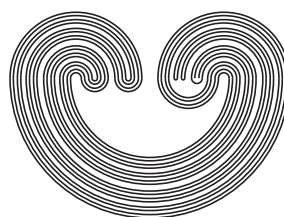


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COMPLETE TOPOLOGIZED POSETS AND SEMILATTICES

by

TARAS BANAKH AND SERHII BARDYLA

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Web: <http://topology.auburn.edu/tp/>

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

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COMPLETE TOPOLOGIZED POSETS AND SEMILATTICES

TARAS BANAKH AND SERHII BARDYLA

ABSTRACT. In this paper we discuss the notion of completeness of a topologized poset and survey some recent results on closedness properties of complete topologized semilattices.

1. INTRODUCTION

In this paper we discuss a notion of completeness for topologized posets and semilattices.

By a *poset* we understand a set X endowed with a partial order $\leq$. A *topologized poset* is a poset endowed with a topology.

A topologized poset X is defined to be *complete* if each nonempty chain C in X has $\inf C \in \bar{C}$ and $\sup C \in \bar{C}$, where $\bar{C}$ stands for the closure of C in X . More details on this definition can be found in Section 2, where we prove that complete topologized posets can be equivalently defined using directed sets instead of chains. In Section 3 we obtain some results on the preservation of completeness by operations over topologized posets.

In Section 4 we study the interplay between complete and chain-compact topologized posets and in Section 5 we study complete topologized semilattices. In Section 6 we survey some known results on the absolute closedness of complete semitopological semilattices and in Section 7 we survey recent results on the closedness of the partial order in (complete) semitopological semilattices.

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2. THE COMPLETENESS OF TOPOLOGIZED POSETS

In this section we define the notion of a complete topologized poset, which is a topological counterpart of the standard notion of a complete poset, see [17]. First we recall some concepts and notations from the theory of partially ordered sets.

A subset C of a poset $(X, \leq)$ is called a *chain* if any two points $x, y \in C$ are comparable in the partial order of X . This can be written as $y \in \uparrow x$ where

$$\uparrow x := \{y \in C : x \leq y\}, \quad \downarrow x := \{y \in C : y \leq x\}, \quad \text{and} \quad \updownarrow x := (\uparrow x) \cup (\downarrow x).$$

A poset X is called *linear* if X is a chain in X .

A subset D of a poset $(X, \leq)$ is called *up-directed* (resp. *down-directed*) if for any elements $x, y \in D$ there exists an element $z \in D$ such that $x \leq z$ and $y \leq z$ (resp. $z \leq x$ and $z \leq y$). It is clear that each chain is both up-directed and down-directed.

A poset X is defined to be

- *up-complete* if any nonempty up-directed set $D \subseteq X$ has $\sup D$ in X ;
- *down-complete* if any nonempty down-directed set $D \subseteq X$ has $\inf D$ in X ;
- *complete* if X is up-complete and down-complete;
- *a complete lattice* if any nonempty subset $A \subseteq X$ has $\sup A$ and $\inf A$ in X .

In the following definition we introduce topological counterparts of these notions.

Definition 2.1. Let κ be a cardinal. A topologized poset X is defined to be

- *up-complete* (resp. $\uparrow\kappa$ -*complete*) if each nonempty up-directed set $D \subseteq X$ (of cardinality $|D| \leq \kappa$) has $\sup D \in \bar{D}$ in X ;
- *down-complete* (resp. $\downarrow\kappa$ -*complete*) if each nonempty down-directed set $D \subseteq X$ (of cardinality $|D| \leq \kappa$) has $\inf D \in \bar{D}$ in X ;
- *complete* (resp. $\updownarrow\kappa$ -*complete*) if X is up-complete and down-complete (resp. $\uparrow\kappa$ -complete and $\downarrow\kappa$ -complete).

Observe that a poset X is up-complete (resp. down-complete) if it is up-complete (resp. down-complete) as a topologized poset endowed with the anti-discrete topology $\{\emptyset, X\}$. On the other hand, a poset X endowed with the discrete topology is complete if and only if X is *chain-finite* in the sense that each chain in X is finite.

Now we show that up-complete and down-complete topologized posets can be equivalently defined using chains instead of up-directed and down-directed subsets. The following lemma is a topologized version of Iwamura's Lemma [15] (cf. [12], [20]).

Lemma 2.2. *Let κ be an infinite cardinal. A topologized poset X is $\uparrow\kappa$ -complete if and only if any nonempty chain $C \subseteq X$ of cardinality $|C| \leq \kappa$ has $\sup C \in \bar{C}$.*

Proof. The “only if” part is trivial as each chain is an up-directed subset in X . The “if” part will be proved by transfinite induction.

First we prove the lemma for the cardinal ω . Assume that each countable chain $C \subseteq X$ has $\sup C \in \bar{C}$. To prove that X is $\uparrow\omega$ -complete, take any nonempty countable up-directed subset $D = \{x_n\}_{n \in \omega}$ in X . Put $y_0 := x_0$ and for every $n \in \mathbb{N}$ choose an element $y_n \in D$ such that $y_n \geq x_n$ and $y_n \geq y_{n-1}$ (such an element y_n exists as D is up-directed).

By our assumption, the chain $C := \{y_n\}_{n \in \omega}$ has $\sup C \in \bar{C} \subseteq \bar{D}$. We claim that $\sup C$ is the least upper bound for the set D . Indeed, for any $n \in \omega$ we get $x_n \leq y_n \leq \sup C$ and hence $\sup C$ is an upper bound for the set D . On the other hand, each upper bound b for D is an upper bound for C and hence $\sup C \leq b$. Therefore $\sup D = \sup C \in \bar{C} \subseteq \bar{D}$.

Now assume that for some uncountable cardinal κ we have proved that each up-directed set D of cardinality $|D| < \kappa$ in X has $\sup D \in \bar{D}$ if each nonempty chain $C \subseteq X$ of cardinality $|C| < \kappa$ has $\sup C \in \bar{C}$. Assume that each chain $C \subseteq X$ of cardinality $|C| \leq \kappa$ has $\sup C \in \bar{C}$. To prove that the topologized poset X is a $\uparrow\kappa$ -complete, fix any up-directed subset $D \subseteq X$ of cardinality $|D| \leq \kappa$.

Since D is up-directed, there exists a function $f : D \times D \rightarrow D$ assigning to any pair $(x, y) \in D \times D$ a point $f(x, y) \in D$ such that $x \leq f(x, y)$ and $y \leq f(x, y)$. Given any subset $A \subseteq D$, consider the increasing sequence $(A_n)_{n \in \omega}$ of subsets of A , defined by the recursive formula $A_0 := A$ and $A_{n+1} := A_n \cup f(A_n \times A_n)$ for $n \in \omega$. Finally, let $\langle A \rangle := \bigcup_{n \in \omega} A_n$ and observe that $\langle A \rangle$ is an up-directed subset of D . By induction it can be proved that $|A_n| \leq \max\{\omega, |A|\}$ for every $n \in \omega$, and hence $|\langle A \rangle| \leq \max\{\omega, |A|\}$. Moreover, it can be also shown that for any subsets $A \subseteq B$ of D we have $\langle A \rangle \subseteq \langle B \rangle$.

Write the set D as $D = \{x_\alpha\}_{\alpha \in \kappa}$. For every $\beta \in \kappa$ let $D_\beta := \langle \{x_\alpha\}_{\alpha \leq \beta} \rangle$ and observe that $D_\beta \subseteq D$ is an up-directed set of cardinality $|D_\beta| \leq \max\{\omega, |\beta|\} < \kappa$. By the inductive assumption, the up-directed set D_β has $\sup D_\beta \in \bar{D}_\beta \subseteq \bar{D}$. For any ordinals $\alpha < \beta$ in κ the inclusion $D_\alpha \subseteq D_\beta$ implies that $\sup D_\alpha \leq \sup D_\beta$. By our assumption, the chain $C := \{\sup D_\alpha : \alpha \in \kappa\} \subseteq \bar{D}$ has $\sup C \in \bar{C} \subseteq \bar{D}$. It remains to observe that $\sup C = \sup D$. $\square$

Corollary 2.3. *A topologized poset X is up-complete if and only if any nonempty chain $C \subseteq X$ has $\sup C \in \bar{C}$ in X .*

Since a poset is up-complete if and only if it is up-complete as a topologized poset with the anti-discrete topology, Corollary 2.3 implies the following known characterization of up-complete posets, see [15], [12], [20].

Corollary 2.4. *A poset X is up-complete if and only if any nonempty chain $C \subseteq X$ has $\sup C$ in X .*

We shall say that an up-directed set D in a topologized poset X *up-converges* to a point $x \in X$ if for each neighborhood $U_x \subseteq X$ of x there exists $d \in D$ such that $D \cap \uparrow d \subseteq U_x$.

Lemma 2.5. *Let κ be an infinite cardinal. If a topologized poset X is $\uparrow\kappa$ -complete, then each up-directed set $D \subseteq X$ of cardinality $|D| \leq \kappa$ up-converges to its $\sup D$.*

Proof. To derive a contradiction, assume that some up-directed set $D \subseteq X$ of cardinality $|D| \leq \kappa$ does not converge to $\sup D$. Then there exists an open neighborhood $U \subseteq X$ of $\sup D$ such that for every $d \in D$ the set $D \cap \uparrow d \setminus U$ is not empty. Then the set $E := D \setminus U$ is directed and cofinal in D . To see that E is directed, take any points $x, y \in E \subseteq D$ and find a point $z \in D$ with $x \leq z$ and $y \leq z$. Since E is cofinal in D , there exists $e \in E$ such that $z \leq e$. Then $x \leq e$ and $y \leq e$ by the transitivity of the partial order. By the up-completeness of X , the directed set E has $\sup E \in \bar{E}$. The inclusion $E \subseteq D$ implies that $\sup E \leq \sup D$ and the cofinality of E in D that $\sup E = \sup D$. Taking into account that $E \cap U = \emptyset$, we obtain a contradiction: $\sup D = \sup E \in \bar{E} \cap U = \emptyset$, completing the proof. $\square$

We shall say that a down-directed set D in a topologized poset X *down-converges* to a point $x \in X$ if for each neighborhood $U_x \subseteq X$ of x there exists $d \in D$ such that $D \cap \downarrow d \subseteq U_x$.

Applying Lemmas 2.2, 2.5 to the opposite partial order on a topologized poset, we obtain the following dual versions of these lemmas.

Lemma 2.6. *Let κ be an infinite cardinal. A topologized poset X is $\downarrow\kappa$ -complete if and only if any nonempty chain $C \subseteq X$ of cardinality $|C| \leq \kappa$ has $\inf C \in \bar{C}$.*

Lemma 2.7. *Let κ be an infinite cardinal. If a topologized poset X is $\downarrow\kappa$ -complete, then each down-directed set $D \subseteq X$ of cardinality $|D| \leq \kappa$ down-converges to its $\inf D$.*

Lemma 2.6 implies the following two characterizations.

Corollary 2.8. *A topologized poset X is down-complete if and only if any nonempty chain $C \subseteq X$ has $\inf C \in \bar{C}$ in X .*

Corollary 2.9. *A poset X is down-complete if and only if any nonempty chain $C \subseteq X$ has $\inf C$ in X .*

Unifying Corollaries 2.3 and 2.8 we obtain the following useful characterization of completeness of topologized posets.

Theorem 2.10. *A topologized poset X is complete if and only if each nonempty chain $C \subseteq X$ has $\sup C \in \bar{C}$ and $\inf C \in \bar{C}$.*

Since a poset is complete if and only if it is complete as a topologized poset with the anti-discrete topology, Corollary 2.10 implies the following (known) characterization of complete posets.

Theorem 2.11. *A poset X is complete if and only if each nonempty chain $C \subseteq X$ has $\sup C$ and $\inf C$ in X .*

3. PRESERVING THE COMPLETENESS BY TYCHONOFF PRODUCTS OF TOPOLOGIZED POSETS

Now we prove that the completeness of topologized posets is preserved by Tychonoff products. On the Tychonoff product $\prod_{\alpha \in A} X_\alpha$ of topologized posets $(X_\alpha, \leq_\alpha)$ we consider the pointwise partial order $\leq$ defined by $(x_\alpha)_{\alpha \in A} \leq (y_\alpha)_{\alpha \in A}$ iff $x_\alpha \leq_\alpha y_\alpha$ for each $\alpha \in A$.

Theorem 3.1. *Let κ be an infinite cardinal. The Tychonoff product $X := \prod_{\alpha \in A} X_\alpha$ of $\uparrow\kappa$ -complete topologized posets X_α , $\alpha \in A$, is an $\uparrow\kappa$ -complete topologized poset.*

Proof. By Lemma 2.2, the $\uparrow\kappa$ -completeness of X will follow as soon as we prove that each nonempty chain $C \subseteq X$ of cardinality $|C| \leq \kappa$ has $\sup C \in \bar{C}$. For every $\alpha \in A$ let $\text{pr}_\alpha : X \rightarrow X_\alpha$ denote the coordinate projection. Since the projection pr_α is monotone, the image $C_\alpha := \text{pr}_\alpha(C)$ of the chain C is a chain in X_α of cardinality $|C_\alpha| \leq |C| \leq \kappa$. By the $\uparrow\kappa$ -completeness of the topologized poset X_α , the chain C_α has $\sup C_\alpha \in \bar{C}_\alpha$.

Consider the elements $c := (\sup C_\alpha)_{\alpha \in A} \in X$. It is clear that c is an upper bound for C . We claim that $c = \sup C$. Indeed, given any other upper bound $b = (b_\alpha)_{\alpha \in A} \in X$ of C , we have $\sup C_\alpha \leq b_\alpha$ for all $\alpha \in A$ and hence $c \leq b$. So, $c = \sup C$.

It remains to show that $c \in \bar{C}$. Assuming that $c \notin \bar{C}$, we could find an open neighborhood $O_c \subseteq X$ such that $O_c \cap C = \emptyset$. Replacing O_c by a smaller neighborhood, we can assume that O_c is of the basic form $O_c = \prod_{\alpha \in A} U_\alpha$, where the set $F = \{\alpha \in A : U_\alpha \neq X_\alpha\}$ is finite. For every $\alpha \in F$ the set U_α is a neighborhood of $\sup C_\alpha$. Applying Lemma 2.5, find a point $u_\alpha \in C_\alpha$ such that $C_\alpha \cap \uparrow u_\alpha \subseteq U_\alpha$. Since $u_\alpha \in C_\alpha = \text{pr}_\alpha(C)$,

there exists an element $v_\alpha \in C$ such that $\text{pr}_\alpha(v_\alpha) = u_\alpha$. Since C is a chain, the finite set $\{v_\alpha : \alpha \in F\}$ has a largest element v . For this element we have $\text{pr}_\alpha(v) \in C_\alpha \cap \uparrow u_\alpha \subseteq U_\alpha$ for all $\alpha \in F$. Consequently, $v \in C \cap O_c$, which contradicts the choice of the neighborhood O_c . $\square$

Applying Theorem 3.1 to the opposite order on a topologized poset, we get the following dual version of Theorem 3.1.

Theorem 3.2. *Let κ be an infinite cardinal. The Tychonoff product $X := \prod_{\alpha \in A} X_\alpha$ of $\downarrow\kappa$ -complete topologized posets X_α , $\alpha \in A$, is a $\downarrow\kappa$ -complete topologized poset.*

Theorems 3.1 and 3.2 imply:

Theorem 3.3. *Let κ be an infinite cardinal. The Tychonoff product $X := \prod_{\alpha \in A} X_\alpha$ of $\uparrow\kappa$ -complete topologized posets X_α , $\alpha \in A$, is an $\uparrow\kappa$ -complete topologized poset.*

Let us also note the following obvious preservation property of complete topologized posets.

Proposition 3.4. *If a topologized poset X is complete (resp. up-complete, down-complete), then so is each closed topologized subposet in X .*

4. INTERPLAY BETWEEN COMPLETENESS AND CHAIN-COMPACTNESS

In this section we establish the relation between the completeness and chain-compactness of topologized posets.

A topologized poset is defined to be *chain-compact* if each closed chain in X is compact.

Lemma 4.1. *Each complete topologized poset X is chain-compact.*

Proof. Given a nonempty closed chain C in a complete topologized poset X , we shall prove that C is compact. Given any open cover $\mathcal{U}$ of C , we should find a finite subfamily $\mathcal{U}' \subseteq \mathcal{U}$ such that $C \subseteq \bigcup \mathcal{U}'$. By the down-completeness of X , the closed chain C has $\inf C \in \bar{C} = C$, which means that $c := \inf C$ is the smallest element of C . Consider the set $A \subseteq C$ consisting of points $a \in C$ such that the closed interval $[c, a] := \{x \in C : c \leq x \leq a\}$ can be covered by a finite subfamily of the cover $\mathcal{U}$. The set A contains the point c and hence is not empty. By the up-completeness of X , the set A has $\sup A \in \bar{A} \subseteq \bar{C}$.

We claim that the point $b := \sup A$ belongs to A . To derive a contradiction, assume that $b \notin A$. Choose any open set $U_b \in \mathcal{U}$ containing the point $b := \sup A \subseteq \bar{C}$. By Lemma 2.5, the chain A contains a point $a \in A$ such that $A \cap \uparrow a \subseteq U_b$. Taking into account that for any $x \in A$ the interval $[c, x] \subseteq \bar{C}$ is contained in A , we see that $A \cap \uparrow a = [a, b] \setminus \{b\}$.

Then $[a, b] = (A \cap \uparrow a) \cup \{b\} \subseteq U_b$. Now the definition of the set $A \ni a$ yields a finite subfamily $\mathcal{V} \subset \mathcal{U}$ such that $[c, a] \subset \bigcup \mathcal{V}$. Then for the finite subfamily $\mathcal{U}' = \mathcal{V} \cup \{U_b\}$ of $\mathcal{U}$ we have $[c, b] = [c, a] \cup [a, b] \subset \bigcup \mathcal{U}'$, which means that $b \in A$.

To complete the proof, it suffices to show that $C \subseteq \bigcup \mathcal{U}'$. Assuming that the (closed) set $E := C \setminus \bigcup \mathcal{U}'$ is not empty, we can apply the completeness of X and find $\inf E \in \bar{E} = E$. Choose any open set $U_e \in \mathcal{U}$ containing the point $e := \inf E$ and observe that $\mathcal{U}' \cup \{U_e\}$ is a finite subfamily covering the set $[c, e]$, which means that $e \in A$. On the other hand, the (non)inclusion $[c, b] \subseteq \bigcup \mathcal{U}' \not\ni e$ implies that $b < e$, which contradicts the equality $b = \sup A$. $\square$

Lemma 4.1 can be reversed for $\uparrow\downarrow$ -closed topologized posets. We define a topologized poset X to be

- $\uparrow$ -closed if the upper set $\uparrow x$ of any point $x \in X$ is closed in X ;
- $\downarrow$ -closed if the lower set $\downarrow x$ of any point $x \in X$ is closed in X ;
- $\uparrow\downarrow$ -closed if it is $\uparrow$ -closed and $\downarrow$ -closed;
- $\downarrow\uparrow$ -closed if for any point $x \in X$ the set $\downarrow\uparrow x := \downarrow x \cup \uparrow x$ is closed in X ;
- a *pospace* if the partial order $\leq$ is a closed subset of $X \times X$;
- *chain-closed* if the closure of each chain in X is a chain.

For topologized posets we have the following implications:

$$\text{pospace} \Rightarrow \uparrow\downarrow\text{-closed} \Rightarrow \downarrow\uparrow\text{-closed} \Rightarrow \text{chain-closed}.$$

The last implication is not entirely trivial and is proved in the following lemma.

Lemma 4.2. *Each $\downarrow\uparrow$ -closed topologized poset is chain-closed.*

Proof. Given a chain $C \subseteq X$, we should prove that its closure $\bar{C}$ in X is a chain. Assuming that $\bar{C}$ contains two incomparable elements x and y , observe that $V_x := X \setminus \downarrow\uparrow y$ is an open neighborhood of x . Since $x \in \bar{C}$, there exists an element $z \in C \cap V_x$. It follows from $z \notin \downarrow\uparrow y$ that $V_y = X \setminus \downarrow\uparrow z \subseteq X \setminus C$ is an open neighborhood of y , disjoint with C , which is not possible as $y \in \bar{C}$. $\square$

Theorem 4.3. *An $\uparrow\downarrow$ -closed topologized poset X is complete if and only if it is chain-compact.*

Proof. The “only if” part follows from Lemma 4.1. To prove the “if” part, assume that an $\uparrow\downarrow$ -closed topologized poset X is chain-compact. To prove that X is complete, take any nonempty chain C . By Lemma 4.2, the closure $\bar{C}$ of C is a chain in X . By the chain-compactness of X , the closed chain $\bar{C}$ is compact. By the compactness of $\bar{C}$, the centered family $\mathcal{F} = \{\bar{C} \cap \downarrow x : x \in \bar{C}\}$ of closed subsets of $\bar{C}$ has nonempty intersection,

which is a singleton, containing the smallest element s of the compact chain $\bar{C}$. It is clear that s is a lower bound for the set C . On the other hand, for any other lower bound b for the set C , we get $C \subseteq \uparrow b$ and hence $s \in \bar{C} \subseteq \overline{\uparrow b} = \uparrow b$ and finally, $b \leq s$. So, $s = \inf C$.

By analogy we can prove that the compact chain $\bar{C}$ has the greatest element which coincides with $\sup C$. $\square$

Example 4.4. There exists a Hausdorff compact topologized poset which is not complete.

Proof. Pick any distinct points $a, b \notin \omega$ and consider the space $X = \omega \cup \{a, b\}$ endowed with the topology τ consisting of sets $U \subseteq X$ satisfying the following conditions:

- if $a \in U$, then $\{2n : n > k\} \subseteq U$ for some $k \in \omega$;
- if $b \in U$, then $\{2n + 1 : n > k\} \subseteq U$ for some $k \in \omega$.

Endow X with the partial order

$$\leq_X := (\omega \times \{a, b\}) \cup \{(a, a), (b, b)\} \cup \{(n, m) : n \leq m\}$$

that extends the standard well-order on ω . Observe that a and b are incomparable upper bounds of the set ω , which implies that the chain ω has no supremum in X . Consequently, the topologized poset X is not complete. $\square$

Remark 4.5. Example 4.3 shows that the $\uparrow$ -closedness is essential in Theorem 4.3.

5. COMPLETE TOPOLOGIZED SEMILATTICES

In this section we study the notion of completeness in the framework of topologized semilattices.

By a *semilattice* we understand a commutative semigroup X of idempotents (the latter means that $xx = x$ for all $x \in X$). Each semilattice X carries a natural partial order $\leq$ defined by $x \leq y$ iff $xy = x$. So, we can consider a semilattice X as a poset such that each nonempty finite subset $A = \{a_1, \dots, a_n\} \subseteq X$ has $\inf A = a_1 \cdots a_n$.

By a *topologized semilattice* we understand a semilattice endowed with a topology. A topologized semilattice X is called a *(semi)topological semilattice* if the semilattice operation $X \times X \rightarrow X$, $(x, y) \mapsto xy = \inf\{x, y\}$, is (separately) continuous.

A topologized semilattice is *complete* (resp. *up-complete*, *down-complete*) if it is complete (resp. up-complete, down-complete) as a topologized poset endowed with the natural order, induced by the semilattice operation. By Theorem 2.10, a topologized semilattice X is *complete* if and only if each nonempty chain $C \subseteq X$ has $\inf C \in \bar{C}$ and $\sup C \in \bar{C}$.

Observe that a discrete topological semilattice is complete if and only if it is *chain-finite* in the sense that each chain in X is finite.

The completeness of topologized semilattices is preserved by many operations. The following two propositions are partial cases of Proposition 3.4 and Theorem 3.3.

Proposition 5.1. *Let X be a closed subsemilattice of a topologized semilattice Y . If Y is complete (resp. up-complete, down-complete), then so is the topologized semilattice X .*

Proposition 5.2. *If topologized semilattices X_α , $\alpha \in A$, are complete (resp. up-complete, down-complete), then so is their Tychonoff product $\prod_{\alpha \in A} X_\alpha$.*

A topologized poset X is defined to be *weakly $\uparrow$ -closed* if for every $x \in X$ we have $\overline{\{x\}} \subseteq \uparrow x$. It is easy to see that a topologized poset is weakly $\uparrow$ -closed if it is $\uparrow$ -closed or satisfies the separation axiom T_1 .

Lemma 5.3. *Let κ be an infinite cardinal. Let $h : X \rightarrow Y$ be a continuous surjective homomorphism from a topologized semilattice X to a weakly $\uparrow$ -closed topologized semilattice Y . If the topologized semilattice X is down-complete (and $\uparrow\kappa$ -complete), then so is the topologized semilattice Y .*

Proof. Assume that the topologized semilattice X is down-complete (and $\uparrow\kappa$ -complete). By Corollary 2.8 (and Lemma 2.2), the down-completeness (and $\uparrow\kappa$ -completeness) of X will follow as soon as we show that each nonempty chain $C \subseteq X$ (of cardinality $|C| \leq \kappa$) has $\inf C \in \bar{C}$ (and $\sup C \in \bar{C}$).

Observe that for every $c \in C$ the preimage $h^{-1}(c)$ is a subsemilattice in X and hence is a down-directed set in X . By the down-completeness of X , it has $\inf h^{-1}(c) \in \overline{h^{-1}(c)}$. Let $b_c := \inf h^{-1}(c)$. By the continuity of h and the weak $\uparrow$ -closedness of Y ,

$$h(b_c) \in h(\overline{h^{-1}(c)}) \subseteq \overline{h(h^{-1}(c))} = \overline{\{c\}} \subseteq \uparrow c$$

and hence $c \leq h(b_c)$. On the other hand, for any $x \in h^{-1}(c)$, we get $b_c \leq x$ and hence $h(b_c) \leq h(x) = c$ and finally

$$h(b_c) = c.$$

Let us show that $b_c = \inf h^{-1}(\uparrow c)$. Indeed, for any $x \in h^{-1}(\uparrow c)$ we have $h(x \cdot b_c) = h(x) \cdot h(b_c) = h(x) \cdot c = c$ and hence $b_c \leq x \cdot b_c \leq x$. So, b_c is a lower bound for the set $h^{-1}(\uparrow c)$. On the other hand, any lower bound b of the set $h^{-1}(\uparrow c)$ is a lower bound of the set $h^{-1}(c) \subseteq h^{-1}(\uparrow c)$ and hence $b \leq b_c$, which means that $b_c = \inf h^{-1}(\uparrow c)$.

For every elements $x \leq y$ in C the inclusion $h^{-1}(\uparrow y) \subseteq h^{-1}(\uparrow x)$ implies $b_x = \inf h^{-1}(\uparrow x) \leq \inf h^{-1}(\uparrow y) = b_y$. So the set $\bar{D} = \{b_c : c \in C\}$ is a chain (of cardinality $\leq \kappa$). By the down-completeness (and $\uparrow\kappa$ -completeness) of X the chain $\bar{D}$ has $\inf \bar{D} \in \bar{D}$ (and $\sup \bar{D} \in \bar{D}$). The continuity of h implies that $f(\inf D) \in f(\bar{D}) \subseteq \overline{f(D)} = \bar{C}$ (and $f(\sup D) \in f(\bar{D}) \subseteq \overline{f(D)} = \bar{C}$).

It remains to prove that $f(\inf D) = \inf C$ (and $f(\sup D) = \sup C$). The monotonicity of f implies that $f(\inf D)$ is a lower bound (and $f(\sup D)$ is an upper bound) for $f(D) = C$. For any lower bound λ of C , we get $C \subseteq \uparrow\lambda$ and $D \subseteq h^{-1}(C) \subseteq h^{-1}(\uparrow\lambda)$. Then $b_\lambda = \inf h^{-1}(\uparrow\lambda) \leq \inf D$ and $\lambda = h(b_\lambda) \leq h(\inf D)$, which means that $h(\inf D) = \inf C$. (For any upper bound u of C and any $c \in C$ the inequality $c \leq u$ implies $b_c \leq b_u$. So, b_u is an upper bound of the chain $D = \{b_c : c \in C\}$. Then $\sup D \leq b_u$ and $h(\sup D) \leq h(b_u) = u$ by the monotonicity of h , witnessing that $h(\sup D) = \sup C$.) $\square$

Finally, we discuss the relation of the completeness of a topologized semilattice X to the compactness of the the *weak $^\bullet$ topology* $\mathcal{W}_X^\bullet$, which is generated by the subbase consisting of complements to closed subsemilattices in X . A topologized semilattice X is called *$\mathcal{W}^\bullet$ -compact* if its weak $^\bullet$ topology $\mathcal{W}_X^\bullet$ is compact. The weak $^\bullet$ -topology was introduced and studied in [5]. According to Lemmas 5.4, 5.5 of [5], for any topologized semilattice we have the implications:

$$\text{complete} \Rightarrow \mathcal{W}^\bullet\text{-compact} \Rightarrow \text{chain-compact}.$$

These implications combined with Theorem 4.3 yield the following characterization.

Theorem 5.4. *For an $\uparrow\downarrow$ -closed topologized semilattice X the following conditions are equivalent:*

- (1) X is complete;
- (2) X is $\mathcal{W}_X^\bullet$ -compact;
- (3) X is chain-compact.

6. ABSOLUTE CLOSEDNESS OF COMPLETE TOPOLOGIZED SEMILATTICES

Quite often the notion of completeness is connected with the absolute closedness, understood in an appropriate sense, see e.g. [1], [3], [10], [11], [14], [18], [19], [21]. For example, a metric space X is complete if and only if it is closed in each metric space Y , containing X as a metric subspace. A uniform space X is complete if and only if X is closed in each uniform space, containing X as a uniform subspace. A topological group X is complete in its two-sided uniformity if and only if X is closed in

any topological group, containing X as a topological subgroup. A similar phenomenon happens in the category of (semi)topological semilattices.

Historically the first result in this direction belongs to J.W. Stepp [22], [23] who proved that any chain-finite semilattice X is closed in each Hausdorff topological semilattice, containing X as a subsemilattice. This result of Stepp was improved to the following characterizations, which can be found in [2].

Theorem 6.1. *A discrete topological semilattice X is chain-finite if and only if X is closed in any Hausdorff zero-dimensional topological semilattice Y containing X as a subsemilattice.*

Theorem 6.2. *A Hausdorff topological semilattice X is complete if and only if each closed subsemilattice Z of X is closed in any Hausdorff topological semilattice containing Z as a topological subsemilattice.*

The “only if” parts of these characterizations are partial cases of the following theorems on closedness of (chain-finite) complete topologized semilattices under continuous homomorphisms.

Theorem 6.3. *For any homomorphism $h : X \rightarrow Y$ from a chain-finite semilattice X to a Hausdorff semitopological semilattice Y , the image $h(X)$ is closed in Y .*

Theorem 6.4. *For any continuous homomorphism $h : X \rightarrow Y$ from a complete topologized semilattice X to a Hausdorff topological semilattice Y , the image $h(X)$ is closed in Y .*

In fact, Theorems 6.3 and 6.4 are corollaries of more general results related to upper semicontinuous multimorphisms between topologized semilattices.

A *multi-valued map* $\Phi : X \multimap Y$ between sets X, Y is a function assigning to each point $x \in X$ a subset $\Phi(x)$ of Y . For a subset $A \subseteq X$ we put $\Phi(A) := \bigcup_{x \in A} \Phi(x)$. A multi-valued map $\Phi : X \multimap Y$ between semigroups is called a *multimorphism* if $\Phi(x) \cdot \Phi(y) \subseteq \Phi(xy)$ for any $x, y \in X$. Here $\Phi(x) \cdot \Phi(y) := \{ab : a \in \Phi(x), b \in \Phi(y)\}$.

A multi-valued map $\Phi : X \multimap Y$ between topological spaces is called *upper semicontinuous* if for any closed subset $F \subseteq Y$ the preimage $\Phi^{-1}(F) := \{x \in X : \Phi(x) \cap F \neq \emptyset\}$ is closed in X .

A subset F of a topological space X is called *θ -closed* if each point $x \in X \setminus F$ has a closed neighborhood, disjoint with F .

The following two theorems (implying Theorems 6.3 and 6.4) are proved in [2].

Theorem 6.5. *Let X be a chain-finite semilattice, Y a semitopological semilattice and $\Phi : X \multimap Y$ a multimorphism such that $\Phi(x)$ is a closed subset of Y for any $x \in X$. Then the image $\Phi(X)$ is closed in Y .*

Theorem 6.6. *Let X be a complete topologized semilattice, Y a topological semilattice and $\Phi : X \multimap Y$ an upper semi-continuous multimorphism such that $\Phi(x)$ is a θ -closed subset of Y for any $x \in X$. Then the image $\Phi(X)$ is θ -closed in Y .*

Theorems 6.3 and 6.4 motivate the following problem.

Problem 6.7. Let $h : X \rightarrow Y$ be a continuous homomorphism from a complete topologized semilattice X to a Hausdorff semitopological semilattice Y . Is the set $h(X)$ closed in Y ?

Problem 6.7 has affirmative answer for homomorphisms to sequential semitopological semilattices.

We recall that a topological space X is *sequential* if each sequentially closed subset in X is closed. A subset A of a topological space X is called *sequentially closed* if A contains the limit points of all sequences $\{a_n\}_{n \in \omega} \subseteq A$ that converge in X .

A topological space X is *countably tight* if for any subset $A \subseteq X$ and point $a \in \bar{A}$ there exists a countable subset $B \subseteq A$ such that $a \in \bar{B}$. It is well-known [16, 1.7.13(c)] that each subspace of a sequential topological space is countably tight. The following (non-trivial) results were proved in [4].

Theorem 6.8. *For any continuous homomorphism $h : X \rightarrow Y$ from a countably tight complete topologized semilattice X to a Hausdorff semitopological semilattice Y , the image $h(X)$ is sequentially closed in Y .*

Corollary 6.9. *For any continuous homomorphism $h : X \rightarrow Y$ from a complete topologized semilattice X to a sequential Hausdorff semitopological semilattice Y , the image $h(X)$ is closed in Y .*

Also Problem 6.7 has an affirmative answer for semitopological semilattices satisfying the separation axiom $\vec{T}_{2\delta}$ (which is stronger than the Hausdorff axiom T_2 but weaker than axiom $\vec{T}_{3\frac{1}{2}}$ of the functional Hausdorffness).

Let us recall that a topological space X satisfies the separation axiom

- T_1 if for any distinct points $x, y \in X$ there exists an open set $U \subseteq X$ such that $x \in U \subseteq X \setminus \{y\}$;
- T_2 if for any distinct points $x, y \in X$ there exists an open set $U \subseteq X$ such that $x \in U \subseteq \bar{U} \subseteq X \setminus \{y\}$;
- T_3 if X is a T_1 -space and for any open set $V \subseteq X$ and point $x \in V$ there exists an open set $U \subseteq X$ such that $x \in U \subseteq \bar{U} \subseteq V$;
- $T_{3\frac{1}{2}}$ if X is a T_1 -space and for any open set $V \subseteq X$ and point $x \in V$ there exists a continuous function $f : X \rightarrow [0, 1]$ such that $x \in f^{-1}([0, 1)) \subseteq V$;

- $T_{2\delta}$ if X is a T_1 -space and for any open set $V \subseteq X$ and point $x \in V$ there exists a countable family $\mathcal{U}$ of closed neighborhoods of x in X such that $\bigcap \mathcal{U} \subseteq V$;
- $\vec{T}_i$ for $i \in \{1, 2, 2\delta, 3, 3\frac{1}{2}\}$ if X admits a bijective continuous map $X \rightarrow Y$ to a T_i -space Y .

Topological spaces satisfying a separation axiom T_i are called T_i -spaces. The separation axioms $T_{2\delta}$ and $\vec{T}_{2\delta}$ were introduced in [7].

The following diagram describes the implications between the separation axioms T_i and $\vec{T}_i$ for $i \in \{1, 2, 2\delta, 3, 3\frac{1}{2}\}$.

$$\begin{array}{ccccccccc}
 T_{3\frac{1}{2}} & \Longrightarrow & T_3 & \Longrightarrow & T_{2\delta} & \Longrightarrow & T_2 & \Longrightarrow & T_1 \\
 \Downarrow & & \Downarrow & & \Downarrow & & \Updownarrow & & \Updownarrow \\
 \vec{T}_{3\frac{1}{2}} & \Longrightarrow & \vec{T}_3 & \Longrightarrow & \vec{T}_{2\delta} & \Longrightarrow & \vec{T}_2 & \Longrightarrow & T_1
 \end{array}$$

Observe that a topological space X satisfies the separation axiom $\vec{T}_{3\frac{1}{2}}$ if and only if it is *functionally Hausdorff* in the sense that for any distinct points $x, y \in X$ there exists a continuous function $f : X \rightarrow \mathbb{R}$ with $f(x) \neq f(y)$. Therefore, each functionally Hausdorff space is a $\vec{T}_{2\delta}$ -space. An example of a Hausdorff space which is not $\vec{T}_{2\delta}$ was constructed in [7].

The following (non-trivial) result was proved in [7].

Theorem 6.10. *For any continuous homomorphism $h : X \rightarrow Y$ from a complete topologized semilattice to a semitopological semilattice Y satisfying the separation axiom $\vec{T}_{2\delta}$, the image $h(X)$ is closed in Y .*

In [6] it was proved that the answer to Problem 6.7 is affirmative for homomorphisms into ω -Lawson semitopological semilattices.

A topologized semilattice is called *Lawson* (see [13, p.12]) it has a base of the topology consisting of open subsemilattices.

For a Hausdorff topologized semilattice X its *Lawson number* $\bar{\Lambda}(X)$ is defined as the smallest cardinal κ such that for any distinct points x, y in X there exists a family $\mathcal{U}$ of closed neighborhoods of x in X such that $|\mathcal{U}| \leq \kappa$ and $\bigcap \mathcal{U}$ is a closed subsemilattice of X which does not contain y . It is easy to see that $\bar{\Lambda}(X) \leq \bar{\psi}(X)$, where $\bar{\psi}(X)$ is the smallest cardinal κ such that for any point $x \in X$ there exists a family $\mathcal{U}$ of closed neighborhoods of x in X such that $|\mathcal{U}| \leq \kappa$ and $\bigcap \mathcal{U} = \{x\}$.

A topologized semilattice X is called κ -*Lawson* for some cardinal κ if it is Hausdorff and $\bar{\Lambda}(X) \leq \kappa$. The Lawson number of a Hausdorff topologized semilattice was introduced and studied in [6].

By [6], each Hausdorff Lawson semitopological semilattice is 1-Lawson. Moreover, a compact Hausdorff semitopological semilattice X is Lawson if and only if it is 1-Lawson. Each Hausdorff topological semilattice X is ω -Lawson. Each Hausdorff linear topologized semilattice is Lawson and 1-Lawson. The following (non-trivial) example was constructed in [6].

Example 6.11. For any infinite cardinal λ there exists a Hausdorff zero-dimensional semitopological semilattice X such that $|X| = \lambda$ and $\bar{\Lambda}(X) = \bar{\psi}(X) = \text{cf}(\lambda)$.

Our next example is a “semilattice” modification of Example 1 [7] of a Hausdorff topological space that does not satisfy the separation axiom $\vec{T}_{2\delta}$.

Example 6.12. There exists a Lawson Hausdorff topological semilattice that does not satisfy the separation axiom $\vec{T}_{2\delta}$.

Proof. Consider the set $L = \{x_\alpha\}_{\alpha \leq \omega_1} \cup \{z\} \cup \{y_\alpha\}_{\alpha \leq \omega_1}$ of pairwise distinct points endowed with the linear order in which $x_\alpha < x_\beta < z < y_\beta < y_\alpha$ for any ordinals $\alpha < \beta \leq \omega_1$. Let $\ddot{L} := L \setminus \{x_{\omega_1}, y_{\omega_1}\}$. On the set

$$X = (\ddot{L} \times [0, \omega_1)) \cup (\{x_{\omega_1}, y_{\omega_1}\} \times \{\omega_1\})$$

consider the semilattice operation

$$(x, \alpha) \cdot (y, \beta) := \begin{cases} (\min\{x, y\}, \min\{\alpha, \beta\}) & \text{if } \alpha, \beta < \omega_1; \\ (\min\{x, z\}, \alpha), & \text{if } \alpha < \omega_1 = \beta; \\ (\min\{z, y\}, \beta), & \text{if } \beta < \omega_1 = \alpha; \\ (\min\{x, y\}, \omega_1), & \text{if } \alpha = \omega_1 = \beta. \end{cases}$$

Endow X with the topology τ consisting of all sets $U \subseteq X$ satisfying the following three conditions:

- if $(z, \alpha) \in U$ for some $\alpha \in [0, \omega_1)$, then $\{(x_\gamma, \alpha), (y_\gamma, \alpha) : \gamma \in [\beta, \omega_1)\} \subseteq U$ for some $\beta \in [0, \omega_1)$;
- if $(x_{\omega_1}, \omega_1) \in U$, then $\{(x_\beta, \gamma) : \beta, \gamma \in [\alpha, \omega_1)\} \subseteq U$ for some $\alpha \in [0, \omega_1)$;
- if $(y_{\omega_1}, \omega_1) \in U$, then $\{(y_\beta, \gamma) : \beta, \gamma \in [\alpha, \omega_1)\} \subseteq U$ for some $\alpha \in [0, \omega_1)$.

It can be shown that (X, τ) is a Lawson (and hence 1-Lawson) Hausdorff topological semilattice, which does not satisfy the separation axiom $\vec{T}_{2\delta}$. $\square$

The following partial answer to Problem 6.7 was obtained in [6].

Theorem 6.13. For any continuous homomorphism $h : X \rightarrow Y$ from a complete topologized semilattice X to an ω -Lawson semitopological semilattice Y , the image $h(X)$ is closed in Y .

Corollary 6.9 and Theorems 6.10, 6.13 imply the following partial answer to Problem 6.7.

Corollary 6.14. *For a continuous homomorphism $h : X \rightarrow Y$ from a complete topologized semilattice X to a Hausdorff semitopological semilattice Y , the image $h(X)$ is closed in Y if one of the following conditions is satisfied:*

- (1) *Y is a topological semilattice;*
- (2) *the topologized semilattice Y is ω -Lawson;*
- (3) *the topological space Y is sequential;*
- (4) *the topological space Y satisfies the separation axiom $\vec{T}_{2\delta}$;*
- (5) *the topological space Y is functionally Hausdorff.*

Comparing Theorems 6.10 and 6.6 we can ask the following problem.

Problem 6.15. Let $\Phi : X \multimap Y$ be an upper semi-continuous multimorphism from a complete topologized semilattice X to a semitopological semilattice Y such that for any $x \in X$, $\Phi(x)$ is θ -closed in Y and Y is ω -Lawson or satisfies the separation axiom $\vec{T}_{2\delta}$. Is the image $\Phi(X)$ closed in Y ?

Problem 6.16. Let X be a complete subsemilattice of a ω_1 -Lawson semitopological semilattice Y . Is X closed in Y ?

7. THE CLOSEDNESS OF THE PARTIAL ORDER IN HAUSDORFF SEMITOPOLOGICAL SEMILATTICES

Observing that the partial order $\leq_X := \{(x, y) \in X \times X : xy = x\}$ of a Hausdorff topological semilattice X is a closed subset of $X \times X$ we can ask the following problem, considered also in [4], [6] and [7].

Problem 7.1. Let X be a complete Hausdorff semitopological semilattice. Is the partial order $\leq_X$ closed in $X \times X$?

In this section we survey some results giving partial answers to Problem 7.1. The following theorem is proved in [4].

Theorem 7.2. *Let X be a countably tight Hausdorff semitopological semilattice. If X is $\downarrow\omega$ -complete and $\uparrow\omega_1$ -complete, then the partial order $\leq_X$ of X is sequentially closed in $X \times X$.*

This theorem implies the following partial answer to Problem 7.1.

Corollary 7.3. *Let X be a Hausdorff semitopological semilattice whose square $X \times X$ is sequential. If X is $\downarrow\omega$ -complete and $\uparrow\omega_1$ -complete, then the partial order $\leq_X$ is closed in $X \times X$.*

Another partial answer to Problem 7.1 was given in [7].

Theorem 7.4. *Let Y be a semitopological semilattice satisfying the separation axiom $\vec{T}_{2\delta}$. For any complete subsemilattice $X \subseteq Y$ the partial order $\leq_X$ of X is closed in $Y \times Y$.*

A similar result holds for complete subsemilattices of ω -Lawson semitopological semilattices, see [6].

Theorem 7.5. *For any complete subsemilattice X of a ω -Lawson semitopological semilattice Y , the partial order $\leq_X$ of X is closed in $Y \times Y$.*

A topologized semilattice X is called

- $\uparrow$ -finite if for any $x \in X$ the upper set $\uparrow x$ is finite;
- $\downarrow$ -finite if for any $x \in X$ the lower set $\downarrow x$ is finite;
- well-founded if every nonempty subset $A \subseteq X$ contains a minimal element $a \in A$ (which means that $x \not\leq a$ for any $x \in A \setminus \{a\}$);
- a U -semilattice if for any open set $U \subseteq X$ and point $x \in U$ there exists a point $u \in U$ whose upper set $\uparrow u$ contains x in its interior;
- a V -semilattice if for any point $x \in X$ and $y \in X \setminus \downarrow x$ there exists a point $z \in X \setminus \downarrow x$ whose upper set $\uparrow z$ contains the point y in its interior.

Proposition 7.6. (1) *Each $\downarrow$ -finite semilattice is well-founded.*
 (2) *Each well-founded semitopological semilattice is a U -semilattice.*
 (3) *Each V -semilattice is $\downarrow$ -closed;*
 (4) *A topologized U -semilattice is a V -semilattice if and only if it is $\downarrow$ -closed.*
 (5) *Each semitopological V -semilattice is 1-Lawson.*

Proof. 1. Assume that the semilattice X is $\downarrow$ -finite. Given any nonempty subset $A \subseteq X$, fix any point $b \in A$ and consider the finite set $\downarrow b$ and its nonempty subset $A \cap \downarrow b$. Being finite, this set contains a minimal element $a \in A \cap \downarrow b$, which remains minimal in the set A , witnessing that A is well-founded.

2. Assume that X is a well-founded semitopological semilattice. To show that X is a U -semilattice, fix any open set $U \subseteq X$ and point $x \in U$. Since X is well-founded, the nonempty set $U \cap \downarrow x \ni x$ contains a minimal element $u \in U \cap \downarrow x$. Since $ux = u \in U$ and X is a semitopological semilattice, the point x has a neighborhood $O_x \subseteq X$ such that $uO_x \subseteq U$. Observe that for any $z \in O_x$ we have $uz \in U \cap \downarrow u \subseteq U \cap \downarrow x$. The minimality of the element u in the set $U \cap \downarrow x$ ensures that $u \leq uz \leq z$ and hence $z \in \uparrow u$ and $O_x \subseteq \uparrow u$. Consequently, $\uparrow u$ contains x in its interior, witnessing that X is a U -semilattice.

3. Assume that X is a V -semilattice. To show that X is $\downarrow$ -closed, we need to check that for any $x \in X$ the lower set $\downarrow x$ is closed in X . Since X is a V -semilattice, for any $y \in X \setminus \downarrow x$ there exists an element $z \in X \setminus \downarrow x$ whose upper set $\uparrow z$ contains some neighborhood O_y of y in X . It follows that $y \in O_y \subset \uparrow z \subseteq X \setminus \downarrow x$, witnessing that the set $\downarrow x$ is closed in X .

4. Assume that X is a $\downarrow$ -closed U -semilattice. Then for any $x \in X$ and $y \in X \setminus \downarrow x$, the set $W := X \setminus \downarrow x$ is an open neighborhood of y . Since X is a U -semilattice, the set W contains an element $z \in X \setminus \downarrow x$ whose upper set $\uparrow z$ contains the point y in its interior, witnessing that X is a V -semilattice.

5. Assume that X is a semitopological V -semilattice. To prove that X is 1-Lawson, fix any distinct elements $x, y \in X$. If $x \notin \downarrow y$, then we can find an element $z \in X \setminus \downarrow y$ whose upper set $\uparrow z$ contains x in its interior. Then the closed subsemilattice $\uparrow z$ is a neighborhood of x that does not contain y . If $x \in \downarrow y$, then $y \notin \downarrow x$ and we can find a point $v \in X \setminus \downarrow x$ whose upper set $\uparrow v$ contains y in its interior. Since X is a semitopological semilattice, the closure $\overline{X \setminus \uparrow v}$ of the subsemilattice $X \setminus \uparrow v$ is a closed subsemilattice in X , which is a neighborhood of x that does not contain y . In both cases we have found a closed subsemilattice of X which is a neighborhood of x that does not contain y . Therefore, the semitopological semilattice X is 1-Lawson. $\square$

Proposition 7.7. *For any $\uparrow$ -closed topologized V -semilattice X , the partial order $\leq_X$ of X is closed in $X \times X$.*

Proof. Given any pair $(x, y) \notin \leq_X$, we conclude that $x \not\leq y$ and hence $x \notin \downarrow y$. Since X is a V -semilattice, there exists a point $z \in X \setminus \downarrow y$ such that $\uparrow z$ contains x in its interior $(\uparrow z)^\circ$. Then $U_x := (\uparrow z)^\circ$ and $U_y = X \setminus \uparrow z$ are two open sets in X such that $U_x \times U_y$ is disjoint with P . So $\leq_X$ is closed in $X \times X$. $\square$

Propositions 7.7 and 7.6 imply

Corollary 7.8. *For a semitopological U -semilattice X the following conditions are equivalent:*

- (1) *the partial order $\leq_X$ of X is closed in $X \times X$;*
- (2) *X is Hausdorff;*
- (3) *X is $\downarrow\uparrow$ -closed.*

Proof. The implication (1) $\Rightarrow$ (2) is well-known and its proof can be found in [17, VI-1.4].

(2) $\Rightarrow$ (3) If X is Hausdorff, then for any $x \in X$ the sets $\downarrow x = \{y \in X : xy = y\}$ and $\uparrow x = \{y \in X : xy = x\}$ are closed by the continuity of the shift $s_x : X \rightarrow X$, $s_x : y \mapsto xy$.

(3) $\Rightarrow$ (1) Assume that X is $\Downarrow$ -closed. By Proposition 7.6(4), X is a V -semilattice and by Proposition 7.7, the partial order $\leq_X$ is closed in $X \times X$. $\square$

Corollary 7.8 and Proposition 7.6 imply the following corollaries.

Corollary 7.9. *For any well-founded semitopological semilattice X the following conditions are equivalent:*

- (1) *the partial order $\leq_X$ of X is closed in $X \times X$;*
- (2) *X is Hausdorff;*
- (3) *X is $\Downarrow$ -closed.*

Corollary 7.10. *For any $\downarrow$ -finite semitopological semilattice X the following conditions are equivalent:*

- (1) *the partial order $\leq_X$ of X is closed in $X \times X$;*
- (2) *X is Hausdorff;*
- (3) *X is $\Downarrow$ -closed.*

Corollary 7.3, Theorems 7.4, 7.5 and Proposition 7.7 imply the following partial answer to Problem 7.1.

Corollary 7.11. *For a complete semitopological semilattice X , its partial order $\leq_X$ is closed in $X \times X$ if one of the following conditions is satisfied:*

- (1) *X is a Hausdorff topological semilattice;*
- (2) *the topologized semilattice X is ω -Lawson;*
- (3) *the topological space $X \times X$ is sequential and Hausdorff;*
- (4) *the topological space X satisfies the separation axiom $\bar{T}_{2\delta}$;*
- (5) *the topological space X is functionally Hausdorff.*

The following examples constructed in [8] and [9] show that the completeness of X in Corollary 7.11 is essential, and Corollary 7.10 has no counterparts for $\uparrow$ -finite semitopological semilattices.

Example 7.12 ([8]). There exists an $\uparrow$ -finite metrizable semitopological semilattice X whose partial order $\leq_X$ is a dense non-closed subset of $X \times X$.

Example 7.13 ([9]). There exists a $\uparrow$ -finite metrizable Lawson semitopological semilattice X whose partial order $\leq_X$ is not closed in $X \times X$.

Corollary 7.11(2) motivates the following problem first posed in [6].

Problem 7.14. Is the partial order $\leq_X$ of an ω_1 -Lawson complete semitopological semilattice X closed in $X \times X$?

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IVAN FRANKO NATIONAL UNIVERSITY OF LVIV (UKRAINE) AND JAN KOCHANOWSKI
UNIVERSITY IN KIELCE (POLAND)
Email address: `t.o.banakh@gmail.com`

UNIVERSITY OF VIENNA, INSTITUTE OF MATHEMATICS, KURT GÖDEL RESEARCH
CENTER, VIENNA (AUSTRIA)
Email address: `sbardyla@yahoo.com`