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by

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# COARSE SPACES, ULTRAFILTERS AND DYNAMICAL SYSTEMS

IGOR PROTASOV

Abstract. For a coarse space $(X, \mathcal{E})$, $X^\sharp$ denotes the set of all unbounded ultrafilters on $X$ endowed with the parallelity relation: $p||q$ if there exists $E \in \mathcal{E}$ such that $E[P] \in q$ for each $P \in p$. If $(X, \mathcal{E})$ is finitary then there exists a group $G$ of permutations of $X$ such that the coarse structure $\mathcal{E}$ has the base $\{\{(x, gx) : x \in X, g \in F\} : F \in [G]^{<\omega},\ id \in F\}$. We survey and analyze interplays between $(X, \mathcal{E})$, $X^\sharp$ and the dynamical system $(G, X^\sharp)$.

The dynamical Švarc-Milnor Theorem and Gromov Theorem arose at the dawn of *Geometric Group Theory.* In both cases, a group or a pair of groups act on some locally compact spaces, see [22, Chapter 1]. The Gromov coupling criterion was transformed into the powerful tool in coarse equivalences (see references in [23]), however some natural questions on the coarse equivalence of groups need more delicate combinatorial technique, see [4].

In this paper, we describe and survey the dynamical approach to coarse spaces originated in the algebra of the Stone-Čech compactification. We identify the Stone-Čech compactification $\beta G$ of a discrete group $G$ with the set of all ultrafilters on $G$. The left regular action $G$ on $G$ gives rise to the action of $G$ on $\beta G$ by $(g, p) \mapsto gp$, $gp = \{gP : P \in p\}$. In turn, the dynamical system $(G, \beta G)$ induces on $\beta G$ the structure of a right topological semigroup. The product $pq$ of ultrafilters $p, q$ is defined by $A \in pq$ if and only if $\{g \in G : g^{-1}A \in q\} \in p$. The semigroup $\beta G$ has a very rich algebraic structure and plenty of combinatorial applications; see nice paper [5], capital book [6] or booklet [9].

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Let $(X, \mathcal{E})$ be a coarse space. We denote by $X^\sharp$ the set of all ultrafilters $p$ on $X$ such that each member $P \in p$ is unbounded in $(X, \mathcal{E})$. Then we define the parallelity equivalence $||$ on $X^\sharp$ by $p||q$ if and only if there exists $E \in \mathcal{E}$ such that $E[P] \in q$ for each $P \in p$. For $p \in X^\sharp$, the orbit $\overline{\overline{p}} = \{q \in X^\sharp : q||p\}$ looks like a smile apart of some hidden cat. This cat appears if $(X, \mathcal{E})$ is finitary. By Theorem 3.1, there exists a group $G$ of permutations of $X$ such that $\mathcal{E}$ has the base $\{\{(x, gx) : x \in F\} : F \in [G]^{<\omega}, \ id \in F\}$. In this case, $X^\sharp = X^*$, $X^* = \beta X \setminus X$ and $\overline{\overline{p}} = Gp$. But even $(X, \mathcal{E})$ is not finitary, $X^\sharp$ contains some counterpart of the kernel of a dynamical system.

The short Section 1 contains definitions of coarse structures, coarse spaces and a classification of subsets of a coarse space by their size and arrangement. In particular, a coarse space $X$ is called $\delta$-tight ($\lambda$-tight) if any two subsets of $X$ are close (linked).

The notion of the parallelity of ultrafilters is discussed in Section 2. We characterize uniformly recurrent points (Theorem 2.2) and the kernel of $X^\sharp$ (Theorem 2.3).

In Section 3, our goal is to clarify interplays between a finitary space $(X, \mathcal{E})$ and the dynamical system $(G, X^*)$ in order to understand the dynamical nature of some extremal coarse spaces. The results are in the line

$$\delta\text{-tight} \Longrightarrow \text{topologically point transitive} \Longrightarrow \lambda\text{-tight} \Longrightarrow \text{indiscrete}$$

and a coarse space $(X, \mathcal{E})$ is $\delta$-tight ($\lambda$-tight) if and only if $(G, X^*)$ is minimal (topologically transitive).

## 1. Coarse spaces

Given a set $X$, a family $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure* on $X$ if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X = \{(x, x) \in X : x \in X\}$;
- if $E$, $E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x, y) : \exists z((x, z) \in E, \quad (z, y) \in E')\}$, $E^{-1} = \{(y, x) : (x, y) \in E\}$;
- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$;
- $\bigcup \mathcal{E} = X \times X$.

Since $\Delta_X \subseteq E$, $\Delta_X \subseteq E'$, we have $E \subseteq E \circ E'$, $E' \subseteq E \circ E'$, so a coarse structure is closed under finite unions.

A subfamily $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* for $\mathcal{E}$ if, for every $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $E \subseteq E'$. For $x \in X$, $A \subseteq X$ and $E \in \mathcal{E}$, we denote

$$E[x] = \{y \in X : (x, y) \in E\}, \ E[A] = \bigcup_{a \in A} E[a], \quad E_A[x] = E[x] \cap A$$

and say that $E[x]$ and $E[A]$ are *balls of radius* $E$ *around* $x$ and $A$.

The pair $(X, \mathcal{E})$ is called a *coarse space* [22] or a ballean [16], [21].

For a coarse space $(X, \mathcal{E})$, a subset $B \subseteq X$ is called *bounded* if $B \subseteq E[x]$ for some $E \in \mathcal{E}$ and $x \in X$. The family $\mathcal{B}_{(X,\mathcal{E})}$ of all bounded subsets of $(X, \mathcal{E})$ is called the *bornology* of $(X, \mathcal{E})$.

A coarse space $(X, \mathcal{E})$ is called *finitary,* if for each $E \in \mathcal{E}$ there exists a natural number $n$ such that $|E[x]| < n$ for each $x \in X$.

We classify subsets of a coarse space $(X, \mathcal{E})$ by their size. A subset $A$ of $X$ is called

- *large* if $E[A] = X$ for some $E \in \mathcal{E}$;
- *small* if $L \backslash A$ is large for each large subset $L$;
- *thick* if, for each $E \in \mathcal{E}$, there exists $a \in A$ such that $E[a] \subseteq A$;
- *prethick* if $E[A]$ is thick for some $E \in \mathcal{E}$;
- *thin (or discrete)* if, for each $E \in \mathcal{E}$, there exists a bounded subset $B$ of $X$ such that $E_A[a] = \{a\}$ for each $a \in A \backslash B$.

For finitary coarse spaces, the dynamical unification of above definitions will be given in Section 3.

Following [17], we say that two subets $A, B$ of $X$ are

- *close* (write $A\delta B$) if there exists $E \in \mathcal{E}$ such that, $A \subseteq E[B]$, $B \subseteq E[B]$;
- *linked* (write $A\lambda B$) if either $A, B$ are bounded or there exist unbounded subsets $A' \subseteq A$, $B' \subseteq B$ such that $A'\delta B'$.

We say that a coarse space $(X, \mathcal{E})$ is

- $\delta$*-tight* if any two unbounded subsets of $X$ are close;
- $\lambda$*- tight* if any two unbounded subsets of $X$ are linked;
- *indiscrete* if $E \in \mathcal{E}$ has no unbounded discrete subsets;
- *ultradiscrete* if $\{X \backslash B : B \in \mathcal{B}_{(X,\mathcal{E})}\}$ is an unltrafilter.

We note that $\lambda$-tight spaces appeared in [2] under the name *utranormal*, $\delta$-tight subsets are called *extremely normal* in [14]and *hypernormal* in [1].

An unbounded coarse space is called *maximal* if it is bounded in every stronger coarse structure. By [18, Theorem 3.1], every maximal coarse space is $\delta$-tight. A ballean $(X, \mathcal{E})$ is $\delta$-tight if and only if every subset of $X$ is large. If a $\lambda$-tight space is not indiscrete then it contains an ultradiscrete subspace [14, Theorem 2.2], so every finitary $\lambda$-tight space is indiscrete.

## 2. Ultrafilters

Let $X$ be a discrete space and let $\beta X$ denote the *Stone–Čech compactification* of $X$. We take the points of $\beta X$ to be the ultrafilter on $X$, with the points of $X$ identified with the principal ultrafilters, so $X^* = \beta X \setminus X$ is the set of all free ultrafilters. The topology of $\beta X$ can be defined by stating that the sets of the form $\bar{A} = \{p \in \beta X : A \in p\}$, where $A$ is a subset of $X$, are base for the open sets. The universal property of $\beta X$ states that every mapping $f : X \longrightarrow Y$, where $Y$ is a compact Hausdorff space, can be extended to the continuous mapping $f^{\beta} : \beta X \longrightarrow X$.

Given a coarse space $(X, \mathcal{E})$, we endow $X$ with the discrete topology and denote by $X^\sharp$ the set of all ultrafilters $p$ on $X$ such that each member $P \in p$ is unbounded. Clearly, $X^\sharp$ is the closed subset of $X^*$ and $X^\sharp = X^*$ if $(X, \mathcal{E})$ is finitary.

Following [10], we say that two ultrafilters $p, q \in X^\sharp$ are *parallel* (and write $p||q$) if there exists $E \in \mathcal{E}$ such that $E[P] \in q$ for each $P \in p$. By [10, Lemma 4.1, 1], $||$ is an equivalence on $X^\sharp$. We denote by $\sim$ the minimal (by inclusion) closed (in $X^\sharp \times X^\sharp$) equivalence on $X^\sharp$ such that $|| \subseteq \sim$. The quotient $\nu(X, \mathcal{E})$ of $X^\sharp$ by $\sim$ is called the Higson corona of $(X, \mathcal{E})$. For $p \in X^\sharp$, we denote

$$\overline{\overline{p}} = \{q \in X^\sharp : q||p\}.$$

The original definition of the Higson corona is in terms of slowly oscillating functions (see [22, Section 2.3]).

A function $f : (X, \mathcal{E}) \longrightarrow \mathbb{R}$ is called *slowly oscillating* if, for every $E \in \mathcal{E}$ and $\epsilon > 0$, there exists a bounded subset $B$ of $X$ such that $diam f(E[x]) < \epsilon$ for each $x \in X \setminus B$.

We recall [10] that a coarse space $(X, \mathcal{E})$ is *normal* if any two asymptotically disjoint subsets $A, B$ of $X$ have disjoint asymptotic neighbourhoods. Two subsets $A, B$ of $X$ are called *asymptotically disjoint* if $E[A] \cap E[B]$ is bounded for each $E \in \mathcal{E}$. A subset $U$ of $X$ is called an *asymptotic neighbourhood* of a subset $A$ if $E[A] \setminus U$ is bounded for each $E \in \mathcal{E}$. By [10, Theorem 2.2], $(X, \mathcal{E})$ is normal if and only if, for any two disjoint and asymptotically disjoint subsets $A, B$ of $X$, there exists a slowly oscillating function $f : X \longrightarrow [0, 1]$ such that $f|_A = 0$, $f|_B = 1$.

By [11, Proposition 1], $p \sim q$ if and only if $h^{\beta}(p) = h^{\beta}(q)$ for every slowly oscillating function $h : (X, \mathcal{E}) \longrightarrow [0, 1]$.

By [4, Theorem 7], $(X, \mathcal{E})$ is normal if and only if $\sim = cl\ ||$.

By [17, Theorem 9 and Corollary 10], if $\lambda_{(X,\mathcal{E})} = \lambda_{(X,\mathcal{E}')}$ then Higson coronas of $(X, \mathcal{E})$ and $(X, \mathcal{E}')$ coincide and if $(X, \mathcal{E})$ is normal then $(X, \mathcal{E}')$ is normal.

By [21, Theorem 2.1.1] a coarse space $(X, \mathcal{E})$ is metrizable if $\mathcal{E}$ has a countable base. If $\lambda_{(X,\mathcal{E})} = \lambda_{(X,\mathcal{E}')}$ and $(X, \mathcal{E})$ is metrizable then $(X, \mathcal{E}')$ needs not to be metrizable [17, Theorem 3].

**Question 2.1 [17].** *Let $\delta_{(X,\mathcal{E})} = \delta_{(X,\mathcal{E}')}$ and $(X, \mathcal{E})$ is metrizable. Is $(X, \mathcal{E}')$ metrizable?*

If the answer to Question 2.1 would be positive then $\mathcal{E} = \mathcal{E}'$,

Let $(X, \mathcal{E})$ be a coarse space. We say that a subset $S$ of $X^{\sharp}$ is *invariant* if $\overline{\overline{p}} \subseteq S$ for each $p \in S$. Every non-empty closed invariant subset of $X^{\sharp}$ contains a minimal by inclusion closed invariant subset. We denote

$$K(X^{\sharp}) = \bigcup\{M : M \quad \text{is minimal closed invariant subset of} \quad X^{\sharp}\}.$$

**Theorem 2.2.** *For $p \in X^{\sharp}$, $cl\overline{\overline{p}}$ is a minimal closed invariant subset if and only if, for every $P \in p$, there exists $E \in \mathcal{E}$ such that $\overline{\overline{p}} \in (E[P])^{\sharp}$.*

*Proof.* Apply arguments proving this statement for metric spaces [13, Theorem 3.1]. □

**Theorem 2.3.** *For $q \in X^{\sharp}$, $q \in clK(X^{\sharp})$ if and only if each subset $Q \in q$ is prethick.*

*Proof.* Apply arguments proving Theorem 3.2 in [13]. □

**Theorem 2.4.** *Let $p, q$ be ultrafilters from $X^{\sharp}$ such that $\overline{\overline{p}}$, $\overline{\overline{q}}$ are countable and $cl\overline{\overline{p}} \cap cl\overline{\overline{q}} \neq \emptyset$. Then either $cl\overline{\overline{p}} \subseteq cl\overline{\overline{q}}$ or $cl\overline{\overline{q}} \subseteq cl\overline{\overline{p}}$.*

*Proof.* Apply arguments proving Theorem 3.4 in [13]. □

**Remark 2.5** Theorem 2.2 and Theorem 2.3 have dynamical analogs, see Theorem 2.10 and Corolary 4.41 in [6].

## 3. Dynamical systems

By a *dynamical system* we mean a pair $(G, T)$, where $T$ is a compact space, $G$ is a group of homeomorphisms of $G$.

The following two theorems make a bridge between coarse spaces and dynamical systems. For usage of Theorem 3.1 in corona constructions see [3].

Let $G$ be a transitive group of permutations of a set $X$. We denote by $X_G$ the set $X$ endowed with the coarse structure with the base.

$$\{\{(x, gx) : g \in F\} : F \in [G]^{<\omega},\ id \in F\}.$$

The following theorem was proved in [12], for more general results see [8], [15].

**Theorem 3.1.** *For every finitary coarse space $(X, \mathcal{E})$, there exists a group $G$ of permutations of $X$ such that $(X, \mathcal{E}) = X_G$.*

*Proof.* For every $E \in \mathcal{E}$, we define a mapping $F_E : X \longrightarrow X^{<\omega}$ by $F_E(x) = E[X]$. By [15, Theorem 1], there exists a finite family $\mathcal{F}_E$ of permutations of $X$ such that $F_E(x) = \{f(x) : f \in \mathcal{F}_E\}$ for each $x \in X$.

We denote by $G$ the group of permutations of $X$ generated by $\{\mathcal{F}_E : E \in \mathcal{E}\}$. By the construction of $\{\mathcal{F}_E : E \in \mathcal{E}\}$, we have $(X, \mathcal{E}) = X_G$. $\square$

**Theorem 3.2.** *If $(X, \mathcal{E})$, $(X, \mathcal{E}')$ are finitary coarse spaces and $||_{(X,\mathcal{E})} = ||_{(X,\mathcal{E}')}$ then $\mathcal{E} = \mathcal{E}'$.*

*Proof.* Theorem 15 in [17]. $\square$

If $(X, \mathcal{E}) = X_G$, we say that $X_G$ is the *$G$-realization* of $(X, \mathcal{E})$. Each $G$-realization of $(X, \mathcal{E})$ defines the dynamical system $(G, X^*)$ with the action $(g, p) \mapsto gp$, $gp = \{gP : P \in p\}$. We note that $\overline{\overline{p}} = Gp$ for each $p \in X^*$. Since the parallelity $||$ is defined by means of entourages, the partition of $X^*$ into $G$-orbits does not depend on $G$-realizations of $(X, \mathcal{E})$. It follows that if some property formulated in terms of $G$-orbits of $(G, X^*)$ is proved for some $G$-realization of $(X, \mathcal{E})$ then it holds for any $G$-realization. Moreover, by Theorem 3.2, every finitary coarse structure can be uniquely recognized by the set of orbits in $X^*$.

Given a finitary coarse space $(X, \mathcal{E})$, its $G$-realization of $X_G$, a subset $A \subseteq X$ and $p \in X^*$, we define the *$p$-companion* of $A$ by

$$\triangle_p(A) = A^* \cap Gp.$$

**Theorem 3.3.** *For a subset $A$ of $(X, \mathcal{E})$, the following statements hold*

(1) *$A$ is large iff $\triangle_p(A) \neq \emptyset$ for each $p \in X^*$;*

(2) *$A$ is thick iff $\triangle_p(A) = Gp$ for some $p \in X^*$;*

(3) *$A$ is thin iff $|\triangle_p(A)| \leq 1$ for each $p \in X^*$.*

*Proof.* Theorem 3.1 and 3.2 in [20] . □

We recall that a dynamical system $(G, T)$ is

- *minimal* if each orbit $Gx$ is dense in $T$;
- *topologically transitive* if, for every non-empty open subset $U \subseteq T$, $GU$ is dense in $T$, $GU = \{g(U) : g \in G\}$;
- *topologically point transitive* if, some orbit $Gx$ is dense in $T$.

For a dynamical system $(G, T)$, $ker(G, T)$ denotes the closure of the union of all minimal closed $G$-invariant subsets of $T$. Theorem 2.3 describes explicitely the kernel of the dynamical system $(G, X^*)$ of $X_G$.

**Theorem 3.4.** *Let $(X, \mathcal{E})$ be a finitary coarse space and $(X, \mathcal{E}) = X_G$. Then $(X, \mathcal{E})$ is $\delta$-tight if and only if the dynamical system $(G, X^*)$ is minimal.*

*Proof.* Apply Theorem 3.3(1). □

**Theorem 3.5.** *Let $(X, \mathcal{E})$ be a finitary coarse space and $(X, \mathcal{E}) = X_G$. Then the following statements are equivalent*

(1) *$(X, \mathcal{E})$ is $\lambda$-tight ;*

(2) *$(G, X^*)$ is topologically transitive;*

(3) *for any family $\{A_n : n \in \omega\}$ of infinite subsets of $X$, there exists $p \in X^*$ such that $A_n^* \cap Gp \neq \emptyset$ for each $n \in \omega$.*

*Proof.* (1) $\Longrightarrow$ (2). We take infinite subsets $A, B$ of $X$ and find $p \in X^*$, $g \in G$ such that $A \in p$, $B \in gp$. Since $A, B$ are linked, there exist $A' \subseteq A$, $B' \subseteq B$ and $H \in [G]^{<\omega}$ such that $A' \subseteq HB'$. We take $p \in X^*$ such $A' \in p$. Then $B' \in h^{-1}p$ for some $h \in H$.

(2) $\Longrightarrow$ (3). We choose inductively a sequence $(g_n)_{n\in\omega}$ in $G$ and a sequence $(C_n)_{n\in\omega}$ of subsets of $G$ such that $C_n \subseteq A_n$, $g_nC_n \subseteq A_{n+1}$, $C_{n+1} \subseteq g_nC_n$. Let $h_n = g_ng_{n-1}\dots g_0$. Then

$$A_0 \cap h_0^{-1}A_1 \cap \dots \cap h_n^{-1}A_{n+1} \neq \emptyset$$

for each $n \in \omega$. We take an arbitrary ultrafilter $p \in X^\sharp$ such that $A_0 \cap h_0^{-1} \cap \dots \cap h_n^{-1}A_{n+1} \in p$ for each $n \in \omega$. Then $Gp \cap A_n^* \neq \emptyset$ for each $n \in \omega$.

(3) $\Longrightarrow$ (1). Evident. □

**Corollary 3.6.** *If $(X, \mathcal{E}) = X_G$ and the dynamical system $(G, X^*)$ is topologically point transitive then $(X, \mathcal{E})$ is $\lambda$-tight.*

**Remark 3.7.** Does there exist a group $G$ of permutations of $\omega$ such that $(G, \omega^*)$ is topologically transitive but $(G, \omega^*)$ is not topologically point transitive? This question can not be answered in ZFC without additional assumptions. Yes, if $\mathfrak{t} < \mathfrak{c}$ and No if $\mathfrak{t} = \mathfrak{c}$, see [1, Theorems 5.2 and 5.3].

**Theorem 3.8.** *Let $K$ be a closed nowhere dense subset of $\omega^*$. Then there exists a transitive group $G$ of permutations of $\omega$ such that $ker(G, \omega^*) = K$ and the orbit $Gp$ is dense in $\omega^*$ for each $p \notin K$.*

*Proof.* We take a filter $\phi$ on $\omega$ such that $K = \overline{\phi}$ and $\overline{\phi}$ has the base $\{\overline{A} : A \in \phi\}$. We denote by $G$ the group of all permutations $g$ of $\omega$ such that there exists $A_g \in \phi$ such that $g(x) = x$ for each $x \in A_g$. Clearly, $G$ is transitive.

If $q \in K$ then $g(q) = q$ for each $g \in G$ so $K \subseteq ker(G, \omega^*)$.

We fix $p \in \omega^* \setminus K$ and take an arbitrary $q \in \omega^*$, $p \neq q$. Let $P \in p$, $Q \in q$ and $P \cap Q = \emptyset$. Since $K$ is nowhere dense, there exists $A \in \phi$ such that $P \setminus A \in p$ and $Q \setminus A$ is infinite. By the definition of $G$, there exists $g \in G$ such that $g(P \setminus A) = Q \setminus A$. Hence, $Gp$ is dense in $\omega^*$ and $K = ker(G, \omega^*)$. □

**Corollary 3.9.** *There are $2^{\mathfrak{c}}$ $\lambda$-tight finitary coarse spaces on $\omega$ which are not $\delta$-tight.*

*Proof.* In light of Corollary 3.6, it suffices to notice that there are $2^{\mathfrak{c}}$ free ultrafilters on $\omega$. □

Each orbit of a dynamical system from the proof of Theorem 3.8 is either dense or a singleton. We construct a topologically point transitive $(G', \omega^*)$ having an infinite discrete orbit.

**Example 3.10.** We partition $\omega$ into infinite subsets $\{W_n : n \in \mathbb{Z}\}$, fix a bijection $f_n : W_n \longrightarrow W_{n+1}$ and denote by $f$ a bijection of $\omega$ such that $f|_{W_n} = f_n$.

For each $n \in \mathbb{Z}$, we put $p_n = f^n(p_0)$ and denote by $K$ the closure of $\{p_n : n \in \mathbb{Z}\}$. By Theorem 3.8, there exists a group $G$ of permutations of $\omega$ such that $(G, \omega^*)$ is topologically point transitive and $g(q) = q$ for all $g \in G$, $q \in K$.

We denote by $G'$ the group of permutations generated by $G \cup \{f\}$. Then $(G', \omega^*)$ remains topologically point transitive and the orbit $G'p_0 = \{p_n : n \in \mathbb{Z}\}$ is discrete. □

**Remark 3.11.** If $(X, \mathcal{E})$ is a finitary coarse space, $(X, \mathcal{E}) = X_G$ then, by Theorem 2.3, every infinite subset of $(X, \mathcal{E})$ is prethick if and only if $ker(G, X^*) = X^*$.

If every infinite subset of a finitary coarse space $(X, \mathcal{E})$ is thick then $(X, \mathcal{E})$ is discrete. Indeed, if $(X, \mathcal{E})$ is not discrete then, by Theorem 3.3(3), there exists $q \in X^*$ and $g \in G$ such that $gq \neq q$. We take $Q \in q$ such that $gQ \cap Q = \emptyset$. It follows that $Q$ is not thick.

We say that a subset $A$ of $X_G$ is

- *sparse* if $\triangle_p(A)$ is finite for each $p \in G^*$;
- *scattered* if, for each infinite subset $Y$ of $A$ there exists $p \in Y^*$ such that $\triangle_p(Y)$ is finite.

**Theorem 3.12.** *Let $(X, \mathcal{E})$ be a finitary coarse space and $(X, \mathcal{E}) = X_G$. Then the following statements are equivalent:*

(1) *$(X, \mathcal{E})$ is indiscrete;*

(2) *for every infinite subset $A$ of $X$, there exist $p \in X^*$ and $g \in G$ such that $A \in p$, $A \in gp$ and $p \neq gp$;*

(3) *every infinite subset $A$ of $X$ is not sparse.*

*Proof.* The equivalence of (1) and (2) follows from Theorem 3.3(3), (3) $\Longrightarrow$ (1) is evident.

To show (2) $\Longrightarrow$ (3), we choose a sequence $(g_n)_{n\in\omega}$ in $G$ and sequence $(A_n)_{n\in\omega}$ of subsets of $A$ such that

$$A_{n+1} \subset A_n, \quad g_n A_{n+1} \cap A_{n+1} = \emptyset, \quad n \in \omega$$

and choose $p \in X^*$ such that $A_n \in p$ for each $n \in \omega$. Then $Gp \cap A^*$ is infinite and $A$ is not sparse. □

**Theorem 3.13.** *A subset $A$ of $X_G$ is scattered if and only if $Gp$ is discrete for each $p \in A^*$.*

*Proof.* Theorem 5.4 in [20] . □

Let $G$ be a group of permutations of a set $X$. Let $(g_n)_{n\in\omega}$ be a sequence in $G$ and let $(x_n)_{n\in\omega}$ be a sequence in $X$ such that

(1) $\{g_0^{\epsilon_0} \dots g_n^{\epsilon_n}\ x_n : \epsilon_i \in \{0,1\}\} \cap \{g_0^{\epsilon_0} \dots g_m^{\epsilon_m}\ x_m : \epsilon_i \in \{0,1\}\} = \emptyset$ for all distinct $n, m \in \omega$;

(2) $\{g_0^{\epsilon_0} \dots g_n^{\epsilon_n}\ x_n : \epsilon_i \in \{0,1\}\}| = 2^{n+1}$ for every $n \in \omega$.

Following [20], we say that a subset $Y$ of $X$ is a *piece-wise shifted $FP$-set* if there exist $(g_n)_{n\in\omega}$, $(x_n)_{n\in\omega}$ satisfying (1), (2) and such that

$$Y = \{g_0^{\epsilon_0} \dots g_n^{\epsilon_n}\ x_n : \epsilon_i \in \{0,1\}\},\ n \in \omega\}.$$

**Theorem 3.14.** *A subset $A$ of $X_G$ is scattered if and only if $A$ does not contain piece-wise shifted $FP$-sets.*

*Proof.* Theorem 4.4 in [20]. □

**Theorem 3.15.** *Let $(X, \mathcal{E})$ be a finitary indiscrete space, $(X, \mathcal{E}) = X_G$. Then there exists $p \in X^*$ such that the orbit $G_p$ is not discrete.*

*Proof.* We may suppose that $G$ consists of all permutations $g$ of $X$ such that $(x, gx) \in E$ for some $E \in \mathcal{E}$ and all $x \in X$.

In light of Theorem 3.13 and Theorem 3.14, it suffices to find a piece-wise shifted $FP$-set in $X$ defined by some sequence $(g_n)_{n\in\omega}$ in $G$ and some sequence $(x_n)_{n\in\omega}$ in $X$.

Since $X$ is not discrete, there are an infinite subset $A_0 \subset X$ and an involution $g_0 \in G$ such that $A_0 \cap g_0 A_0 = \emptyset$ and $g_0 x = x$ for each $x \in X \setminus (A_0 \cup g A_0)$. Pick $x_0 \in A_0$.

Suppose that $A_0, \dots, A_n$, $g_0, \dots, g_n$ and $x_0, \dots, x_n$ have been chosen. Since $A_n$ is not discrete, we can find an infinite subset $A_{n+1}$ of $A_n$ and an involution $g_{n+1} \in G$ such that $g_{n+1} A_{n+1} \subset A_n$, $A_{n+1} \cap g_{n+1} A_{n+1} = \emptyset$ and $g_{n+1} x = x$ for each $x \in X \setminus (A_{n+1} \cup g_{n+1} A_{n+1})$. Pick $x_{n+1} \in A_{n+1}$.

After $\omega$ steps, we get the desired sequences $(g_n)_{n\in\omega}$, $(x_n)_{n\in\omega}$. □

**Example 3.16.** We show that an indiscrete space needs not to be $\lambda$-tight. To this end, we take the set $X$ of all rational number on $[0, 1]$, denote by $G$ the group of all homeomorphisms of $X$ and consider the finitary space $X_G$. Let $A = \{a_n : n \in \omega\}$, $B = \{b_n : n \in \omega\}$ be subsets of $X$ such that $(a_n)_{n\in\omega}$ converges to 0 and $(b_n)_{n\in\omega}$ converges to some irrational number. Then $A, B$ are not linked, so $X_G$ is not $\lambda$-tight. On the other hand, let $A$ be an infinite subset of $X$. We take two disjoint sequences $(a_n)_{n\in\omega}$, $(b_n)_{n\in\omega}$ in $A$ which converge to some point from $[0, 1]$. Then there is a homeomorphism $g$ of $X$ such that $g(a_n) = b_n$, $n \in \omega$. Applying Theorem 3.12, we see that $X_G$ is indiscrete.

As the results, we have got the following line

$\delta$-tight $\Longrightarrow$ topologically point transitive $\Longrightarrow$ $\lambda$-tight $\Longrightarrow$ indiscrete, in which the first and the third arrow can not be reversed, and the second arrow can be reversed but only under some assumptions additional to ZFC.

**Question 3.17.** *Is a dynamical system $(G, \omega^*)$ minimal provided that $ker(G, \omega^*) = \omega^*$ and $(G, \omega^*)$ is topologically point transitive?*

The Higson corona of every $\lambda$-tight space is a singleton. The following example suggested by Taras Banakh shows that the Higson corona of finitary indiscrete space needs not to be a singleton.

**Example 3.18.** Let $(X_1, \mathcal{E}_1)$, $(X_2, \mathcal{E}_2)$ be infinite indiscrete finitary spaces. We endow the union $X$ of $X_1$ and $X_2$ with the smallest coarse structure $\mathcal{E}$ such that the restrictions of $\mathcal{E}$ to $X_1$ and $X_2$ coincide with $\mathcal{E}_1$ and $\mathcal{E}_2$. Then the Higson corona of $(X, \mathcal{E})$ is not a singleton because the function $f$ defined by $f(x) = 0$, $x \in X_1$ and $f(x) = 1$, $x \in X_2$ is slowly oscillating.

Given a coarse space $X$, it is naturally to ask a characterization of the smallest ideal in the Boolean algebra of subsets of $X$ containing all discrete subsets. In other words, how can one detect whether a given subset $A$ of $X$ can be partitioned into finitely many discrete? For metric spaces, this question was answered in [7] with usage of the following notion.

A subset $A$ of a coarse space $(X, \mathcal{E})$ is called *$n$-thin*, $n \in \mathbb{N}$ if for each $E \in \mathcal{E}$ there exists a bounded subset $B$ of $(X, \mathcal{E})$ such that $|E_A[a]| \leq n$ for each $a \in A \setminus B$. Clearly, the union of $n$ discrete subsets is $n$-thin. Every $n$-thin metrizable coarse space can be partitioned into $\leq n$ thin subsets [7], but the Bergman's construction from [19] gives a finitary $n$-thin space which can not be partitioned into $\leq n$ thin subsets.

**Theorem 3.19.** *There exists a group $G$ of permutations of $\omega$ such that the coarse space $\omega_G$ is 2-thin but $G$ can not be finitely partitioned into discrete subsets.*

*Proof.* Theorem 6.1 in [1]. □

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