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by

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π–PSEUDOCOMPLETE SPACES

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Abstract. In this paper we introduce the class of π–pseudocomplete spaces, which contains the class of π–complete spaces. Among others, we show that assuming MA+¬CH, π–pseudocomplete CCC spaces satisfy the strong Baire property and π–pseudocomplete CCC spaces with small pseudobases are separable (Theorem 3.12 and Theorem 4.2, respectively). Finally, we prove, also assuming MA+¬CH, that if X is a T_1 quasiregular compact space with $c(X) = \omega$, $t(X) = \omega$ and $|\operatorname{cl}_X(A)| \leq 2^\omega$ for every $A \in [X]^{\leq\omega}$, then $|X| \leq HW(X)2^\omega$ (Corollary 4.17). This result provides a partial positive answer to the following question due to Arhangel'skii ([1]): Does MA+¬CH imply that every compact space with $c(X) = \omega$ and countable tightness has cardinality $\leq 2^\omega$?

1. Introduction

A topological space X has countable cellularity (or X is CCC), denoted by $c(X) \leq \omega$, if every disjoint family of open sets in X is at most countable. It is well known that Martin's Axiom (MA) with the negation of the Continuum hypothesis (¬CH) imply the Suslin hypothesis, that is, every ordered space with countable cellularity is separable.

The following question posed by Hajnal and Juhász, is natural (see [10]).

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Problem 1.1. *Assume that* MA $+\neg$CH *holds. For which classes of spaces, separability and countable cellularity are equivalent?*

Juhász defined the class of π–complete spaces and he showed that under MA and certain additional conditions, countable cellularity and separability are equivalent in this class.

In this paper, using Todd's pseudobase notion [22] we introduce the class of π–pseudocomplete spaces and we show for instance, that countable cellularity and separability are also equivalent in this class (under MA and certain additional conditions).

Finally, in Proposition 4.16, we establish an upper bound (using a cardinal function introduced recently in [3]) for the cardinality of π–pseudocomplete spaces which satisfy certain additional conditions. As a consequence of this proposition, we obtain a result (Corollary 4.17) related to the open problem due to Arhangel'skii (see Problem 4.13).

2. Notations and definitions

The notations and definitions about topological spaces are the ones from [5]. Let X be a topological space and κ be a cardinal number. A subset Y of X is G_κ or *G_κ–type* (for some cardinal κ) if Y is the intersection of $\leq \kappa$ open subsets of X. We recall that a Tychonoff space X is an *absolute–G_κ* if it is a G_κ subset of its Stone–Čech compactification. A space X is Čech–complete if it is $G_{\aleph_0}$ in its Stone–Čech compactification (see [21]).

Let κ be an infinite cardinal and X be a topological space. $A \subseteq X$ is *nowhere dense* if $X \setminus \overline{A}$ is dense. We say that X is *κ–Baire* or satisfies the *κ–Baire property*, if it is not a union of less than κ of its nowhere dense subsets. A space X is strongly Baire, or satisfies the strong Baire property, if it is 2^ω–Baire. A space X is *quasiregular* if each nonempty open set U in X contains the closure of a nonempty open set V in X.

We recall that a filter base $\mathcal{F}$ in a space X is called *regular* if for any $F_1,\ F_2 \in \mathcal{F}$ there exists an element $F \in \mathcal{F}$ such that $\overline{F} \subseteq int(F_1 \cap F_2)$.

We refer the reader to [11] and [13] for definitions and terminology on cardinal functions not explicitly given here. Let X be a topological space. By $c(X), d(X), \chi(X), t(X), hd(X), nw(X), s(X)$ we denote the *cellularity, density, character, tightness, hereditary density, net weight* and *spread* of X respectively. For a topological space X, $t(X)^+$ is the smallest regular cardinal κ with the property that for any $A \subseteq X$ and each $x \in \overline{A}$, there exists $B \subseteq A$, with $|B| < \kappa$ such that $x \in \overline{B}$.

Let us recall some well–known notions that are necessary in what follows in this article.

- Let X be a topological space and $x \in X$. A family of nonempty open sets $\mathcal{U}$ is said to be a *local π–base* at the point $x \in X$ if for every open set $V \subseteq X$ such that $x \in V$ there is a set $U \in \mathcal{U}$ such that $U \subseteq V$. The *π–character of the point* x in X ($\pi\chi(x, X)$) is the least cardinality of a local π–base at x. The *π–character* of the space X is defined as $\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$.
- Let X be a topological space and $x \in X$. A family of open nonempty sets $\mathcal{U}$ is said to be a *π–base* for X if for every open set $V \subseteq X$ there is a set $U \in \mathcal{U}$ such that $U \subseteq V$. The *π–weight* of X ($\pi w(X)$), is the least cardinality of a π–base for X.

The proof of Theorem 4.5 in this paper uses the κ–closure operator, which is defined as follows:

For $A \subseteq X$ and κ an infinite cardinal, the *κ–closure* of A is defined as $[A]_\kappa = \bigcup\{\mathrm{cl}_X(C) : C \in [A]^{\leq\kappa}\}$. A set A is *κ–closed* if $[A]_\kappa = A$. It is straightforward to verify the following basic properties of the κ–closure operator (see [4]).

(a) $[A]_\kappa \subseteq \mathrm{cl}_X(A)$ for every $A \subseteq X$;
(b) if $|A| \leq \kappa$, then $[A]_\kappa = \mathrm{cl}_X(A)$;
(c) every closed set is κ–closed;
(d) $[[A]_\kappa]_\kappa = [A]_\kappa$; and
(e) $t(X) \leq \kappa$ if and only if for all $A \subseteq X$, $[A]_\kappa = \mathrm{cl}_X(A)$.

3. π–PSEUDOCOMPLETENESS

Juhász introduced the notion of π–completeness in [12]: A regular space X is called *π–complete* if it admits a family $\{\mathcal{B}_\alpha : \alpha < \lambda\}$ of π–bases for some $\lambda < 2^\omega$, such that whenever $\mathcal{G} \subseteq \bigcup\{\mathcal{B}_\alpha : \alpha < \lambda\}$ is a regular filter base with $|\mathcal{G}| < 2^\omega$ and such that $\mathcal{G} \cap \mathcal{B}_\alpha \neq \emptyset$ for each $\alpha < \lambda$, then $\bigcap \mathcal{G} \neq \emptyset$.

The difference between the notions of π–completeness and π–pseudocompleteness is that the first one uses π–bases and the second one uses the notion of pseudobase. The concept of pseudobase is due to Todd and we recall it in the next definition (see [22]).

Definition 3.1. In a topological space, a *pseudobase* is a family of sets whose interiors are nonempty, and every nonempty open subset of the space contains a set of this family.

Clearly every base and π–base are pseudobases.

Definition 3.2. A space X is called *π–pseudocomplete* if X is quasiregular and it has a family $\{\mathcal{B}_\alpha : \alpha < \lambda\}$ of pseudobases with $\lambda < 2^\omega$ such that whenever $\mathcal{G} \subseteq \bigcup\{\mathcal{B}_\alpha : \alpha < \lambda\}$ is a regular filter base with $|\mathcal{G}| < 2^\omega$ and $\mathcal{G} \cap \mathcal{B}_\alpha \neq \emptyset$, for each $\alpha < \lambda$, then $\bigcap \mathcal{G} \neq \emptyset$.

The idea behind the use of pseudobases is that, even though the π–completeness definition involves regular filter bases, the use of π–bases forces the elements of the filter base to be open, while the notion of pseudobase retrieves the general case.

Since every π–base is a pseudobase, it is immediate that every π–complete space is a π–pseudocomplete space. Moreover, it is clear that every pseudobase induces a π–base. So far the authors do not know an answer to the following question:

Problem 3.3. *Is every π–pseudocomplete space a π–complete space?*

Remark: The pseudocompleteness property was first introduced by Oxtoby [16]. Todd came up with a modification of the pseudocompleteness property (which employs pseudobases instead of π–bases) [22]. He observed that this modification generate results similar to Oxtoby's. Formally, the pseudocompleteness in the sense of Oxtoby is stronger than that of Todd. However, whether the two properties are equivalent remains an open question (see [18] or [22]).

Let us see that the notions of π–completeness and π–psudocompleteness generate similar results as well.

Juhász states that the class of π–complete spaces contains the class of the Čech–complete spaces (see [12]). The following result shows directly that the class of π–pseudocomplete spaces includes all Čech–complete spaces.

Proposition 3.4. *Let X be a compact Hausdorff space and let $\kappa < 2^\omega$. If Y is a dense subspace of X which is G_κ, then Y is π–pseudocomplete.*

Proof. By hypothesis there exists a collection $\{V_\alpha : \alpha \in \kappa\}$ of open subsets of X such that $Y = \bigcap\{V_\alpha : \alpha \in \kappa\}$.

For each $\alpha < \kappa$, let $\mathcal{B}_\alpha = \{U \cap Y : U \subseteq X, \ \mathrm{int}_X(U) \neq \emptyset \text{ and } \mathrm{cl}_X(U) \subseteq V_\alpha\}$.

It is easy to see that for each $\alpha < \kappa$, $\mathcal{B}_\alpha$ is a pseudobase in Y.

We will show that if $\mathcal{G} \subseteq \bigcup\{\mathcal{B}_\alpha : \alpha \in \kappa\}$ is a regular filter base such that for each $\alpha \in \kappa$, $\mathcal{G} \cap \mathcal{B}_\alpha \neq \emptyset$, then $\bigcap \mathcal{G} \neq \emptyset$. For every $G \in \mathcal{G}$, there exists $\alpha \in \kappa$ such that $G \in \mathcal{B}_\alpha$. Hence there exists $U_{(\alpha,G)} \subseteq X$ such that $G = Y \cap U_{(\alpha,G)}$ and $\mathrm{cl}_X(U_{(\alpha,G)}) \subseteq V_\alpha$. Since $\mathcal{G}$ is a regular filter base, the collection $\mathcal{G}' = \{\mathrm{cl}_X(U_{(\alpha,G)}) : G \in \mathcal{G}\}$ is a family of nonempty closed subsets which satisfies the finite intersection property. Then by the compactness of X, $\bigcap \mathcal{G}' \neq \emptyset$.

Clearly, if $x_0 \in \bigcap \mathrm{cl}_X(U_{(\alpha,G)})$, then $x_0 \in \bigcap \mathcal{G}$. We conclude that Y is π–pseudocomplete. $\square$

Corollary 3.5. *The class of absolute–G_κ (for any $\kappa < 2^\omega$) is a subclass of the π–complete class (and therefore of the π–pseudocomplete class). In particular each Čech–complete space is a π–pseudocomplete space.*

Todd proved that every pseudocompact space is pseudocomplete, so it appears natural to ask whether every weakly pseudocompact space is pseudocomplete [22]. In that relation, in [18], the authors show that every weakly pseudocompact space is T–pseudocompact. (For the definition of T-pseudocompact spaces see [18]).

There exists another class of topological spaces that belong to π–pseudocomplete spaces, namely:

(a) A regular topological space X is called *basis compact* if it has a base $\mathcal{B}$ such that for any $\mathcal{G} \subseteq \mathcal{B}$ with the finite intersection property, we have $\bigcap\{\overline{G} : G \in \mathcal{G}\} \neq \emptyset$ (see [21]). It is not difficult to see that every basis compact space is π–pseudocomplete.

(b) A Tychonoff space X, is called *scattered* if every nonempty subset of X has an isolated point. In [7], the authors proved that every scattered space is subcompact, i.e., it admits a base $\mathcal{B} \subseteq \tau_X \setminus \{\emptyset\}$ such that whenever $\mathcal{G} \subseteq \mathcal{B}$ is a regular filter base, then $\bigcap \mathcal{G} \neq \emptyset$. Hence, every subcompact space is, trivially, π–pseudocomplete. Hence, every scattered space is also π–pseudocomplete.

Proposition 3.4 naturally motivates the next question: Let X be a π–pseudocomplete space and let $\kappa < 2^\omega$. Is it true that if Y is a dense subspace of X (which is G_κ), then Y is π–pseudocomplete?

As we will see in the next proposition, the answer to the reverse of the previous question is affirmative.

Proposition 3.6. *Let X be a quasiregular space. If Y is a π–pseudocomplete dense subset of X, then X is a π–pseudocomplete space.*

Proof. Let $\{\mathcal{B}^Y_\alpha : \alpha < \lambda\}$; with $\lambda < 2^\omega$ be a family of pseudobases in Y which witnesses the π–pseudocompleteness of Y.

For each $\alpha < \lambda$, let $\mathcal{B}_\alpha = \mathcal{B}^Y_\alpha$. Note that, for every $\alpha < \lambda$, $\mathcal{B}_\alpha$ is a pseudobase in X.

Now, let $\mathcal{F} \subseteq \bigcup\{\mathcal{B}_\alpha : \alpha \in \lambda\}$ be a regular filter base with $|\mathcal{F}| < 2^\omega$ such that for each $\alpha \in \lambda$, $\mathcal{F} \cap \mathcal{B}_\alpha \neq \emptyset$. We need to show that $\bigcap \mathcal{F} \neq \emptyset$. It will be enough to show that $\mathcal{F}$ is a regular filter base in Y.

Indeed, let $B_1, B_2 \in \mathcal{F}$. Then there is $B_3 \in \mathcal{F}$ such that $\mathrm{cl}_X(B_3) \subseteq \mathrm{int}_X(B_1 \cap B_2) = \mathrm{int}_X(B_1) \cap \mathrm{int}_X(B_2)$. Hence $\mathrm{cl}_X(B_3) \cap Y \subseteq \mathrm{int}_X(B_1 \cap B_2) \cap Y$ implies that $\mathrm{cl}_Y(B_3) \subseteq \mathrm{int}_Y(B_1) \cap \mathrm{int}_Y(B_2)$, and so $\bigcap \mathcal{F} \neq \emptyset$. Thus, X is π–pseudocomplete. □

The following result states that the property of π–pseudocompleteness is inherited by nonempty open subsets.

Proposition 3.7. *Let X be a quasiregular space. If X is π–pseudocomplete, then every nonempty open subset of X is π–pseudocomplete.*

Proof. Let U be a nonempty open set in X. By hypothesis, there exists a family $\{\mathcal{B}_\alpha : \alpha < \lambda\}$ of pseudobases in X which testifies that X is a π–pseudocomplete space with $\lambda < 2^\omega$. Let us consider for each $\alpha < \lambda$ the family $\mathcal{B}_\alpha^U = \{B \in \mathcal{B}_\alpha : \overline{B} \subseteq U\}$. Note that $\mathcal{B}_\alpha^U$ is a family of pseudobases in U.

Now, let $\mathcal{F} \subseteq \bigcup\{\mathcal{B}_\alpha^U : \alpha \in \lambda\}$ be a regular filter base, in U with $|\mathcal{F}| < 2^\omega$ such that for each $\alpha \in \lambda$, $\mathcal{F} \cap \mathcal{B}_\alpha^U \neq \emptyset$. We need to show that $\bigcap \mathcal{F} \neq \emptyset$.

It is enough to show that $\mathcal{F}$ is a regular filter base in X. To see it, notice that if $B_1, B_2 \in \mathcal{F}$, there exists $B_3 \in \mathcal{F}$ such that $\mathrm{cl}_U B_3 \subseteq \mathrm{int}_U(B_1 \cap B_2) \subseteq \mathrm{int}_X(B_1 \cap B_2)$. But, $\mathrm{cl}_U B_3 = \mathrm{cl}_X(B_3) \cap U = \mathrm{cl}_X(B_3)$. Thus, $\mathrm{cl}_X(B_3) \subseteq \mathrm{int}(B_1 \cap B_2)$. So $\mathcal{F}$ is a regular filter base in U and therefore $\bigcap \mathcal{F} \neq \emptyset$. □

Our next proposition shows that the π–pseudocompleteness is preserved under finite products.

Proposition 3.8. *If (X, τ_X) and (Y, τ_Y) are π–pseudocomplete spaces, then $X \times Y$ is a π–pseudocomplete space.*

Proof. Let $\{\mathcal{B}_\alpha^X : \alpha < \lambda_X\}$ and $\{\mathcal{B}_\alpha^Y : \alpha < \lambda_Y\}$ be two family of pseudobases in X and Y respectively. For each $(\alpha, \beta) \in \lambda_X \times \lambda_Y$, we define

$$\mathcal{B}(\alpha, \beta) := \{B \times V : B \in \mathcal{B}_\alpha^X \text{ and } V \in \mathcal{B}_\beta^Y\}.$$

It is easy to see that $\mathcal{B}(\alpha, \beta)$ is a pseudobase in $X \times Y$.

Now, let $\mathcal{F}^+ \subseteq \bigcup\{\mathcal{B}(\alpha, \beta) : (\alpha, \beta) \in \lambda_X \times \lambda_Y\}$ be a regular filter base in $X \times Y$ with $|\mathcal{F}^+| < 2^\omega$ and $\mathcal{F}^+ \cap \mathcal{B}(\alpha, \beta) \neq \emptyset$ for every $(\alpha, \beta) \in \lambda_X \times \lambda_Y$.

Let us define

$\mathcal{F}_X = \{B \in \bigcup_{\alpha \in \lambda_X} \mathcal{B}_\alpha^X :$ there exists $V \in \bigcup_{\beta \in \lambda_Y} \mathcal{B}_\beta^Y$ such that $B \times V \in \mathcal{F}^+\}$, and

$\mathcal{F}_Y = \{V \in \bigcup_{\beta \in \lambda_Y} \mathcal{B}_\beta^Y :$ there exists $B \in \bigcup_{\alpha \in \lambda_X} \mathcal{B}_\alpha^X$ such that $B \times V \in \mathcal{F}^+\}$.

Note that $\mathcal{F}_X \subseteq \bigcup \mathcal{B}_\alpha^X$ and $\mathcal{F}_Y \subseteq \bigcup \mathcal{B}_\beta^Y$ with $|\mathcal{F}_X| < 2^\omega$ and $|\mathcal{F}_Y| < 2^\omega$. Note also that $\mathcal{F}_X \cap \mathcal{B}_\alpha^X \neq \emptyset$ and $\mathcal{F}_Y \cap \mathcal{B}_\beta^Y \neq \emptyset$ for every $(\alpha, \beta) \in \lambda_X \times \lambda_Y$.

Now, let $B_1, B_2 \in \mathcal{F}_X$ and since, $\mathcal{F}^+$ is a regular filter base, there are $V_1, V_2 \in \mathcal{F}_Y$ such that $B_1 \times V_1, B_2 \times V_2 \in \mathcal{F}^+$, and so there exists $B_3 \times V_3 \in \mathcal{F}^+$ such that $\mathrm{cl}_{(X \times Y)}(B_3 \times V_3) \subseteq \mathrm{int}((B_1 \times V_1) \cap (B_2 \times V_2))$, implying that $\mathrm{cl}_X(B_3) \subseteq \mathrm{int}(B_1) \cap \mathrm{int}(B_2)$.

Analogously, if $V_1, V_2 \in \mathcal{F}_Y$, there exists $B_1, B_2 \in \mathcal{F}_X$ and $B_3 \times V_3 \in \mathcal{F}^+$ such that $\mathrm{cl}_{(X \times Y)}(B_3 \times V_3) \subseteq \mathrm{int}((B_1 \times V_1) \cap (B_2 \times V_2))$, implying that $\mathrm{cl}_Y(V_3) \subseteq \mathrm{int}(V_1) \cap \mathrm{int}(V_2)$.

Therefore, $\mathcal{F}_X$ and $\mathcal{F}_Y$ are regular filter bases. It follows that there exist $x \in \bigcap \mathcal{F}_X$ and $y \in \bigcap \mathcal{F}_Y$, and therefore $(x, y) \in \bigcap \mathcal{F}^+$. Hence $X \times Y$ is π–pseudocomplete. □

Corollary 3.9. *The Cartesian product of a finite family of π–pseudocomplete spaces is a π–pseudocomplete space.*

At the moment, the authors do not know if the π–pseudocompleteness is a property preserved under an open continuous function and arbitrary products. However, our next theorem shows that if X is a π–pseudocomplete space, then the Cartesian product X^κ is π-pseudocomplete for each $\kappa < 2^\omega$.

Theorem 3.10. *Let X be a π–pseudocomplete space. The space X^κ is π–pseudocomplete for each $\kappa < 2^\omega$.*

Proof. Let $\{\mathcal{B}_\alpha : \alpha < \lambda\}$ be a collection of pseudobases in X which witnesses the π–pseudocompleteness of X. For each $(\alpha, \beta) \in \lambda \times \kappa$, we define $\mathcal{B}(\alpha, \beta)$ as follows:

$$\mathcal{B}(\alpha,\beta) = \left\{ \bigcap_{i\in I} \pi_i^{-1}(B_i) : \forall i \in I,\ B_i \in \mathcal{B}_\alpha \text{ where } I \in [\kappa]^{<\omega} \text{ and } \beta \in I \right\}$$

where π_i is the natural projection onto the i^{th} factor of X^κ.

We claim that $\mathcal{B}(\alpha, \beta)$ is a pseudobase in X^κ. Indeed, notice that

(a) If $B \in \mathcal{B}(\alpha, \beta)$, then $I \in [\kappa]^{<\omega}$ exists, such that $B = \bigcap_{i\in I} \pi^{-1}(B_i)$. It is clear that $\bigcap_{i\in I} \pi_i^{-1}(int(B_i)) \subseteq \bigcap_{i\in I} \pi_i^{-1}(B_i)$.

(b) If U is an open subset of X^κ and $p \in U$, then there exits $I \in [\kappa]^{<\omega}$ and a family $\{U_i : i \in I\}$ of open sets such that $p \in \bigcap_{i\in I} \pi_i^{-1}(U_i) \subseteq U$. For each $i \in I \in [\kappa]^{<\omega}$ we fix $B_i \in \mathcal{B}_\alpha$ such that $B_i \subseteq U_i$. Hence $\bigcap \pi_i^{-1}(B_i) \subseteq \bigcap \pi_i^{-1}(U_i) \subseteq U$ (we may assume that $\beta \in I$).

Thus, for each $(\alpha, \beta) \in \lambda \times \kappa$, the family $\mathcal{B}(\alpha, \beta)$ is a pseudobase in X^κ.

An immediate consequence of the last condition is that if $\mathcal{F} \subseteq \bigcup\{\mathcal{B}(\alpha, \beta) : (\alpha, \beta) \in \lambda \times \kappa\}$ is a regular filter base with $|\mathcal{F}| < 2^\omega$ such that for any $(\alpha, \beta) \in \lambda \times \kappa$, $\mathcal{F} \cap \mathcal{B}(\alpha, \beta) \neq \emptyset$, then the family $\mathcal{F}_{\beta_0} = \{\pi_{\beta_0}(B) : B \in \mathcal{F}\}$ satisfies the following conditions:

(1) the family $\mathcal{F}_{\beta_0}$ is a regular filter base in X_{β_0},
(2) $|\mathcal{F}_{\beta_0}| < 2^\omega$, $\mathcal{F}_{\beta_0} \subseteq \bigcup\{\pi_{\beta_0}(B_\alpha) : B_\alpha \in \mathcal{B}_\alpha\}$ and
(3) for each $\alpha < \lambda$, $\mathcal{F}_{\beta_0} \cap \mathcal{B}_\alpha \neq \emptyset$.

We have, $\bigcap \mathcal{F}_{\beta_0} \neq \emptyset$, and thus $\bigcap \mathcal{F} \neq \emptyset$. Therefore, the space X^κ is a π–pseudocomplete. □

Now we will show that every π–pseudocomplete space with $c(X) = \omega$ has the strong Baire property. First, we present an auxiliary result concerning dense subsets and generic filters on partial orders.

We refer the reader to [14] (Chapter II, section 2) for definitions and notations not explicitly mentioned in the statement or in the proof of our next result.

Proposition 3.11. *Let $\mathbb{P}$ be a partially ordered set and $\mathcal{D} = \{D_\alpha : \alpha < \gamma\}$ be a collection of dense subsets of cardinality γ. For every $\mathcal{D}$–generic filter, $\mathcal{G}$, in $\mathbb{P}$, there exists $\mathcal{G}' \subseteq \mathcal{G}$ such that:*

(a) $|\mathcal{G}'| \leq \gamma$;
(b) $D_\alpha \cap \mathcal{G}' \neq \emptyset$, for any $\alpha \in \lambda$, and
(c) for each $p,\ q \in \mathcal{G}'$, there is $r \in \mathcal{G}'$ such that $r \leq p$ and $r \leq q$.

Proof. For each $\alpha < \gamma$, we fix a point $p_\alpha \in D_\alpha \cap \mathcal{G}$, and we denote with A_0 the set of all such points. Then we define, recursively, for every $0 < \alpha < \gamma$, $A_{\alpha+1}$, the set of points $r \in \mathcal{G}$, as follows:

If $\alpha = \beta + 1$, for each $p,\ q \in \bigcup\{A_\beta : \beta \leq \alpha\}$, we take $r_{p,q} \in \mathcal{G}$ such that $r_{p,q} \leq p$ and $r_{p,q} \leq q$ and define:

$$A_{\alpha+1} = \bigcup\{A_\beta : \beta \leq \alpha\} \cup \{r_{p,q} : p, q \in A_\alpha\}.$$

If α is a limit ordinal, we put $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$.

Let $\mathcal{G}' = \bigcup\{A_\alpha : \alpha < \gamma\}$. By the construction, the family $\mathcal{G}'$ satisfies conditions (a) and (b). It remains to prove that $\mathcal{G}'$ also satisfies (c). Let p and $q \in \mathcal{G}'$ and suppose that $p \neq q$ (in other case we are done).

Then there are $\alpha_1, \alpha_2 \in \gamma$ such that $p \in A_{\alpha_1}$ and $q \in A_{\alpha_2}$. Without loss of generality, suppose that $\alpha_1 \leq \alpha_2$, then $p,\ q \in A_{\alpha_2}$. Hence, there is $r_{p,q} \in A_{\alpha_2+1}$ such that $r_{p,q} \leq p$ and $r_{p,q} \leq q$. The proof is complete. □

Theorem 3.12. (MA+¬CH) *If X is a π–pseudocomplete space with $c(X) = \omega$, then X has the strong Baire property.*

Proof. Let $\{\mathcal{B}_\alpha : \alpha < \lambda\}$ be a collection of pseudobases that testifies to the π–pseudocompleteness of X, with $\lambda < 2^\omega$. Suppose that $\{U_\xi : \xi < \kappa\}$ is a collection of open dense subsets of X with $\kappa < 2^\omega$.

To see that $\bigcap\{U_\xi : \xi < \kappa\} \neq \emptyset$, let $\mathbb{P} = \bigcup\{\mathcal{B}_\alpha : \alpha \in \lambda\}$ and for each U, $V \in \mathbb{P}$, we say $U \leq V$ if and only if $\overline{U} \subseteq int(V)$.

It is not difficult to prove that $\mathbb{P}$ satisfies the countable antichain condition. Now, for each $(\alpha, \xi) \in \lambda \times \kappa$, we define the set:

$$\mathcal{D}(\alpha, \xi) = \{\mathrm{int}_X(B) \cap U_\xi : B \in \mathcal{B}_\alpha\}.$$

We claim that for every $(\alpha, \xi) \in \lambda \times \kappa$, $\mathcal{D}(\alpha, \xi)$ is a dense subset in $\mathbb{P}$. Indeed, let us fix $(\alpha, \xi) \in \lambda \times \kappa$, and let $V \in \mathbb{P}$. Since U_ξ is dense, $\mathrm{int}(V) \cap U_\xi \neq \emptyset$. Since $\mathcal{B}_\alpha$ is pseudobase and X is quasiregular, there exists

$V' \in \mathcal{B}_\alpha$ such that $\overline{V'} \subseteq \text{int}(V) \cap U_\xi$. Clearly $(\text{int}_X(V') \cap U_\xi) \in \mathcal{D}(\alpha, \xi)$ and $(\text{int}_X(V') \cap U_\xi) \leq V$. Therefore we conclude that $\mathcal{D}(\alpha, \xi)$ is dense in $\mathbb{P}$. Let $D = \{\mathcal{D}(\alpha, \xi) : (\alpha, \xi) \in \lambda \times \kappa\}$. By MA, there exists $\mathcal{D}$–generic filter $\mathcal{G} \subseteq \mathbb{P}$, of cardinality less than 2^ω. Hence, by Proposition 3.11 applied to $\gamma = |\lambda \times \kappa| < 2^\omega$ there exists $\mathcal{G}' \subseteq \mathcal{G}$ such that (a), (b) and (c) in Proposition 3.11 hold.

Clearly $\mathcal{G}'$ is a regular filter base with $|\mathcal{G}'| < 2^\omega$, such that for all $\alpha < \lambda$, $\mathcal{G}' \cap \mathcal{B}_\alpha \neq \emptyset$. Hence $\bigcap \mathcal{G}' \neq \emptyset$. Notice that if $p \in \bigcap \mathcal{G}'$, then $p \in \bigcap\{U_\xi : \xi < \lambda\}$. The proof is complete. □

Corollary 3.13. (MA+¬CH) *Let X be a topological space with $c(X) = \omega$.*

(1) If X is an absolute G_κ, $\kappa < 2^\omega$, then X has the strong Baire property [21].

(2) If X is a π–complete space, then X has the strong Baire property [12].

Recall that Martin's Axiom implies that any product of spaces with countable cellularity has countable cellularity (see [10]). Hence, from Theorem 3.10 and Theorem 3.12 we obtain the following result.

Corollary 3.14. (MA+¬CH) *Let X be a topological space with $c(X) = \omega$ and $\lambda < 2^\omega$.*

(1) *If X is a π–pseudocomplete space, then X^λ has the strong Baire property.*

(2) *If X is a π–complete space, then X^λ has the strong Baire property* [12].

4. π–PSEUDOCOMPLETENESS AND SEPARABILITY

In [10] and [21], the authors analyze the following question: Assuming MA+¬CH, for which classes of spaces, separability and countable cellularity are equivalent?

For instance, regarding this question, a couple of results establish that (assuming MA+¬CH) if X is a topological space with $c(X) = \omega$ and $\pi w(X) < 2^\omega$ then:

- If X is a basis compact space, then X is separable [10].
- If X is an absolute G_κ, $\kappa < 2^\omega$, then X is separable [21].

Before establishing the main theorem of this section, we present a key result to prove it, which is the version in terms of pseudobases of Theorem 1.3 from [12].

Proposition 4.1. *Let X be a topological space and $\kappa < 2^\omega$. If X^ω is κ–Baire with a pseudobase of cardinality $< \kappa$, then $d(X) = \omega$.*

Proof. Let $\mathcal{P}$ be a pseudobase of X with $|\mathcal{P}| < \kappa$. For each $P \in \mathcal{P}$ we define

$$\mathcal{G}_P = \left\{ \bigcap_{i\in I} \pi^{-1}(G_i) : I \in [\omega]^{<\omega}, \forall i \in I : G_i \text{ is open and } \exists i \in I : G_i = P \right\}.$$

It is clear that each element of $\mathcal{G}_P$ is a pseudobase for X^ω; hence, $D_P = \bigcup \mathcal{G}_P$ is a dense subset of X^ω. Since $|\mathcal{P}| < \kappa$ and X is κ-Baire, we conclude that $D = \bigcap\{D_P : P \in \mathcal{P}\} \neq \emptyset$.

Let $x \in D$. Since $S = \{x_n = \pi_n(x) : n \in \omega\}$ is a countable dense subset of X, we conclude that $d(X) = \omega$. $\square$

From the previous result and Corollary 3.14 we obtain the following:

Theorem 4.2. (MA+$\neg$CH) *Let X be a π–pseudocomplete space with $c(X)=\omega$. If X has a pseudobase $\mathcal{B}$ with $|\mathcal{B}| < \kappa$ ($\kappa < 2^\omega$), then X is separable.*

We say that a collection $\mathcal{B}_x$ of subsets of a topological space (X, τ) is a *local π–pseudobase* at $x \in X$ if it satisfies:

(a) for each $B \in \mathcal{B}_x$, $int(B) \neq \emptyset$, and
(b) for every $U \in \tau \setminus \{\emptyset\}$ such that $x \in U$, there is a $V \in \mathcal{B}_x$ such that $V \subseteq U$.

Corollary 4.3. (MA+$\neg$CH) *Let X be a π–pseudocomplete space with $c(X) = \omega$. If there is a dense subspace D of X such that*

(a) *$|D| < 2^\omega$, and*
(b) *for every $x \in D$ there is a local π–pseudobase $\mathcal{B}_x$ at x such that $|\mathcal{B}_x| < 2^\omega$,*

then X is separable.

Proof. It is clear that $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in D\}$ is a pseudobase for X and $|\mathcal{B}| < 2^\omega$; hence, by Theorem 4.2, the space X is separable. $\square$

Corollary 4.4. (MA+$\neg$CH) *Let X be a topological space with $c(X) = \omega$, a dense subset $D \subseteq X$ such that $|D| < 2^\omega$ and $\chi(p, X) < 2^\omega$ for each $p \in D$.*

(1) *If X is a regular basis compact, then X is separable* [10].
(2) *If X is an absolute G_κ, $\kappa < 2^\omega$, then X is separable* [21].

Proof. In both cases it is enough to see that $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in D\}$ is a pseudobase of X and $|\mathcal{B}| < 2^\omega$; where $\mathcal{B}_x$ is a local base at x in X such that $|\mathcal{B}_x| = \chi(p, X) < 2^\omega$. $\square$

Now we turn to presenting our main theorem of this section, which is inspired by the result above.

Theorem 4.5. (MA+$\neg$CH) *Let X be a topological space and let λ be an infinite cardinal such that $\lambda^+ < 2^\omega$. Suppose that*

(1) Every λ–closed subspace of X is π–pseudocomplete,
(2) $c(X) = \omega$ and
(3) the space X admits a subset $Y \subseteq X$ such that $\overline{Y} = X$ and for each $y \in Y$, there exist a local π–pseudobase $\mathcal{B}_y$ of y in X such that $\sup\{|\mathcal{B}_y| : y \in Y\} < 2^\omega$.

Then X is separable.

Proof. For each open subset G of X we choose $y_{(G)} \in G \cap Y$ and we let $y_{(\emptyset)}$ be any element of Y. For each $A \subset Y$ let

$$A' := A \cup \{y_{(X\setminus\overline{A})}\} \cup \{y_{(U\cap V)} : U, V \in \bigcup\{\mathcal{B}_z : z \in A\}\}.$$

We construct an increasing sequence $\{F_\alpha : \alpha < \lambda^+\}$ of subsets in Y, as follows:

(a) $F_0 = \emptyset$,
(b) for each $\xi < \lambda^+$, $F_{\xi+1} = F'_\xi$ and
(c) if $0 < \xi \leq \lambda^+$ is a limit ordinal, then $F_\xi = \bigcup\{F_\alpha : \alpha < \xi\}$.

Let $F_{\lambda^+} = \bigcup\{F_\xi : \xi < \lambda^+\}$. Notice that $|F_{\lambda^+}| < 2^\omega$. To finish we will check that F_{λ^+} satisfies the following properties:

(1) $c(F_{\lambda^+}) = \omega$. Let $\mathcal{H}$ be an uncountable family of open subsets of F_{λ^+}. For each $H \in \mathcal{H}$ we choose a point $x_H \in H$ and let G_H be an open subset of X, such that $H = F_{\lambda^+} \cap G_H$.

Then, for each $H \in \mathcal{H}$, there exists $B_H \in \mathcal{B}_{x_H}$ such that $B_H \subseteq G_H$. Let $\mathcal{C} = \{B_H : H \in \mathcal{H}\}$. We have two cases:

(i) If $|\mathcal{C}| \leq \omega$, there are $H \in \mathcal{H}$ and $\mathcal{H}' \subseteq \mathcal{H}$, with $|\mathcal{H}'| = |\mathcal{H}|$ such that $B_{H'} = B_H$ for every $H' \in \mathcal{H}'$.

(ii) Assume that $|\mathcal{C}| > \omega$. Since $c(X) = \omega$, the family $\mathcal{C}$ is not cellular in X. Hence, there are $H_1,\ H_2 \in \mathcal{H}$ such that $B_{H_1} \cap B_{H_2} \neq \emptyset$.

In both cases there are $H_1,\ H_2 \in \mathcal{H}$ such that $B_{H_1} \cap B_{H_2} \neq \emptyset$.

Since λ^+ is regular, there exists $\xi < \lambda^+$ such that $x_{H_1}, x_{H_2} \in F_\xi$; hence, $B_{H_1}, B_{H_2} \in \bigcup\{B_y : y \in F_\xi\}$; which implies that $y_{(B_{H_1}\cap B_{H_2})} \in F'_\xi = F_\xi \subseteq F_{\lambda^+}$. Moreover $y_{(B_{H_1}\cap B_{H_2})} \in B_{H_1} \cap B_{H_2} \cap F_{\lambda^+} \subseteq G_{H_1} \cap G_{H_2} \cap F_{\lambda^+} = (G_{H_1} \cap F_{\lambda^+}) \cap (G_{H_2} \cap F_{\lambda^+}) = H_1 \cap H_2$. So $H_1 \cap H_2 \neq \emptyset$ and thus, $\mathcal{H}$ is not a cellular family. Therefore, $c(F_{\lambda^+}) = \omega$.

(2) $[F_{\lambda^+}]_\lambda$ have a pseudobase of cardinality $< 2^\omega$. Let $\mathcal{V} = \{U \cap [F_{\lambda^+}]_\lambda : U \in \bigcup\{\mathcal{B}_y : y \in F_{\lambda^+}\}\}$. Now we will see that $\mathcal{V}$ is a pseudobase for $[F_{\lambda^+}]_\lambda$. Let W be an open subset of $[F_{\lambda^+}]_\lambda$. Since F_{λ^+} is dense in $[F_{\lambda^+}]_\lambda$, there is $z \in W \cap F_{\lambda^+}$; hence, there is $\xi < \lambda^+$ such that $z \in F_\xi$.

On the other hand, there exists an open subset $W' \subseteq X$ such that $W = [F_{\lambda^+}]_\lambda \cap W'$. Since $z \in W'$ and $\mathcal{B}_z$ is a local π–pseudobase of z, there exists $B_W \in \mathcal{B}_z$ such that $B_W \subseteq W'$. Then $[F_{\lambda^+}]_\lambda \cap B_W \in \mathcal{V}$ and $[F_{\lambda^+}]_\lambda \cap B_W \subseteq [F_{\lambda^+}]_\lambda \cap W' = W$. Therefore, $\mathcal{V}$ is a pseudobase for $[F_{\lambda^+}]_\lambda$ and $|\mathcal{V}| < 2^\omega$.

(3) The space $[F_{\lambda^+}]_\lambda$ is separable. We have $c([F_{\lambda^+}]_\lambda) = c(F_{\lambda^+}) \leq \omega$. By Theorem 4.2, we obtain that $[F_{\lambda^+}]_\lambda$ is separable.

Now, it only remains to show that $X \subseteq [F_{\lambda^+}]_\lambda$. To see it, notice that $[F_{\lambda^+}]_\lambda = \bigcup\{[F_\xi]_\lambda : \xi < \lambda^+\}$. Since $[F_{\lambda^+}]_\lambda$ is separable, there exists $D \subseteq [F_{\lambda^+}]_\lambda$ which is a countable dense set in $[F_{\lambda^+}]_\lambda$. As λ^+ is regular, there exists $\xi < \lambda^+$ such that $D \subseteq [F_\xi]_\lambda$. Finally, we have that $F'_\xi \subseteq F_{\xi+1} \subseteq F_{\lambda^+} = [F_\xi]_\lambda$. Thus, $[F_\xi]_\lambda = X$ and therefore $[F_{\lambda^+}]_\lambda = X$. The proof is complete. □

Next we derive several interesting consequences of the previous result, but before that, we recall that a space X is said to be *monolithic* if for every infinite subset A of X, the network weight of the closure of A does not exceed the cardinality of A. Arhangel'skii has shown that the weight of every monolithic compact space X with $c(X) = \omega$ does not exceed 2^ω (see [2]). It is well known that for any compact space $X, w(X) \leq |X|$. Hence, it is natural to ask the following question:

Problem 4.6. *Suppose that X is a monolithic compact space with $c(X) = \omega$. Is the cardinality of X not grater than 2^ω?*

From Theorem 4.5, we obtain a positive partial consistent answer for Problem 4.6 and therefore we also respond partially and positively to Problem 15 from [2].

Corollary 4.7. (MA+$\neg$CH) *If X is a monolithic compact with $c(X) = \omega$, $\kappa = t(X)$ and $\kappa^+ < 2^\omega$, then $|X| \leq 2^\omega$.*

Proof. The set Y of points at which the π–character of X is countable is dense in X. This is so by a theorem of Shapirovskii [20]. Thus, by Theorem 4.5, we conclude that $d(X) = \omega$. Finally, since X is monolithic, $nw(X) = \omega$, and therefore $|X| \leq 2^\omega$. □

Corollary 4.8. [19] (MA+$\neg$ CH) *Let X be a T_2 compact space. If $s(X) = \omega$, then X is hereditary separable.*

Proof. Let Y be a subspace of X. To see that Y is separable, consider $cl_X(Y) = Z$. Notice that Z is a compact subspace of X with $s(Z) = \omega$. Thus $t(Z) = \omega$ and $c(Z) = \omega$. Hence, $\pi\chi(Z) \leq h\pi\chi(Z) = t(Z)$

(because Z is compact). Finally, since $t(Z) = \omega$, we have that every ω–closed subspace of Z is a compact subspace of Z and therefore π–pseudocomplete. Hence, we apply Theorem 4.5 to conclude that Z is separable.

Let D be a countable dense subset of Z. Since $t(Z) = \omega$ for each $x \in Z$ there exits $Y_x \in [Y]^{\leq\omega}$ such that $x \in cl_Z(Y_x)$. It is easy to see that

$$Y' = \bigcup\{Y_x : x \in D\}$$

is a countable subset of Y which is dense in Y. Therefore Y is separable. □

Corollary 4.9. [10] (MA+¬ CH) *Let X be a regular basis compact space with $c(X) = \omega$ such that*

(a) *every closed subspace of X is basis compact;*

(b) *$\chi(X) = \alpha$, and $\alpha^+ < 2^\omega$.*

Then X is separable.

Proof. Let $\lambda = t(X)$. It is clear that $\lambda^+ < 2^\omega$ and that every λ–closed subset of X is closed and therefore every λ–closed subspace of X is π–pseudocomplete. Finally, it follows from (b) that $\pi\chi(X) < 2^\omega$. By Theorem 4.5, X is separable. □

Corollary 4.10. [21] (MA+¬ CH) *Let X be absolute G_κ, $\kappa < 2^\omega$ (or regular with closed subset basis compact), and $c(X) = \omega$. If $(\chi(X))^+ < 2^\omega$, then X is separable.*

Now we present a result which generalizes a theorem given by Juhász in [12]. The proof is similar to the one given in Theorem 4.5, therefore we omit some steps.

Theorem 4.11. (MA+¬CH) *Let X be a topological space such that:*

(a) *every closed subspace of X is π–pseudocomplete;*

(b) *$c(X) = \omega$, $t(X)^+ < 2^\omega$;*

(c) *X admits a dense subset Y such that* $\sup\{\pi\chi(y, X) : y \in Y\} < 2^\omega$.

Then, X is separable.

Proof. Let $\lambda = t(X)^+$. For every $y \in Y$ let $\mathcal{B}_y$ be a local π–base of y in X with $|\mathcal{B}_y| < 2^\omega$.

Then for each open subset G of X we choose $y_{(G)} \in Y$ and we let $y_{(\emptyset)}$ be any element of Y. Let $A \subseteq Y$ and A' be defined as in the proof of Theorem 4.5.

Let $F_\lambda = \bigcup\{F_\xi : \xi < \lambda\}$, where the family $\{F_\alpha : \alpha < \lambda\}$ is constructed as in the proof of Theorem 4.5.

The folllowing claims about F_λ could be easily verified:

(1) $c(F_\lambda) = \omega$.
(2) $\pi w(\mathrm{cl}_X(F_\lambda)) < 2^\omega$. (Note that $\mathcal{V} = \{U \cap \mathrm{cl}_X(F_\lambda) : U \in \bigcup\{\mathcal{B}_y : y \in F_\lambda\}\}$ is a π–base of $\mathrm{cl}_X(F_\lambda)$.)
(3) $\mathrm{cl}_X(F_\lambda)$ is separable and $\mathrm{cl}_X(F_\lambda) = X$.

Therefore X is separable. □

From our last theorem we obtain as a corollary the following result of Juhász [12].

Corollary 4.12. ([12]) (MA+¬CH) *Suppose X is such that*

(a) *every closed subspace of X is π–complete;*
(b) *$c(X) = \omega$, $t(X)^+ < 2^\omega$, and*
(c) *$\pi\chi(p, X) < 2^\omega$ for every $p \in X$.*

Then X is separable.

In [1], Arhangel'skii asks the following (see also [17]):

Problem 4.13. *Does* MA+¬CH *imply that every compact space X with countable cellularity and countable tightness has cardinality $\leq 2^\omega$?*

To conclude this work, in Corollary 4.17 we will show the following: (MA+¬CH) if X is a T_1 quasiregular compact space with $c(X) = \omega$, $t(X) = \omega$, and $|\mathrm{cl}_X(A)| \leq 2^\omega$ for every $A \in [X]^{\leq\omega}$, then $|X| \leq HW(X)2^\omega$. But before that, we will recall some definitions and establish a couple of auxiliary results.

We start with some notions recently defined by Bonanzinga, Stavrova and Staynova in [3].

If X is a T_1 topological space, then for each $x \in X$ we put

$$Hw(x) = \bigcap\{\overline{U} \mid x \in U \text{ and } U \text{ is open in } X\}.$$

The *Hausdorff width* is $HW(X) = \sup\{|Hw(x)| \mid x \in X\}$. Moreover, for every $x \in X$, $\psi w(x) = \min\{|\mathcal{U}_x| : \bigcap\{\overline{U} \mid U \in \mathcal{U}_x\} = Hw(x)$, where $\mathcal{U}_x$ is a family of open neighbourhoods of x. Finally, $\psi w(X) = \sup\{\psi w(x) \mid x \in X\}$.

It is not difficult to show that for any T_1–space X we have $\psi w(X) \leq \chi(X)$.

The following cardinal function was introduced in [15].

Definition 4.14. For a given topological space X and an infinite cardinal κ, the *almost Lindelöf degree* of X with respect to κ-closed sets, denoted by $aL_\kappa(X)$, is defined as follows

$$aL_\kappa(X) = \sup\{L(A) : A \text{ is } \kappa-\text{closed}\},$$

where $L(A)$ is the Lindelöf degree of A.

The proof of the next theorem uses elementary submodels. We use the notations and definitions about elementry submodels given in [6].

Theorem 4.15. *Let X be a T_1–space, and let κ be an infinite cardinal such that*

(1) $aL_\kappa(X) \leq \kappa$;
(2) $\psi w(X) \leq 2^\kappa$, and
(3) for each $A \in [X]^{\leq\kappa}$, we have $|\overline{A}| \leq 2^\kappa$.

Then $|X| \leq HW(X)2^\kappa$.

Proof. For every $x \in X$, let $\mathcal{U}_x$ be a family of open neigborhoods of x in X such that $|\mathcal{U}_x| \leq 2^\kappa$ and $\bigcap\{\overline{U} \mid U \in \mathcal{U}_x\} = Hw(x)$ (it is posible because $\psi w(X) \leq 2^\kappa$). Let $A = 2^\kappa \cup \{2^\kappa, \tau, X\}$, and let $\mathcal{M}$ be an elementary submodel such that $A \subseteq \mathcal{M}$ and $|\mathcal{M}| = 2^\kappa$.

It is not difficult to show that $X \cap \mathcal{M}$ is a κ–closed set in X (the condition 3 is used here).

Since $|(X \cap \mathcal{M})^*| \leq HW(X)2^\kappa$, where $(X \cap \mathcal{M}) = \bigcup\{Hw(x) : x \in X \cap \mathcal{M}\}$, the proof will be complete if we show that $X \subseteq (X \cap \mathcal{M})^*$. Suppose not, fix a $p \in X \setminus (X \cap \mathcal{M})^*$. Then for every $y \in X \cap \mathcal{M}$ there is $U_y \in \mathcal{U}_y$ such that $p \notin \overline{U}_y$.

Clearly, the collection $\{U_x : x \in X\}$ covers $X \cap \mathcal{M}$. Hence, there exists $E \in [X \cap \mathcal{M}]^{\leq\kappa}$ such that $X \cap \mathcal{M} \subseteq \bigcup\{U_x : x \in E\} = B$.

By the properties of $\mathcal{M}$, we have that $B \in \mathcal{M}$, and by (5), we conclude that $X = B$, a contradiction with $p \notin B$. Therefore $X \subseteq (X \cap \mathcal{M})^*$ and thus $|X| \leq HW(X)2^\kappa$. □

Proposition 4.16. (MA+¬CH) *Let X be a T_1 space and let λ be an infinite cardinal such that $\lambda^+ < 2^\omega$. Suppose that*

(1) Every λ–closed subspace of X is π–pseudocomplete;
(2) $c(X) = \omega$;
(3) $aL_\lambda(X) \leq \omega$;
(4) for each $A \in [X]^{\leq\omega}$, we have $|\overline{A}| \leq 2^\omega$.
(5) the space X admits a subset $Y \subseteq X$ such that $\overline{Y} = X$ and $\pi\chi(y, X) < 2^\omega$ for every $y \in Y$.

Then $|X| \leq HW(X)2^\omega$.

Proof. By Theorem 4.5, we have that X is separable. To conclude the proof, by Theorem 4.15, it is enough to show that $\psi w(X) \leq 2^\omega$. To see it, let D be a countable dense subset of X. Let $x \in X$ and let $\mathcal{V}_x$ be the collection of all open neighbourhoods of x. Let $\mathcal{D}$ be the collection of all $A \subseteq D$ such that there exists $V \in \mathcal{V}_x$ such that $V \cap D = A$. Now, for each $A \in \mathcal{D}$ we fix $V_A \in \mathcal{V}_x$, such that $V_A \cap D = A$.

It is not difficult to show that $\bigcap\{\overline{V}_A : A \in \mathcal{D}\} = \bigcap\{\overline{V} : V \in \mathcal{V}_x\} = Hw(x)$ and $|\{V_A : A \in \mathcal{D}\}| \leq 2^\omega$. Hence $\psi w(X) \leq 2^\omega$ and therefore, $|X| \leq HW(X)2^\omega$. □

Now we are ready to establish the final result of this work.

Corollary 4.17. (MA+¬CH) *Let X be a T_1 quasiregular compact space with $c(X) = \omega$, $t(X) = \omega$. If for every $A \in [X]^{\leq\omega}$, $|\overline{A}| \leq 2^\omega$, we have $|X| \leq HW(X)2^\omega$.*

Remark 4.18. The Hausdorff with $HW(X)$ of a space X is related to a cardinal invariant, introduced in [9], called *non-Hausdorff number of X with respect to singletons* and denoted by $nh_s(X)$. Both cardinal functions are related as follows: $nh_s(X) = 1+HW(X)$ and when $HW(X)$ is infinite, both cardinal functions are equal (see [9]). Hence, evidently the results given above holds when $HW(X)$ is replaced with $nh_s(X)$. In [8], Gotchev introduced the cardinal function *non-Hausdorff number*, denoted by $nh(X)$. It is not difficult to show that $nh(X) \leq nh_s(X)$, for any topological space X. The referee asks if the previous results (Theorem 4.15, Proposition 4.16 and Corollary 4.17) are satisfied when $HW(X)$ is replaced by $nh(X)$. At the moment the authors do not know the answer to this question.

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