class of which is that of all convergences ([18]). Of course, a surjective map $f : |\xi| \to |\tau|$ is an I-*quotient map,* whenever

$$\tau \geq f\xi. \tag{4.6}$$

Traditionally, I-quotient maps are called *almost open* [2]. It follows from the definition that ([19])

Proposition 15. *A surjective map $f : |\xi| \to |\tau|$ is almost open if and only if for each filter $\mathcal{H}$ and each $y \in \lim_\tau \mathcal{H}$, there exists $x \in f^-(y)$ and a filter $\mathcal{F}$ such that $x \in \lim_\xi \mathcal{F}$ and $f[\mathcal{F}] = \mathcal{H}$.*

Recall that a surjective map $f : |\xi| \to |\tau|$ is called *open* provided that $f(O)$ is τ-open for each ξ-open subset O of $|\xi|$. We shall see that each open map (between topologies) is almost open.

Proposition 16. *A surjective map $f : |\xi| \to |\tau|$ is open provided that for each filter $\mathcal{H}$ and each $y \in \lim_\tau \mathcal{H}$ and for every $x \in f^-(y)$ there exists a filter $\mathcal{F}$ such that $x \in \lim_\xi \mathcal{F}$, and $f[\mathcal{F}] = \mathcal{H}$.*

Proof. Suppose that the condition holds, and O is ξ-open. If $y \in f(O) \cap \lim_\tau \mathcal{H}$, then there exists $x \in f^-(y) \cap O$. On the other hand, there exists a filter $\mathcal{F}$ such that $x \in \lim_\xi \mathcal{F}$, and $f[\mathcal{F}] = \mathcal{H}$. As O is ξ-open, $O \in \mathcal{F}$ and thus $f(O) \in f[\mathcal{F}] = \mathcal{H}$, proving that $f(O)$ is τ-open. □

If ξ is a topology, then the converse also holds [15, p. 398].

5. Characterizations in terms of covers

5.1. Covers, inherences, adherences.

Open covers are not an adequate tool in general convergences, because open sets do not determine non-topological convergences. Therefore, the notion of cover need be extended.

Let ξ be a convergence. A family $\mathcal{P}$ of sets is a *ξ-cover* of A if $\mathcal{P} \cap \mathcal{F} \neq \varnothing$ for every filter $\mathcal{F}$ such that $A \cap \lim_\xi \mathcal{F} \neq \varnothing$.

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[18] I is not adherence-determined, because the greatest adherence-determined reflector is the pseudotopologizer S, which adherence-determined by the class $\mathbb{F}$ of all filters. Indeed, there exist convergences that are not pseudotopologies (Example 48).

[19] Indeed, as $f\xi$ is the finest convergence on $|\tau|$, for which f is continuous, $y \in \lim_{f\xi} \mathcal{H}$ whenever there exists $x \in f^-(y)$ and a filter $\mathcal{F}$ such that $x \in \lim_\xi \mathcal{F}$ and $f[\mathcal{F}] = \mathcal{H}$.

The concept of *cover* is closely related to those of *adherence* and *inherence*. If $\mathcal{P}$ be a family of subsets of X, then $\mathcal{P}_c := \{X \setminus P : P \in \mathcal{P}\}$. The inherence $\operatorname{inh}_\xi \mathcal{P}$ of a family $\mathcal{P}$ is defined by

$$\operatorname{inh}_\xi \mathcal{P} := (\operatorname{adh}_\xi \mathcal{P}_c)^c. \tag{5.1}$$

Observe that if $\mathcal{P}$ a family and $\mathcal{F}$ is a filter, then

$$\mathcal{P} \cap \mathcal{F} = \varnothing \Longleftrightarrow \mathcal{P}_c \# \mathcal{F}. \tag{5.2}$$

The following theorem constitute a special case of [7, Theorem 3.1].

Theorem 17 ([15, p. 69])**.** *The following statements are equivalent:*

1. *$\mathcal{P}$ is a ξ-cover of A,*
2. *$A \subset \operatorname{inh}_\xi \mathcal{P}$,*
3. *$\operatorname{adh}_\xi \mathcal{P}_c \cap A = \varnothing$.*

Proof. By definition, (1) holds whenever $\mathcal{P} \cap \mathcal{F} \neq \varnothing$ for every filter $\mathcal{F}$ such that $A \cap \lim_\xi \mathcal{F} \neq \varnothing$. Equivalently, if $\mathcal{P} \cap \mathcal{F} = \varnothing$ then $A \cap \lim_\xi \mathcal{F} = \varnothing$ for each filter $\mathcal{F}$. By (5.2), $\mathcal{P}_c \# \mathcal{F}$ implies that $A \cap \lim_\xi \mathcal{F} = \varnothing$, hence (1) and (3) are equivalent. Now (3) amounts to $(\operatorname{inh}_\xi \mathcal{P})^c \cap A = \varnothing$, that is, to (2). □

Consequently, we shall use the symbol

$$A \subset \operatorname{inh}_\xi \mathcal{P}$$

for "$\mathcal{P}$ is a ξ-cover of A".

Remark 18. In order to relate this notion to open covers, observe that if ξ is a topology, then $\mathcal{P}$ is a ξ-cover of A whenever $A \subset \bigcup_{P \in \mathcal{P}} \operatorname{int}_\xi P$, where $\operatorname{int}_\xi P$ is the ξ-*interior* of P.

5.2. **Characterizations.** Proposition 19 below rephrases Definition 10 in terms of the original definitions, introduced in [5, 7].

Proposition 19. *A surjective map f is $\mathbb{H}$-quotient if and only if, for each $\mathcal{H} \in \mathbb{H}$ and each y,*

$$y \in \operatorname{adh}_\tau \mathcal{H} \Longrightarrow f^-(y) \cap \operatorname{adh}_\xi f^-[\mathcal{H}] \neq \varnothing; \tag{5.3}$$

it is $\mathbb{G}$-perfect if and only if, for each $\mathcal{G} \in \mathbb{G}$ and each y,

$$y \in \operatorname{adh}_\tau f[\mathcal{G}] \Longrightarrow f^-(y) \cap \operatorname{adh}_\xi \mathcal{G} \neq \varnothing. \tag{5.4}$$

We are now in a position to characterize quotient-like and perfect-like maps in terms of covers.

We shall see that for a transferable class $\mathbb{H}$, a surjective map f is $\mathbb{H}$-quotient if and only if the image under f of every cover (from the dual class of $\mathbb{H}$) of $f^-(y)$ is a cover of y. More precisely,

Proposition 20. *Let $\mathbb{H}$ be transferable. A surjective map f is $\mathbb{H}$-quotient if and only if, for each $\mathcal{Q}$ such that $\mathcal{Q}_c \in \mathbb{H}$ and each y,*

$$\operatorname{inh}_\xi \mathcal{Q} \supset f^-(y) \Longrightarrow y \in \operatorname{inh}_\tau f[\mathcal{Q}]. \tag{5.5}$$

Proof. Indeed, set $\mathcal{Q} = f^-[\mathcal{H}_c] = f^-[\mathcal{H}]_c$ in (5.3). Then $\mathcal{H}_c = f[f^-[\mathcal{H}_c]] = f[\mathcal{Q}]$.

By (5.1), $y \in \operatorname{adh}_\tau f[\mathcal{Q}]_c$ implies $f^-(y) \cap \operatorname{adh}_\xi \mathcal{Q}_c \neq \varnothing$, we infer that $y \notin \operatorname{inh}_\tau f[\mathcal{Q}]$ implies $f^-(y) \cap (\operatorname{inh}_\xi \mathcal{Q})^c \neq \varnothing$, which is equivalent to (5.4). □

We shall now see that if $\mathbb{G}$ is transferable, then f is $\mathbb{G}$-perfect if and only if $\mathcal{P}$ is a cover of y provided that the preimage under f of a family $\mathcal{P}$ (from the dual class of $\mathbb{G}$) is a cover of $f^-(y)$. More precisely,

Proposition 21. *Let $\mathbb{G}$ be transferable. A surjective map f is $\mathbb{G}$-perfect if and only if, for each $\mathcal{P}$ such that $\mathcal{P}_c \in \mathbb{G}$ and each y,*

$$\operatorname{inh}_\xi f^-[\mathcal{P}] \supset f^-(y) \Longrightarrow y \in \operatorname{inh}_\tau \mathcal{P}. \tag{5.6}$$

Proof. In fact, by setting $\mathcal{G} = f^-[\mathcal{P}_c] = f^-[\mathcal{P}]_c$, hence $f[\mathcal{G}] = f[f^-[\mathcal{P}_c]] = \mathcal{P}_c$. Thus (5.4) becomes $y \in \operatorname{adh}_\tau \mathcal{P}_c \Longrightarrow f^-(y) \cap \operatorname{adh}_\xi f^-[\mathcal{P}]_c \neq \varnothing$. By (5.1), $y \notin \operatorname{inh}_\tau \mathcal{P}$ implies $f^-(y) \cap (\operatorname{inh}_\xi f^-[\mathcal{P}])^c \neq \varnothing$, that is, (5.6). □

Remark 22. Notice that in (5.6) and (5.5) above, covers are either ideals (closed under finite unions and subsets) or ideal bases.

6. Variants of quotient maps

Many variants of quotient maps were defined in topology, the profusion being due to a search for most adequate classes of maps that preserve various important types of topological spaces [21]. I will list the traditional names of $\mathbb{H}$-quotients (see Proposition 19) for specific classes $\mathbb{H}$ of filters (in the brackets):

(1) *quotient* (principal filters of $f\xi$-closed sets),
(2) *hereditarily quotient* (principal filters),
(3) *countably biquotient* (countably based filters),
(4) *biquotient* (all filters).

In topology, these notions were originally defined in terms of covers. Moreover, they comprised continuity. Remember that we do not ask that the considered maps be continuous!

We shall list below (in topological context) cover definitions of quotient-like maps. By Theorem 17 and Proposition 20, a surjective map $f : |\xi| \to |\tau|$ is

(1) *quotient* if a set is τ-open, provided that its preimage under f is ξ-open,

(2) *hereditarily quotient* if whenever an open set P includes $f^{-}(y)$, then $f(P)$ is a neighborhood of y ([20]),
(3) *countably biquotient* if, for each countable open cover $\mathcal{P}$ of $f^{-}(y)$, there exists a finite subfamily $\mathcal{P}_0$ such that $\bigcup_{P\in\mathcal{P}_0} f(P)$ is a neighborhood of y ([21]),
(4) *biquotient* if for each open cover $\mathcal{P}$ of $f^{-}(y)$, there exists a finite subfamily $\mathcal{P}_0$ such that $\bigcup_{P\in\mathcal{P}_0} f(P)$ is a neighborhood of y ([21]).

Recall that a generalization of H-quotient maps to arbitrary (concrete) reflectors resulted in *almost open* maps (with $H = \mathrm{I}$, the identity reflector). On the other hand, in topology, *open maps* are almost open.

7. Mixed properties and functorial inequalities

Numerous classical properties of topologies can be expressed in terms of inequalities, involving two or more functors, typically, a coreflector and a reflector. These properties make sense for an arbitrary convergence, and will be introduced here in the convergence framework. Let us say that a convergence ξ is JE if

$$\xi \geq JE\xi, \tag{7.1}$$

where J is a (concrete) reflector and E is a (concrete) coreflector (see subsections 2.3 and 11.7). A composition of two functors is, of course, a functor.

Let us give a couple of examples of JE-properties. In the examples below, we shall employ a reflector S_1 on the class of convergences that are determined by adherences pf countably based filters (see Appendix 11.4), and a coreflector I_1 on the subclass of convergences of *countable character* (see Appendix 11.7).

Example 23. ([21]) A topology is called *sequential* if each sequentially closed set is closed. If ξ is a topology, then $\mathrm{Seq}\,\xi$ is the coarsest *sequentially based convergence* (see Appendix 11.7) in general non-topological, that is finer than ξ.

Then $\mathrm{T}\,\mathrm{Seq}\,\xi$ stands for the topology, for which the open sets and the closed sets are determined by sequential filters, that is, are sequentially open and closed, respectively. Therefore, a topology ξ is sequential if it coincides with $\mathrm{T}\,\mathrm{Seq}\,\xi$, which is equivalent to

$$\xi \geq \mathrm{T}\,\mathrm{Seq}\,\xi. \tag{7.2}$$

[20] for each $y \in |\tau|$.
[21] Mind that *sequential topology* and *sequentially based convergence* are different notions!

Incidentally, $\mathrm{T}\,\mathrm{Seq} = \mathrm{T}\,\mathrm{I}_1$, hence the inequality above amounts to $\xi \geq \mathrm{T}\,\mathrm{I}_1\,\xi$.

Example 24. A convergence ξ is called *Fréchet* if $x \in \mathrm{adh}_\xi A$ implies the existence of a sequential filter $\mathcal{E}$ such that $A \in \mathcal{E}$ and $x \in \lim_\xi \mathcal{E}$. It is straightforward that ξ is Fréchet whenever

$$\xi \geq \mathrm{S}_0\,\mathrm{Seq}\,\xi = \mathrm{S}_0\,\mathrm{I}_1\,\xi.$$

Example 25. A convergence ξ is called *countably bisequential* ([22]) if $x \in \bigcap_n \mathrm{adh}_\xi A_n$ for a decreasing sequence $(A_n)_n$ implies the existence of a sequence $(x_n)_n$ such that $x_n \in A_n$ for each n, and $x \in \lim_\xi (x_n)_n$. It is straightforward that ξ is countably bisequential if and only if $\mathcal{A}$ is a countably based filter, and $x \in \mathrm{adh}_\xi \mathcal{A}$, then there is a countably based filter $\mathcal{E}$ such that $\mathcal{A}\#\mathcal{E}$ and $x \in \lim \mathcal{E}$, that is, whenever

$$\xi \geq \mathrm{S}_1\,\mathrm{Seq}\,\xi = \mathrm{S}_1\,\mathrm{I}_1\,\xi.$$

Example 26. A convergence ξ is called *bisequential* if for every ultrafilter $\mathcal{U}$ such that $x \in \lim_\xi \mathcal{U}$, there exists a countably based filter $\mathcal{H} \leq \mathcal{U}$ such that $x \in \lim_\xi \mathcal{H}$. In other words, ξ is bisequential whenever $\lim_\xi \mathcal{U} \subset \lim_{\mathrm{I}_1 \xi} \mathcal{U}$ for each ultrafilter $\mathcal{U}$, that is, whenever

$$\xi \geq \mathrm{S}\,\mathrm{I}_1\,\xi.$$

It has been long well-known that (topologically) continuous quotient maps preserve (topologically) sequential topologies, but do not preserve Fréchet topologies. On the other hand, continuous hereditarily quotient maps preserve Fréchet topologies, but not countably bisequential topologies, which are preserved by continuous countably biquotient maps, and so on.

Theorem 28 below explains the mechanism of preservation of mixed properties. Roughly speaking, a JE-property is preserved by continuous L-quotient maps provided that L is a concrete reflector such that $J \leq L$.

It follows directly from the definition that

Proposition 27. *A convergence ξ is JE if and only if the identity from $E\xi$ to ξ is J-quotient.*

On the other hand, the following observation generalizes several classical theorems concerning particular preservation cases by E. Michael [21], A. V. Arhangel'skii [1], V. I. Ponomarev [25], S. Hanai [18], F. Siwiec [26], and others.

Theorem 28. *If $f \in C(\xi, \tau)$ is a J-quotient map, and ξ is a JE-convergence, then τ is also a JE-convergence. Conversely, each JE-convergence is a continuous J-quotient image of an E-convergence.*

[22] Also called *strongly Fréchet*.

Proof. By definition, f is a J-quotient map whenever $\tau \geq J(f\xi)$, and ξ is a JE-map whenever $\xi \geq JE\xi$. Putting these two inequalities together, yields

$$\tau \geq J(f\xi) \geq J(f(JE\xi)),$$

and, by (functorial), $J(f(JE\xi)) \geq JE(f\xi)$, which is greater than $JE\tau$, in view of continuity. By concatenating these formulas, we get $\tau \geq JE\tau$.

Conversely, $\tau \geq JE\tau = J(i(E\tau))$, where $i \in C(\xi, \tau)$ is the identity (see Proposition 27). $\square$

The mentioned classical theorems, however, assert that each JE topology is a J-quotient image of a topology (sometimes even metrizable or paracompact) θ such that $\theta \geq E\theta$. In our proof, $E\tau$ is in general not a topology. This topologicity requirement might provide technical difficulties (see [5, Section 5], for example).

Below, we shall give an example of the discussed preservation scheme only in case of $E = \mathrm{I}_1$ (the coreflector of *countable character*). The interested readers will find similar, but more exhaustive, tables in [5, p. 11] and in [15, p. 400]. The first column indicates the type of quotient, the second the corresponding reflector. A quotient type in a given line preserves the properties of the same line and of the lines that are below. The strength of the listed properties decreases downwards.

quotient type	reflector J	mixed property	type $J\mathrm{I}_1$
almost open	I	countable character	I_1
biquotient	S	bisequential	SI_1
countably biquotient	S_1	countably bisequential	$\mathrm{S}_1\mathrm{I}_1$
hereditarily quotient	S_0	Fréchet	$\mathrm{S}_0\mathrm{I}_1$
quotient	T	sequential	TI_1

8. Various classes of perfect-like maps

As for quotient-like maps, traditional definitions of perfect-like maps incorporate continuity. In most situations, however, continuity is superfluous, and no continuity is assumed here. Recall the definition of $\mathbb{G}$-perfect maps: a map f from $|\xi|$ to $|\tau|$ whenever (4.2)

$$\mathrm{adh}_\tau\, f[\mathcal{G}] \subset f(\mathrm{adh}_\xi\, \mathcal{G})$$

holds for each filter $\mathcal{G} \in \mathbb{G}$. This can be also reformulated as

$$y \in \mathrm{adh}_\tau\, f[\mathcal{G}] \Longrightarrow f^-(y) \cap \mathrm{adh}_\xi\, \mathcal{G} \neq \varnothing \tag{8.1}$$

for each filter $\mathcal{G} \in \mathbb{G}$.

I list below the names of surjective maps f corresponding to (8.1) for specific classes $\mathbb{G}$ of filters (written in brackets):

(1) *closed* (principal filters of closed sets),
(2) *adherent* (principal filters),
(3) *countably perfect* (countably based filters),
(4) *perfect* (all filters).

Perfect maps are *countably perfect*, which are *adherent*, which, in turn, are *closed* ([23]). *Adherent maps* were introduced in [5] in the context of general convergences; they were not detected before, because they coincide with *closed maps* in the framework of topologies. Here we show that they do not coincide in general.

Example 29. Let $\xi = \mathrm{S}_0\xi > \mathrm{T}\xi = \tau$. Then the identity $i \in C(\xi, \tau)$ is a quotient map, that is, $\tau \geq \mathrm{T}\xi$, hence a closed map by Remark 12. As $\mathrm{S}_0\xi > \tau$, it is not a hereditarily quotient map, hence not an adherent map by Remark 12.

A surjective map is *closed* if it maps closed sets onto closed sets. Indeed, if $\mathcal{G} = \{\mathrm{cl}_\xi G\}^\uparrow$ then $\mathrm{adh}_\xi G \subset G$, hence (4.2) becomes $\mathrm{adh}_\tau f(G) \subset f(G)$, that is, whenever $f(G)$ is τ-closed.

The following two propositions characterize perfect and countably perfect maps in topological spaces in traditional terms. They are particular cases of a more abstract situation, discussed in [15, Proposition XV.3.6].

A map between topological spaces is called *perfect* [16, p. 236] if it is continuous, closed and all its fibers are compact (in a Hausdorff domain of the map [24]). In this paper we consider only surjective maps. On the other hand, continuity is irrelevant in these propositions.

Proposition 30. *A surjective map between topological spaces is* perfect *if and only if it is closed and all its fibers are compact.*

Proposition 31. *A surjective map between topological spaces is* countably perfect *if and only if it is closed and all its fibers are countably compact.*

We shall provide proofs to these propositions in Section 9.

The following table details Proposition 11. The strength of the listed properties decreases downwards.

[23]To see that *adherent* implies *closed*, take a ξ-closed set G, that is, $\mathrm{adh}_\xi G = G$ and set $\mathcal{G} = \{G\}^\uparrow$ in the definition of adherent map above, with $\mathbb{G}$ the class of principal filters, and get $\mathrm{adh}_\tau f[G] \subset f(G)$, that is, $f(G)$ is τ-closed.

[24]Hausdorffness is assumed, because compactness in [16] is defined only for Hausdorff topologies.

perfect-like	$\Rightarrow$	quotient-like
		open
		almost open
perfect	$\Rightarrow$	biquotient
countably perfect	$\Rightarrow$	countably biquotient
adherent	$\Rightarrow$	hereditarily quotient
closed	$\Rightarrow$	topologically quotient

No arrow can be reversed. Indeed,

Example 32. Let $f : \mathbb{R} \to S^1$ be given by $f(x) := (\cos 2\pi x, \sin 2\pi x)$ ([25]). It follows immediately from Proposition 16 that f is open, hence, has all the properties from the right-hand column. Notice that the set $\{n + \frac{1}{n} : n \in \mathbb{N}\}$ is closed, but its image under f is not closed, so that f is not closed, and thus has no property from the left-hand column.

9. Compactness characterizations

In Appendix 11.5, a relation $R \subset W \times Z$ (with convergences θ on W and σ on Z) is said to be $\mathbb{J}$-*compact* at w if for every $\mathcal{F}$ such that $w \in \lim_\theta \mathcal{F}$ implies that $R(w) \# \operatorname{adh}_\sigma \mathcal{J}$ for each $\mathcal{J} \# R[\mathcal{F}]$ such that $\mathcal{J} \in \mathbb{J}$. Now I will characterize the classes of maps discussed in terms of such relations. These characterizations enable us to see new interrelations (see Propositions 30 and 31 and Section 10).

9.1. Continuous maps.

Theorem 33. *If J is a $\mathbb{J}$-adherence-determined reflector, then $f \in C(J\xi, J\tau)$ if and only if f is $\mathbb{J}$-compact from ξ to τ.*

Proof. By definition,

$$\lim_{J\tau} f[\mathcal{F}] = \bigcap_{\mathbb{J} \ni \mathcal{H} \# f[\mathcal{F}]} \operatorname{adh}_\tau \mathcal{H}. \tag{9.1}$$

Let $x \in \lim_\xi \mathcal{F}$. If $f \in C(J\xi, J\tau) = C(\xi, J\tau)$ ([26]), then $f(x)$ belongs to (9.1), equivalently, $\{f(x)\} \# \operatorname{adh}_\tau \mathcal{H}$ for each $\mathcal{H} \in \mathbb{J}$ such that $\mathcal{H} \# f[\mathcal{F}]$, that is, f is $\mathbb{J}$-compact from ξ to τ. □

[25] We consider the usual topologies on the real line $\mathbb{R}$ and on the unit circle S^1.

[26] $C(\xi, J\tau) \subset C(J\xi, J\tau)$, because $\xi \geq J\xi$. On the other hand, by applying J to $f\xi \geq J\tau$, we get $J(f\xi) \geq JJ\tau = J\tau$. As J is a functor, $f(J\xi) \geq J(f\xi)$ by (functorial 2).

9.2. **Perfect-like maps.** Perfect-like maps f were characterized in [7, 9] in terms of various types of compactness of f^-.

Theorem 34. *A surjective map f is $\mathbb{J}$-perfect if and only if the relation f^- is $\mathbb{J}$-compact.*

Proof. Recall (5.4) that a map $f : |\xi| \to |\tau|$ is $\mathbb{J}$-perfect whenever, for each $\mathcal{J} \in \mathbb{J}$ and each $y \in |\tau|$,

$$y \in \operatorname{adh}_\tau f[\mathcal{J}] \Longrightarrow f^-(y) \# \operatorname{adh}_\xi \mathcal{J}.$$

Assume that f is $\mathbb{J}$-perfect and let $y \in \lim_\tau \mathcal{F}$ and consider $\mathcal{J} \# f^-[\mathcal{F}]$ such that $\mathcal{J} \in \mathbb{J}$. Accordingly, $f[\mathcal{J}] \# \mathcal{F}$ hence $y \in \operatorname{adh}_\tau f[\mathcal{J}]$, and thus, by assumption, $f^-(y) \# \operatorname{adh}_\xi \mathcal{J}$, which proves that f^- is $\mathbb{J}$-compact.

Conversely, let f^- be $\mathbb{J}$-compact, and $y \in \operatorname{adh}_\tau f[\mathcal{J}]$ with $\mathcal{J} \in \mathbb{J}$. Consequently, there exists $\mathcal{F} \# f[\mathcal{J}]$ and such that $y \in \lim_\tau \mathcal{F}$. By the compactness assumption, $f^-(y) \# \operatorname{adh}_\xi \mathcal{J}$ for each $\mathcal{J} \in \mathbb{J}$ such that $\mathcal{J} \# f^-[\mathcal{F}]$. □

By Proposition 52,

Proposition 35. *Let $\mathbb{J}$ be transferable, and f be $\mathbb{J}$-perfect. Then $f^-(K)$ is $\mathbb{J}$-compact, provided that K is $\mathbb{J}$-compact.*

Proof. The notion of graph-closedness (Definition 42) will be used in the following □

Corollary 36. *If J is $\mathbb{J}$-adherence-determined, then a surjective map f (from $|\xi|$ to $|\tau|$) is* $\mathbb{J}$-perfect *provided that f is graph-closed and*

$$f^-\tau \geq J(\chi_\xi), \tag{9.2}$$

where χ_ξ is the characteristic convergence of ξ (11.3).

We are now in a position to easily prove Propositions 30 and 31.

Proof of Proposition 30. Let $f \in C(\xi, \tau)$ be perfect. Then the inverse image $f^-[\{y\}^\uparrow]$ is ξ-compact at $f^-(y)$ (Definition 49) for each $y \in |\tau|$, because singletons are compact. This means that $f^-(y)$ is a ξ-compact set. On the other hand, since (8.1) holds, in particular, for principal filters $\mathcal{G}$ of ξ-closed sets, f is closed.

To show the converse, let $\mathcal{G} \# f^-[\mathcal{F}]$, equivalently, $G \# f^-[\mathcal{F}]$ for each $G \in \mathcal{G}$, where $y \in \lim_\xi \mathcal{F}$. As f is a closed map, hence, in particular, f^- is an $\mathbb{F}_0$-compact relation, $\operatorname{cl}_\xi G \cap f^-(y) \neq \varnothing$ for every $G \in \mathcal{G}$. Because $f^-(y)$ is assumed to be ξ-compact,

$$\operatorname{adh}_\xi \mathcal{G} \cap f^-(y) = \bigcap\nolimits_{G \in \mathcal{G}} \operatorname{cl}_\xi G \cap f^-(y) \neq \varnothing,$$

which concludes the proof. □

Proposition 31 can be demonstrated analogously *mutatis mutandis.*

9.3. **Quotient-like maps.** It turns out that also quotient-like maps admit characterizations in terms of various types of compactness, as was shown in [22, Theorem 3.8] by F. Mynard.

Theorem 37. *A surjective map f is $\mathbb{J}$-quotient from ξ to τ if and only if f is $\mathbb{J}$-compact from $f^{-}\tau$ to $f\xi$.*

Proof. ($\Rightarrow$) If $x \in \lim_{f^{-}\tau} \mathcal{F}$, then

$$f(x) \in f(\lim_{f^{-}\tau} \mathcal{F}) = f(f^{-}(\lim_{\tau} f[\mathcal{F}])) = \lim_{\tau} f[\mathcal{F}],$$

because f is surjective. Consider $\mathcal{J} \# f[\mathcal{F}]$ such that $\mathcal{J} \in \mathbb{J}$. By (4.3), $f^{-}(f(x)) \# \operatorname{adh}_{\xi} f^{-}[\mathcal{J}]$, hence

$$f(x) \in f(\operatorname{adh}_{\xi} f[\mathcal{J}]) = \operatorname{adh}_{f\xi} \mathcal{J}.$$

Because $f^{-}[\mathcal{J}] \# \mathcal{F}$, we infer that $f(x) \in \operatorname{adh}_{f\xi} \mathcal{J}$, so that $f[\mathcal{F}]$ is $\mathbb{J}$-compact at $\{f(x)\}$ in $f\xi$.

($\Leftarrow$) Let $\mathcal{J} \in \mathbb{J}$ and $y \in \operatorname{adh}_{\tau} \mathcal{J}$. Hence, taking into account the equality $f[f^{-}[\mathcal{J}]] = \mathcal{J}$,

$$f^{-}(y) \subset f^{-}(\operatorname{adh}_{\tau} \mathcal{J}) = f^{-}(\operatorname{adh}_{\tau} f[f^{-}[\mathcal{J}]]),$$

we infer that $f^{-}(y) \subset \operatorname{adh}_{f^{-}\tau} f^{-}[\mathcal{J}]$. Let $x \in f^{-}(y), x \in \lim_{f^{-}\tau} \mathcal{F}$ and $\mathcal{F} \# f^{-}[\mathcal{J}]$, equivalently $f[\mathcal{F}] \# \mathcal{J}$. As f is $\mathbb{J}$-compact from $f^{-}\tau$ to $f\xi$, the filter-base $f[\mathcal{F}]$ is $\mathbb{J}$-compact with respect to $f\xi$. In particular, $\operatorname{adh}_{f\xi} \mathcal{J} \# \{f(x)\}$, that is, $y \in \operatorname{adh}_{f\xi} \mathcal{J}$, so that, (4.3) is proved. $\square$

We recall that the topologicity of ξ does not entail that of $f\xi$ ([27]), so that Theorem 37 cannot be addressed in the topological framework.

10. Preservation of completeness

It is well-known that (in topological spaces) continuous maps preserve *compactness*, but neither *local compactness,* nor *topological completeness*, which are preserved by continuous perfect maps.

Definition. We say that a filter is *adherent* if its adherence is non-empty; *non-adherent* if its adherence is empty.

Our representation theorems for perfect maps enable us to perceive the reason for the facts mentioned above ([28]). I hope that you would appreciate a remarkable simplicity of the proof below. The argument uses just two facts:

[27] Any finitely deep convergence can be represented as the quotient of a topology.

[28] To establish $\geq$ in (10.1) we need that the map be perfect, to assure that the images of non-adherent filters be non-adherent. This is not needed if ξ is compact, because then all filters on $|\xi|$ are ξ-adherent.

(1) the image of each adherent filter under a continuous map is adherent ([29]), and
(2) the preimage of each adherent filter under a perfect map is adherent ([30]).

Recall (Appendix 11.6) that the completeness number $\mathrm{compl}(\theta)$ of a convergence θ is the least cardinal κ, for which θ is κ-complete.

Theorem 38 ([15, XI.6])**.** *If ξ and τ are convergences, and if $f \in C(\xi, \tau)$ is a (surjective) perfect map, then*

$$\mathrm{compl}(\xi) = \mathrm{compl}(\tau). \tag{10.1}$$

Proof. Let $\mathbb{J}$ be a a collection of non-ξ-adherent filters, with $\mathrm{card}(\mathbb{J}) \leq \kappa$, such that a filter $\mathcal{G}$ is ξ-adherent, provided that $\neg(\mathcal{G}\#\mathcal{J})$ for each $\mathcal{J} \in \mathbb{J}$. As f is perfect, $f[\mathcal{J}]$ is non-τ-adherent for each $\mathcal{J} \in \mathbb{J}$, so that $\mathrm{card}(\{f[\mathcal{J}] : \mathcal{J} \in \mathbb{J}\}) \leq \kappa$. If now $\neg(\mathcal{L}\#f[\mathcal{J}])$ then $\neg(f^-[\mathcal{L}]\#\mathcal{J})$, and by completeness assumption above, $\mathrm{adh}_\xi f^-[\mathcal{L}] \neq \varnothing$. As f is continuous, $\varnothing \neq \mathrm{adh}_\tau f[f^-[\mathcal{L}]]$, and $f[f^-[\mathcal{L}]] = \mathcal{L}$, because f is surjective. We conclude that $\mathrm{compl}(\xi) \geq \mathrm{compl}(\tau)$.

Conversely, let $\mathbb{M}$ be a collection of non-τ-adherent filters, with $\mathrm{card}(\mathbb{M}) \leq \kappa$, such that a filter $\mathcal{G}$ is τ-adherent, provided that $\neg(\mathcal{G}\#\mathcal{M})$ for each $\mathcal{M} \in \mathbb{M}$. As f is continuous, $f^-[\mathcal{M}]$ is non-ξ-adherent for each $\mathcal{M} \in \mathbb{M}$, and $\mathrm{card}(\{f^-[\mathcal{M}] : \mathcal{M} \in \mathbb{M}\}) \leq \kappa$.

If now $\neg(\mathcal{G}\#f^-[\mathcal{M}])$ then $\neg(f[\mathcal{G}]\#\mathcal{M})$, thus, by the completeness assumption above, $\varnothing \neq \mathrm{adh}_\tau f[\mathcal{G}]$, hence there is a filter $\mathcal{F}$ such that $\varnothing \neq \lim_\tau \mathcal{F}$ and $\mathcal{F}\#f[\mathcal{G}]$, equivalently $f^-[\mathcal{F}]\#\mathcal{G}$. As f is perfect, $f^-[\mathcal{F}]$ is ξ-compactoid, and thus $\mathrm{adh}_\xi \mathcal{G} \neq \varnothing$. As a result, $\mathrm{compl}(\tau) \geq \mathrm{compl}(\xi)$. □

We infer that

Corollary 39. *If $f \in C(\xi, \tau)$ is a (surjective) perfect map, then ξ is compact (resp. locally compact, topologically complete) if and only if τ is compact (resp. locally compact, topologically complete).*

11. Appendix

11.1. **Relations.** It is well-known that maps and their converses are special cases of binary relations; equivalence and order are other familiar examples of such relations. A purpose of this section is not that of recalling basic facts about relations, but rather to fix a notation and present a *calculus of relations*, which, though elementary, is very convenient.

[29]Equivalently, the preimage of a non-adherent filter under a continuous is non-adherent.

[30]Equivalently, the image of a non-adherent filter under a perfect map is non-adherent.

The framework of relations is more flexible than that of maps, and their calculus simplifies and clarifies reasoning. If $R \subset W \times Z$ is a relation, then $Rw := \{z \in Z : (w,z) \in R\}$, and $R^- z := \{w \in W : (w,z) \in R\}$, which defines the *inverse relation* R^- of R. For $A \subset W, B \subset Z$, we denote the *image* of A under R and the *preimage* of B under R, respectively, by

$$RA := \bigcup\nolimits_{w \in A} Rw, \text{ and } R^- B := \bigcup\nolimits_{z \in B} R^- z.$$

Notice that, for a relation $R \subset W \times Z, A \subset W$ and $B \subset Z$, the following expressions are equivalent:

$$RA \cap B \neq \emptyset,$$

$$A \cap R^- B \neq \emptyset,$$

$$(A \times B) \cap R \neq \emptyset.$$

If X is a set, the relation of *grill* $\#$ on $2^X := \{A : A \subset X\}$ is defined by

$$A \# B \Longleftrightarrow A \cap B \neq \varnothing.$$

Now, if $R \subset W \times Z$ is a relation, then the grill interplays with R and R^-. It can be used to express the equivalences above:

$$RA \# B \Longleftrightarrow A \# R^- B \Longleftrightarrow (A \times B) \# R.$$

The relation of grill is extended to families of subsets, if $\mathcal{A}$ and $\mathcal{B}$ are families of subsets (of a given set), then

$$\mathcal{A} \# \mathcal{B} \Longleftrightarrow \underset{A \in \mathcal{A}}{\forall} \ \underset{B \in \mathcal{B}}{\forall} A \# B.$$

If $R \subset W \times Z, \mathcal{A} \subset 2^W$ and $\mathcal{B} \subset 2^Z$, then $R[\mathcal{A}] := \{RA : A \in \mathcal{A}\}$ and $R^-[\mathcal{B}] := \{R^- B : B \in \mathcal{B}\}$. Notice that

$$R[\mathcal{A}] \# \mathcal{B} \Longleftrightarrow \mathcal{A} \# R^-[\mathcal{B}]. \tag{11.1}$$

Definition 40. A *relation* $R \subset W \times Z$ is said to be *injective* if $Rw_0 \cap Rw_1 \neq \varnothing$ implies $w_0 = w_1$; it is called *surjective* if $RW = Z$ ([11][15]).

In particular, a map $f \subset X \times Y$ is injective if $f(x_0) = f(x_1)$ implies $x_0 = x_1$, and it is surjective if $f^-(y) \neq \varnothing$ for each $y \in Y$. It is straightforward that

Proposition 41. *A relation $R \subset X \times Y$ is a map from X to Y if and only if the inverse relation R^- is injective and surjective* ([31]).

[31]Indeed, R is surjective, that is, $R^- Y = X$ whenever for each $x \in X$ there exists $y \in Rx$. On the other hand, R^- is injective if for each $x \in X$ there is at most one $y \in Rx$.
If one of the involved sets X, Y is empty, then the product $X \times Y = \varnothing$, hence there is only one relation $R = \varnothing$. There are two cases: if $X = \varnothing$, then R is a map; if $Y = \varnothing$ and $X \neq \varnothing$, then R is not a map [15, p. 506].

Definition 42. Let θ and σ be convergences on W and Z, respectively. A relation $R \subset W \times Z$ is called *graph-closed at* w if $\mathrm{adh}_\sigma R[\mathcal{F}] \subset Rw$ for each filter $\mathcal{F}$ such that $w \in \lim_\theta \mathcal{F}$.

Notice that a relation R is graph-closed at each point if and only if R is a closed subset of the product. In fact, $(w, z) \in \mathrm{adh}_{\theta\times\sigma} R$ whenever there exist $\mathcal{F}$ and $\mathcal{G}$ such that $R\#(\mathcal{F}\times\mathcal{G}), w \in \lim_\theta \mathcal{F}$ and $z \in \lim_\sigma \mathcal{G}$. By (11.1), $R\#(\mathcal{F} \times \mathcal{G})$ is equivalent to $\mathcal{G}\#R[\mathcal{F}]$, so that $z \in \lim_\sigma \mathcal{G} \subset \mathrm{adh}_\sigma R[\mathcal{F}]$. On the other hand, $(w, z) \in R$ if and only if $z \in Rw$.

It follows that R is graph-closed at every point if and only if R^- is graph-closed at every point.

Finally, notice that if a map f is continuous and valued in a Hausdorff convergence ([32]), then the relation f is graph-closed, hence also f^- is graph-closed. Indeed, if $f \in C(\xi, \tau)$ and $x \in \lim_\xi \mathcal{F}$ then $\{f(x)\} = \lim_\tau f[\mathcal{F}] = \mathrm{adh}_\tau f[\mathcal{F}]$, both equalities are a consequence of the fact that τ is Hausdorff.

11.2. **Filters.** A non-empty family $\mathcal{F}$ of subsets of a set X is called a *filter* whenever $F_0 \in \mathcal{F}$ and $F_1 \in \mathcal{F}$ if and only if $F_0 \cap F_1 \in \mathcal{F}$. A filter $\mathcal{F}$ is called *non-degenerate* if $\varnothing \notin \mathcal{F}$.

Let $\overline{\mathbb{F}}X$ stand for the set of all, and by $\mathbb{F}X$ of all non-degenerate filters on X. If $\mathcal{F}$ is a filter, then $\mathcal{B} \subset \mathcal{F}$ is called a *base* of $\mathcal{F}$ (or $\mathcal{F}$ is said to be *generated* by $\mathcal{B}$) if for each $F \in \mathcal{F}$ there is $B \in \mathcal{B}$ such that $B \subset F$.

A filter is called *countably based* if it admits a countable base. We write $\mathbb{F}_1X$ for the set of *countably based filters* on X.

A filter is called *principal* if it admits a finite base, equivalently, a one element base. The principal filters on a set X are of the form $\{A\}^\uparrow := \{F \subset X : F \supset A\}$. A filter $\mathcal{F}$ is said to be *free* whenever $\bigcap_{F\in\mathcal{F}} F = \varnothing$.

Ordered by inclusion, $\overline{\mathbb{F}}X$ is a complete lattice, that is, $\bigcap \mathbb{G}$ and $(\bigcup \mathbb{G})^\cap$ ([33]) is a (possibly degenerate) filter for each $\varnothing \neq \mathbb{G} \subset \overline{\mathbb{F}}X$. The least element of $\overline{\mathbb{F}}X$ is $\{X\}$, and the greatest one is 2^X, the family of all subsets of X ([34]). Restricted to $\mathbb{F}X = \overline{\mathbb{F}}X \setminus \{2^X\}$, this order admits arbitrary infima and $\bigwedge \mathbb{G} = \bigcap \mathbb{G}$; however, $\bigvee \mathbb{G}$ exists in $\mathbb{F}X$ if and only if $(\bigcup \mathbb{G})^\cap \neq 2^X$, that is, whenever $G_1 \cap G_2 \cap \ldots \cap G_n \neq \varnothing$ for each finite subfamily $\{G_1, G_2, \ldots, G_n\}$ of $\bigcup \mathbb{G}$, in which case, $\bigvee \mathbb{G} = (\bigcup \mathbb{G})^\cap$.

The elements of $\mathbb{F}X$ that are maximal for the order defined by inclusion are called *ultrafilters*. The set of ultrafilters that include a filter $\mathcal{F}$ is denoted by $\beta\mathcal{F}$. For each filter $\mathcal{F}$,

$$\mathcal{F} = \bigcap_{\mathcal{U}\in\beta\mathcal{F}} \mathcal{U}.$$

[32] A convergence θ is said to be *Hausdorff* if $\{y_0, y_1\} \subset \lim_\theta \mathcal{F}$ implies that $y_0 = y_1$.

[33] $\mathcal{A}^\cap$ stands for the set of all the finite intersections of the elements of $\mathcal{A}$.

[34] Which is the only degenerate filter on X.

An ultrafilter is either free or principal; in the latter case, the ultrafilter on a (non-empty) set X is of the form $\{A \subset X : x \in A\}$ for some $x \in X$.

Let us recall a very useful link between a filter $\mathcal{F}$ and the set $\beta\mathcal{F}$ of its ultrafilters [15, Proposition II.6.5].

Theorem 43 (Compactness property of filters)**.** *Let $\mathcal{F}$ be a filter, and for each $\mathcal{U} \in \beta\mathcal{F}$ let $F_{\mathcal{U}} \in \mathcal{U}$. Then there is a finite subset $\boldsymbol{F}$ of $\beta\mathcal{F}$ such that $\bigcup_{\mathcal{U}\in\boldsymbol{F}} F_{\mathcal{U}} \in \mathcal{F}$.*

Every filter $\mathcal{F}$ on X admits a unique decomposition

$$\mathcal{F} = \mathcal{F}^* \wedge \mathcal{F}^\bullet \text{ and } \mathcal{F}^* \vee \mathcal{F}^\bullet = 2^X,$$

to two (possibly degenerate) filters such that $\mathcal{F}^*$ is free, $\mathcal{F}^\bullet$ is principal. Indeed, if $\mathcal{F} \in \mathbb{F}X$, then let $F_\bullet := \bigcap_{F\in\mathcal{F}} F$ and $\mathcal{F}^\bullet := \{A \subset X : F_\bullet \subset A\}$, and $\mathcal{F}^* := \{F \setminus F_\bullet : F \in \mathcal{F}\}^\uparrow$.

Definition 44. Let $A \subset B \subset X$. The family

$$({}^B\!/\!{}_A)_0 := \{F \subset X : A \subset F, \operatorname{card}(B \setminus F) < \infty\}$$

is a filter, called a *cofinite filter of B centered at A*.

If $A = \varnothing$ *we drop "centered at A" and write* $(B)_0$. If $B \setminus A$ is infinite, then the *free part* of $({}^B\!/\!{}_A)_0$ *is* $(B \setminus A)_0$, and the *principal part* is $\{A\}^\uparrow := \{F \subset X : A \subset F\}$.

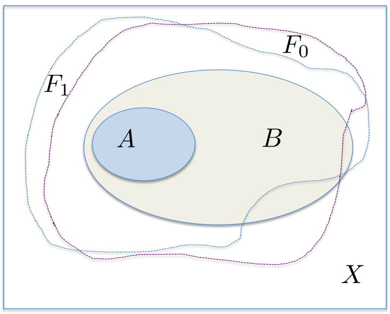

FIGURE 11.1. The *cofinite filter* of B centered at A consists of all those F that include A and such that $B \setminus F$ is finite.

Observe that if X is an infinite set, then the cofinite filter $(X)_0$ of X is the coarsest free filter on X. In particular, $(X)_0$ is the infimum of all free ultrafilters on X.

11.3. **Sequential filters.** A filter $\mathcal{E}$ on X is called *sequential* if there exists a sequence $(x_n)_n$ of elements of X such that $\{\{x_n : n > m\} : m \in \mathbb{N}\}$ is a base of $\mathcal{E}$. Sequential filters are non-degenerate, countably based and each contains a countable set, either infinite or finite. Moreover

Proposition. *A filter is sequential if and only if it is a cofinite filter of a countable set B centered at a subset A of B.*

Example. The sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ generates a cofinite filter of $\{\frac{1}{n} : n \in \mathbb{N}_0\}$ ([35]) centered at the empty set $\varnothing$; therefore it is a free filter. A constant sequence with $x_n = 1$ for each n, generates a cofinite filter of $\{1\}$ at $\{1\}$; it is a principal filter. The sequence $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots, \underbrace{\frac{1}{n}, \ldots, \frac{1}{n}}_{n \text{ times}}$ generates the same filter as the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$. Finally, the sequence $1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \underbrace{1, \ldots, \frac{1}{n}}_{n \text{ terms}}$ is a cofinite filter of $\{\frac{1}{n} : n \in \mathbb{N}_0\}$ centered at $\{\frac{1}{n} : n \in \mathbb{N}_0\}$; it generates the principal filter $\{F \subset X : \{\frac{1}{n} : n \in \mathbb{N}_0\} \subset F\}$.

11.4. **Adherence-determined subclasses.** Reflective classes of topologies, pretopologies, paratopologies, and pseudotopologies are *adherence-determined*.

If for each convergence θ, there exists a collection of filters $\mathbb{H}(\theta)$ on $|\theta|$, containing all principal filters, and such that $\eta \le \theta$ implies $\mathbb{H}(\eta) \subset \mathbb{H}(\theta)$, $\mathbb{H}(H\theta) = \mathbb{H}(\theta)$, and $\mathcal{H} \in \mathbb{H}(\tau)$ implies $f^-[\mathcal{H}] \in \mathbb{H}(f^-\tau)$ for all η, θ, τ and f, then

$$\lim_{H\theta} \mathcal{F} = \bigcap_{\mathbb{H}(\theta) \ni \mathcal{H} \# \mathcal{F}} \operatorname{adh}_\theta \mathcal{H}$$

defines a concrete reflector H [15, Corollary XIV.3.3]. In Subsection 2.4, we have already discussed and applied adherence-determination by classs of filters independent of the convergence. The class of topologies is adherence-determined by the class of closed principal filters, hence, depending on the convergence.

Here is a table of some important adherence-determined reflectors:

convergence class	reflector H	$\mathbb{H}$	filter class
pseudotopologies	S	$\mathbb{F}$	all filters
paratopologies	S_1	$\mathbb{F}_1$	countably based filters
pretopologis	S_0	$\mathbb{F}_0$	principal filters
topologies	T	$\mathbb{F}_0(\cdot)$	closed principal filters

[35]Where $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$.

It is clear that

$$\mathbb{F}_0(\cdot) \subset \mathbb{F}_0 \subset \mathbb{F}_1 \subset \mathbb{F},$$

and thus,

$$\mathrm{T} \leq \mathrm{S}_0 \leq \mathrm{S}_1 \leq \mathrm{S}.$$

In particular, a convergence ξ is a *pretopology* whenever ([36])

$$\lim_\xi \mathcal{F} = \bigcap\nolimits_{H \in \mathcal{F}^\#} \mathrm{adh}_\xi H,$$

and ξ is a *pseudotopology* whenever

$$\lim_\xi \mathcal{F} = \bigcap\nolimits_{\mathcal{H} \# \mathcal{F}} \mathrm{adh}_\xi \mathcal{H} = \bigcap\nolimits_{\mathcal{U} \in \beta\mathcal{F}} \mathrm{adh}_\xi \mathcal{U} = \bigcap\nolimits_{\mathcal{U} \in \beta\mathcal{F}} \lim_\xi \mathcal{U},$$

where $\beta\mathcal{F}$ stands for the set of all ultrafilters that are finer than $\mathcal{F}$.

Example 45 (Non-topological pretopology)**.** Let $(X_n)_n$ be a sequence of disjoint countably infinite sets, let $x_n \in X_n$ for each n, and $x_\infty \notin X_n$ for each $n \in \mathbb{N}$. Consider the set

$$X := \bigcup\nolimits_{n \in \mathbb{N}} X_n \cup \{x_\infty\}.$$

Before continuing, recollect the notion of *cofinite filter* (Appendix, Definition 44). Consider on X a pretopology θ given by setting $\mathcal{V}_\theta(x_n)$ to be the cofinite filter of X_n centered in x_n, and $\mathcal{V}_\theta(x_\infty)$ to be the cofinite filter of $X_\infty := \{x_\infty, x_0, \ldots\}$ centered at x_∞, while all the other points be isolated.

Example 46.

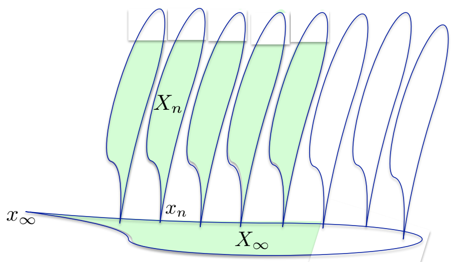

FIGURE 11.2. A typical open set O containing x_∞ is represented by the green area; it has cofinite intersections with X_n for a cofinite subset of x_n in X_∞.

This pretopology is not a topology. Indeed, if O is an open set containing x_∞, then $O \setminus X_\infty$ is finite, and $O \setminus X_n$ is finite for each n such that $x_n \in O \cap X_\infty$. Therefore, $X_\infty \in \mathcal{V}_\theta(x_\infty) \setminus \mathcal{O}_\theta(x_\infty)$, hence θ is not a topology.

[36]This is equivalent to the fact that, for each x, there exists a coarsest filter that ξ-converges to x. It is called the vicinity filter of ξ at x, and is denoted by $\mathcal{V}_\xi(x)$.

Example 47 (A non-pretopological pseudotopology [15, Example VIII.2.6])**.** The *usual sequential convergence* σ of the real line is defined by $r \in \lim_\sigma \mathcal{F}$ if there exists a sequential filter (Appendix 11.3) $\mathcal{E} \leq \mathcal{F}$ so that $\mathcal{E}$ converges to r in the *usual topology.* The convergence σ is not a pretopology, because the infimum of all sequential filters converging to a given point r of $\mathbb{R}$, is the usual neighborhood filter at r. In fact, if there were

$$V \in \mathcal{V}_\sigma(r) \setminus \bigcap\{\mathcal{E} \in \mathbb{E} : \mathcal{E} \supset \mathcal{V}_\sigma(r)\}, \tag{11.2}$$

where $\mathbb{E}$ stands for the class of sequential filters, then there would be a sequential filter $\mathcal{E} \supset \mathcal{V}_\sigma(r)$ such that $V \notin \mathcal{E}$. This means that there exists a sequence $(x_n)_n$ of elements of $\mathbb{R} \setminus V$, and which generates a filter finer than $\mathcal{E}$, hence finer than $\mathcal{V}_\sigma(r)$ in contradiction with (11.2).

The convergence σ is a pseudotopology. Indeed, it is enough to consider a free filter $\mathcal{F}$ such that $r \in \lim_\sigma \mathcal{U}$ for each $\mathcal{U} \in \beta\mathcal{F}$. Accordingly, for each $\mathcal{U} \in \beta\mathcal{F}$ there is a countably infinite set $E_\mathcal{U} \in \mathcal{U}$, hence $\mathcal{E}_\mathcal{U} \subset \mathcal{U}$ and $r \in \lim_\sigma \mathcal{E}_\mathcal{U}$, where $\mathcal{E}_\mathcal{U}$ is the cofinite filter of $E_\mathcal{U}$ (see Appendix 11.2). By Theorem 43, there is a finite subset $\boldsymbol{F}$ of $\beta\mathcal{F}$ such that $\bigcup_{\mathcal{U}\in\boldsymbol{F}} E_\mathcal{U} \in \mathcal{F}$, and thus $\mathcal{E} := \bigwedge_{\mathcal{U}\in\boldsymbol{F}} \mathcal{E}_\mathcal{U}$ is the cofinite filter of $\bigcup_{\mathcal{U}\in\boldsymbol{F}} E_\mathcal{U}$, hence $\mathcal{E} \subset \mathcal{F}$ ([37]) and $r \in \lim_\sigma \mathcal{E} \subset \lim_\sigma \mathcal{F}$.

Example 48 (Non-pseudotopological convergence)**.** Consider a convergence on an infinite set X, for which all the points are isolated, except for one x_∞, to which converges each free ultrafilter on X, as well as $\{x_\infty\}^\uparrow$. This convergence is not pseudotopological, because the cofinite filter $(X)_0$ (see Appendix 11.2), the infimum of all free filters on X, does not converge to x_∞, by the definition of the convergence.

11.5. **Compactness.** In order to provide an opportune framework for further understanding, I shall extend the notion of compactness in several ways. Let me recall that no separation axiom is implicitly assumed!

In topology, a set A is said to be *compact at a set* B if each open cover of B admits a finite subfamily that is a cover of A. This property admits an equivalent formulation in terms of filters: if $\mathcal{F}$ is a filter such that $A \in \mathcal{F}^\#$ then $B \# \operatorname{adh} \mathcal{F}$.

A subset K of a topological space is called *compact* if it is compact at K (compact at itself). In each Hausdorff topology, a compact subset is closed, but without the separation axiom (here the Hausdorff property), this no longer the case ([38]).

---

[37]Indeed, every free filter $\mathcal{F}$ is finer than the cofinite filter of any $F_0 \in \mathcal{F}$.

[38]Let \$ be a Sierpiński topology on $\{0,1\}$, that is, the open sets are $\varnothing, \{0\}, \{0,1\}$. In particular, $\lim_{\$}\{0\}^\uparrow = \{0,1\}$. Each subset of this space is compact, in particular, the set $\{0\}$ is compact, but is not closed.

In general convergences spaces, the two notion of compactness above are no longer equivalent. Cover compactness is stronger than filter compactness, and it turns out that filter compactness is more relevant, because continuous maps preserve filter compactness, but, in general, not cover compactness (see [22]).

The concept of compactness has been extended from sets to families of sets by L. V. Kantorovich and al. [20], M. P. Kac [19], and in [13, 14] by G. H. Greco, A. Lechicki and the author.

A family $\mathcal{A}$ is said to be *compact at* a family $\mathcal{B}$ whenever if a filter meshes $\mathcal{A}$ then its adherence meshes $\mathcal{B}$. A family is called *compact* if it is compact at itself; *compactoid* ([39]) if it is compact at the whole underlying space.

Obviously, the defined concept ([40]) generalizes that of (relative) compactness: a set A is compact at a set B if and only if $\{A\}$ is compact at $\{B\}$.

Remark. If a convergence θ is Hausdorff and A is θ-compact at B, then $\operatorname{adh}_\theta A \subset B$. Indeed, if $x \in \operatorname{adh}_\theta A$, then there exists an ultrafilter $\mathcal{U}$ such that $A \in \mathcal{U} = \mathcal{U}^{\#}$ and $\{x\} = \lim_\theta \mathcal{U}$, beacuse θ is Hausdorff. But assumption, $B \# \operatorname{adh}_\theta \mathcal{U} = \lim_\theta \mathcal{U}$, and thus $x \in B$. If a convergence is not Hausdorff, then $\operatorname{adh}_\theta A \subset B$ need not hold, as we have seen in the case of the Sierpiński topology in Footnote 38.

On the other hand, it extends the notion of convergence. In fact, each convergent filter is compactoid, because $\lim_\xi \mathcal{F} \subset \operatorname{adh}_\xi \mathcal{H}$ for every $\mathcal{H}\#\mathcal{F}$. More significantly, $x \in \lim_{\mathrm{S}\xi} \mathcal{F}$ if and only if $\mathcal{F}$ is ξ-compact at $\{x\}$.

An ulterior extension allows to embrace *countable compactness, Lindelöf property,* and others.

Definition 49. Let $\mathbb{J}$ be a class of filter containing all principal filters. Let ξ be a convergence, and $\mathcal{A}$ and $\mathcal{B}$ be two families of subsets of $|\xi|$. We say that $\mathcal{A}$ is $\mathbb{J}$*-compact at* $\mathcal{B}$ *(in* ξ*) if*

$$\underset{\mathcal{F}\in\mathbb{J}}{\forall}(\mathcal{F}\#\mathcal{A} \Longrightarrow \operatorname{adh}_\xi \mathcal{F} \in \mathcal{B}^{\#}).$$

In particular, if $\mathbb{J}$ stands for the class of all filters, we recover *compactness*, and if $\mathbb{J}$ denotes the class of countably based filters, we get *countable compactness.*

$\mathbb{J}$-compactness for the class $\mathbb{J}$ of principal filters is a rather recent concept [6]. It is called *finite compactness.*

[39] The term *compactoid* was introduced by G. Choquet in [4] for *relatively compact.* Its advantage is its shortness.

[40] The concept of compactness of families of sets proved its usefulness in various situations, for example, open sets in some hyperspaces have been characterized in terms of compact families [14, Theorem 3.1].

Remark. In particular, if A is finitely compact at B for a T_1-convergence (all points are closed), then $A \subset B$. Indeed, in this situation, for every $x \in A$, the set $\{x\}$ meshes A, hence $\mathrm{adh}\{x\} = \{x\} \in B^{\#}$, hence $x \in B$. Footnote 38 shows that this need not be the case without the assumption of T_1.

Let χ_ξ be the *characteristic convergence* of ξ ([41]), defined by

$$\lim_{\chi_\xi} \mathcal{F} := \begin{cases} |\xi|\,, & \text{if } \lim_\xi \mathcal{F} \neq \varnothing, \\ \varnothing, & \text{otherwise.} \end{cases} \tag{11.3}$$

Proposition 50. [7] *If J is $\mathbb{J}$-adherence-determined, then a filter $\mathcal{H}$ is $\mathbb{J}$-compactoid (in ξ), if and only if*

$$\lim_{J(\chi_\xi)} \mathcal{H} \neq \varnothing.$$

Proof. By definition, $\mathcal{H}$ is $\mathbb{J}$-compactoid (in ξ) if and only if $\mathrm{adh}_\xi \mathcal{J} \neq \varnothing$ (equivalently, $\mathrm{adh}_{\chi_\xi} \mathcal{J} = X$) for each $\mathcal{J} \in \mathbb{J}$ such that $\mathcal{J} \# \mathcal{H}$. As

$$\lim_{J(\chi_\xi)} \mathcal{H} = \bigcap\nolimits_{\mathbb{J} \ni \mathcal{J} \# \mathcal{H}} \mathrm{adh}_{\chi_\xi} \mathcal{J},$$

it is non-empty (equivalently, equal to the whole space) if and only if $\mathrm{adh}_{\chi_\xi} \mathcal{J} = X$ for each $\mathcal{J} \in \mathbb{J}$. $\square$

Definition 51. Let θ be a convergence on W and σ be a convergence on Z. A relation R is called $\mathbb{J}$-*compact* if $w \in \lim_\theta \mathcal{F}$ implies that $R(w) \# \mathrm{adh}_\sigma \mathcal{H}$ for each $\mathcal{H} \# R[\mathcal{F}]$ such that $\mathcal{H} \in \mathbb{J}$ ([42]).

Let us observe that ([43])

Proposition 52. *Assume that the class $\mathbb{J}$ of filters is transferable. Then the image under a $\mathbb{J}$-compact relation R of a filter $\mathcal{A}$ that is $\mathbb{J}$-compact at B is $\mathbb{J}$-compact at RB.*

As it was observed after Proposition 11, the classes of all filters, countably based filters, and principal filters are transferable.

---

[41]The relation χ_ξ is a convergence. Indeed, if $x \in \lim_{\chi_\xi} \mathcal{F}$ and $\mathcal{F} \subset \mathcal{G}$, then $\varnothing \neq \lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{G}$, hence $x \in X = \lim_{\chi_\xi} \mathcal{G}$. On the other hand, $x \in \lim_\xi \{x\}^\uparrow \subset \lim_{\chi_\xi} \{x\}^\uparrow$.

[42]A relation $R \subset W \times Z$ is $\mathbb{J}$-*compactoid* if $\lim_\theta \mathcal{F} \neq \varnothing$ implies that $R[\mathcal{F}]$ is $\mathbb{J}$-compactoid for σ. It is straightforward that a graph-closed (Definition 42) $\mathbb{J}$-compactoid relation is $\mathbb{J}$-compact.

[43]Indeed, let $\mathcal{G} \in \mathbb{J}$ be such that $\mathcal{G} \# R[\mathcal{A}]$, equivalently $R^-[\mathcal{G}] \# \mathcal{A}$, hence $\mathrm{adh}\, R^-[\mathcal{G}] \# B$, because $R^-[\mathcal{G}] \in \mathbb{J}$. Therefore, there exists $x \in \mathrm{adh}\, R^-[\mathcal{G}] \cap B$, so that there is a filter $\mathcal{F} \# R^-[\mathcal{G}]$, equivalently, $R[\mathcal{F}] \# \mathcal{G}$, such that $B \# \lim \mathcal{F}$. As R is $\mathbb{J}$-compact, $\mathrm{adh}\, \mathcal{G} \# Rx$, hence $\mathrm{adh}\, \mathcal{G} \# RB$.

11.6. **Completeness.** *Compactness, local compactness*, and *topological completeness* (also called *Čech completeness*) are special cases of κ*-completeness* (where κ is a cardinal number), introduced and studied by Z. Frolík in [17]. In convergence spaces this notion was investigated without any separation axioms, for example, in [8, 12, 15] ([44]).

A convergence θ is called κ*-complete* if there exists a collection $\mathbb{P}$ of covers with $\mathrm{card}(\mathbb{P}) \leq \kappa$ and such that $\mathrm{adh}_\theta \mathcal{G} \neq \varnothing$ for every filter $\mathcal{G}$ such that $\mathcal{G} \cap \mathcal{P} \neq \varnothing$ for each $\mathcal{P} \in \mathbb{P}$. The property is not altered if we consider merely collections $\mathbb{P}$ of *ideals* (families stable under subsets and finite unions).

By virtue of Theorem 17, a convergence θ is κ-complete if and only if there exists a collection $\mathbb{H}$ of filters with $\mathrm{card}(\mathbb{H}) \leq \kappa$ such that $\mathrm{adh}_\theta \mathcal{H} = \varnothing$ for each $\mathcal{H} \in \mathbb{H}$, and $\mathrm{adh}_\theta \mathcal{G} \neq \varnothing$ for every filter $\mathcal{G}$ such that $\neg(\mathcal{G}\#\mathcal{H})$ for every $\mathcal{H} \in \mathbb{H}$.

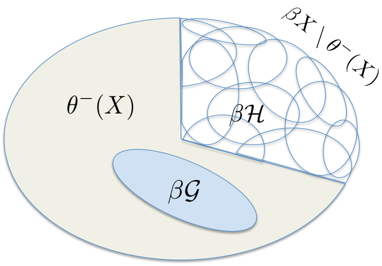

Figure 11.3. The set of non-θ-adherent ultrafilters $\beta X \setminus \theta^-(X)$ is filled by $\beta\mathcal{H} := \{\mathcal{U} \in \beta X : \mathcal{H} \leq \mathcal{U}\}$, where $\mathcal{H} \in \mathbb{H}$. Each filter $\mathcal{G}$ such $\neg(\mathcal{G}\#\mathcal{H})$ for each $\mathcal{H} \in \mathbb{H}$ is θ-adherent.

In other words, a convergence is κ-complete whenever there exists a collection of cardinality κ of non-adherent filters such that an ultrafilter is convergent provided that it is not finer than any filter from the collection.

The least κ for which θ is κ-complete, is called the *completeness number* of θ and is denoted by $\mathrm{compl}(\theta)$.

Proposition 53. *A convergence θ is compact if and only if* $\mathrm{compl}(\theta) = 0$.

Indeed, the condition means that there is no non-θ-adherent filter, that is, each ultrafilter is θ-convergent, equivalently, each filter is θ-adherent.

A convergence is called *locally compactoid* if each convergent filter contains a compactoid set.

[44] There is a slight difference between our concept and that of Frolík who uses filters admitting bases of open sets; the two concepts coincide for regular topologies.

Proposition 54. *A convergence θ is locally compactoid if and only if* $\mathrm{compl}(\theta) \leq 1$ ([45]).

In fact, the condition says that there is a non-θ-adherent filter $\mathcal{H}$ such that a filter $\mathcal{W}$ is θ-adherent provided that there is $H \in \mathcal{H}$ with $H^c \in \mathcal{W}$. Hence, the ideal $\mathcal{H}_c := \{H^c : H \in \mathcal{H}\}$ is composed of θ-compactoid sets.

Therefore, if a filter $\mathcal{F}$ is θ-convergent, then each $\mathcal{U} \in \beta\mathcal{F}$ contains an element of $\mathcal{H}_c$, so that, by Theorem 43, $\mathcal{F} \cap \mathcal{H}_c \neq \varnothing$, because $\mathcal{H}_c$ is an ideal. As a result, $\mathcal{F}$ contains a θ-compactoid set, that is, θ is locally compactoid.

Finally, it follows from the definition of Čech completeness (e. g., [17]) that

Proposition 55. *A convergence θ is Čech complete if and only if* $\mathrm{compl}(\theta) \leq \aleph_0$.

It is a classical result that a topology is completely metrizable if and only if it is metrizable and Čech complete (e. g., [16, p. 343],[15, Corollary XI.11.6]).

11.7. **Examples of corefelective subclasses.** To complete our rough picture of convergence spaces, let us present three basic coreflective properties. A convergence θ is called

1. *sequentially based* if $x \in \lim_\theta \mathcal{F}$ entails the existence of a sequential filter $\mathcal{E}$ (Appendix 11.3) such that $x \in \lim_\theta \mathcal{E}$ and $\mathcal{E} \leq \mathcal{F}$.
2. *of countable character* if $x \in \lim_\theta \mathcal{F}$ entails the existence of a countably based filter $\mathcal{E}$ such that $x \in \lim_\theta \mathcal{E}$ and $\mathcal{E} \leq \mathcal{F}$. M*etrizable topologies* are of countable character.
3. *locally compactoid* if $x \in \lim_\theta \mathcal{F}$ entails the existence of a θ-compactoid set $K \in \mathcal{F}$.

The corresponding coreflectors are denoted by $\mathrm{Seq}, \mathrm{I}_1$, and K. The listed coreflective classes traverse the already discussed reflective classes; they all contain the discrete topologies. For instance, the convergence θ from Example 47 is sequential.

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[45] *If θ is κ-complete for $\kappa < \aleph_0$ then* $\mathrm{compl}(\theta) \leq 1$.

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Contents

Introduction 101
1. **Continuity in topological spaces** 103
   1.1. Convergence of filters in topological spaces 104
   1.2. Some operations on families of sets 105
   1.3. Adherences 105
   1.4. Back to continuity 106
2. **Framework of convergences** 107
   2.1. Open and closed sets for arbitrary convergences 108
   2.2. Continuity in convergence spaces 108
   2.3. Functors, reflectors, coreflectors 109
   2.4. Adherence-determined reflectors 110
3. **Characterizations of continuity** 110
4. **Inverse continuity** 111
5. **Characterizations in terms of covers** 113
   5.1. Covers, inherences, adherences 113
   5.2. Characterizations 114
6. **Variants of quotient maps** 115
7. **Mixed properties and functorial inequalities** 116
8. **Various classes of perfect-like maps** 118
9. **Compactness characterizations** 120
   9.1. Continuous maps 120
   9.2. Perfect-like maps 121
   9.3. Quotient-like maps 122
10. **Preservation of completeness** 122
11. **Appendix** 123
   11.1. Relations 123
   11.2. Filters 125
   11.3. Sequential filters 127
   11.4. Adherence-determined subclasses 127
   11.5. Compactness 129
   11.6. Completeness 132
   11.7. Examples of corefelective subclasses 133
References 133

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