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DESTROYING FRÉCHET-URYSOHN PROPERTY BY FORCING

AKIRA IWASA

Abstract. We study how forcing destroys properties such as Fréchet-Urysohn property, sequentiality and countable tightness. We show that adding a Cohen real destroys Fréchet-Urysohn property of non-strongly Fréchet-Urysohn Hausdorff spaces (Theorem 3.4).

1. Introduction

Let $\mathbf{V}$ be a ground model and let $\mathbb{P}$ be a forcing. Let $\mathbf{V}^{\mathbb{P}}$ denote the forcing extension of $\mathbf{V}$ by $\mathbb{P}$. For a topological space (X, τ) in $\mathbf{V}$, we consider a topological space $(X, \tau^{\mathbb{P}})$ in $\mathbf{V}^{\mathbb{P}}$ such that $\tau^{\mathbb{P}}$ is the topology on X generated by τ in $\mathbf{V}^{\mathbb{P}}$. By definition τ is a base for $\tau^{\mathbb{P}}$. Let φ be a topological property. We say that a forcing $\mathbb{P}$ preserves the property φ, provided whenever a space (X, τ) satisfies φ, $(X, \tau^{\mathbb{P}})$ also satisfies φ. If a forcing $\mathbb{P}$ does not preserve a property φ, then we say that $\mathbb{P}$ destroys the property φ. Grunberg, Junqueira, and Tall in [10] and Fleissner, LaBerge, and Stanley in [6] studied how normality is preserved by adding Cohen reals. In [12], we studied how covering properties are preserved by adding Cohen reals, and in [13], we investigated how forcing preserves countable compactness and pseudocompactness. In this note, we continue this line of research and focus on destroying topological properties by forcing. First, let us observe that forcing preserves first countability.

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Proposition 1.1. *Suppose that (X,τ) is a first countable space. For any forcing $\mathbb{P}$, $(X,\tau^{\mathbb{P}})$ remains first countable.*

Proof. Let $x \in X$ and let $\{U_n : n \in \omega\}$ be a countable base at x. To show that $\{U_n : n \in \omega\}$ remains a countable base at x in $\mathbf{V}^{\mathbb{P}}$, take $W \in \tau^{\mathbb{P}}$ such that $x \in W$. By the definition of $\tau^{\mathbb{P}}$, there is $V \in \tau$ such that $x \in V \subseteq W$. Then $x \in U_n \subseteq V$ for some $n \in \omega$. Thus, we have $x \in U_n \subseteq W$. □

Forcing preserves first countability, but it may not preserve properties weaker than first countability. In this note, we study how forcing destroys properties such as Fréchet-Urysohn property, sequentiality, and countable tightness, which are weaker than first countability. We are interested in the following topological spaces.

Definition 1.2. Let X be a topological space.

X is *countably tight*, provided whenever $A \subseteq X$ and $x \in \overline{A} \setminus A$, there is a set $B \in [A]^{\leq \aleph_0}$ such that $x \in \overline{B}$.

X is *sequential*, provided whenever $A \subseteq X$ is a sequentially closed set, A is a closed subset of X. (A is sequentially closed if every convergent sequence in A has the limit point in A.)

X is *Fréchet-Urysohn*, provided whenever $A \subseteq X$ and $x \in \overline{A} \setminus A$, there is a sequence in A that converges to x.

X is *strongly Fréchet-Urysohn*, provided whenever $\langle A_n : n < \omega \rangle$ is a sequence of subsets of X such that $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ and $x \in \overline{A_n} \setminus A_n$ for each $n \in \omega$, there is a point $a_n \in A_n$ for each $n \in \omega$ such that the sequence $\langle a_n : n < \omega \rangle$ converges to x.

Fact 1.3. Let X be a topological space. In the following, $1 \implies 2 \implies 3 \implies 4 \implies 5$.

(1) X is first countable.
(2) X is strongly Fréchet-Urysohn.
(3) X is Fréchet-Urysohn.
(4) X is sequential.
(5) X is countably tight.

In Section 2, we study destroying strong Fréchet-Urysohn property, and provide an example where a ccc forcing destroys strong Fréchet-Urysohn property but preserves countable tightness (Example 2.8). In Section 3, we show that adding a Cohen real destroys Fréchet-Urysohn property of non-strongly Fréchet-Urysohn Hausdorff spaces (Theorem 3.4). In Section 4, we show that adding a Cohen real destroys sequentiality of non-Fréchet-Urysohn regular spaces in which every point is G_δ (Theorem 4.1), and give an example of a non-Fréchet-Urysohn sequential space which remains sequential in every forcing extension (Example 4.4).

In Section 5, we give an example where a ccc forcing destroys countable tightness (Example 5.1). A diagram which illustrates these results is given at the end of this note.

2. Destroying strong Fréchet-Urysohn property

In [20, paragraph before Problem 82], Watson mentions a strongly Fréchet-Urysohn space which becomes non-strongly Fréchet-Urysohn in some forcing extension. He did not give a proof, so for completeness, let us provide a proof of it. (For terminologies regarding trees, see [14, page 113].)

Example 2.1. ([20, paragraph before Problem 82]) There are a topological space (X, τ) and a forcing $\mathbb{P}$ such that

(1) (X, τ) is strongly Fréchet-Urysohn, and
(2) $(X, \tau^{\mathbb{P}})$ is not countably tight.

Proof. Let ${}^{<\omega_1}2$ be the set of all functions from a countable ordinal to 2. Consider ${}^{<\omega_1}2$ as a tree, where the order is defined by inclusion. Define a topological space

$$X = \{x^*\} \cup {}^{<\omega_1}2,$$

where each point in ${}^{<\omega_1}2$ is isolated and a neighborhood of x^* contains all but finitely many branches of ${}^{<\omega_1}2$. Let $\mathbb{P} = \mathrm{Fn}(\omega_1, 2, \omega_1)$, where $\mathrm{Fn}(\omega_1, 2, \omega_1)$ is the set of all partial functions from ω_1 to 2 whose cardinality is less than $\aleph_1$ ([16, VII Definition 6.1]). We will show that (X, τ) and $\mathbb{P}$ are as required.

To show that (X, τ) is strongly Fréchet-Urysohn, take $A_n \subseteq {}^{<\omega_1}2$ such that $x^* \in \overline{A_n} \setminus A_n$ for each $n \in \omega$. A_n meets infinitely many branches, so we can find an infinite antichain $\{a_{n,i} : i \in \omega\} \subseteq A_n$ for each $n \in \omega$. Then for each $n \in \omega$, we can pick inductively distinct $a_{n,i_n} \in A_n$ so that $\{a_{n,i_n} : n \in \omega\}$ is an antichain. The sequence $\langle a_{n,i_n} : n < \omega \rangle$ converges to x^*. Thus, (X, τ) is strongly Fréchet-Urysohn.

Forcing with $\mathrm{Fn}(\omega_1, 2, \omega_1)$ adds new functions from ω_1 to 2, which are new branches of ${}^{<\omega_1}2$. Let B be such a new branch. Then $x^* \in \overline{B} \setminus B$ because B cannot be a subset of the union of finitely many old branches. To show that B witnesses the fact that $(X, \tau^{\mathbb{P}})$ is not countably tight, take a countable set $C \subseteq B$. Then $C \in \mathbf{V}$ because $\mathrm{Fn}(\omega_1, 2, \omega_1)$ is countably closed so it does not add new countable sets ([16, VII Lemma 6.13, Theorem 6.14]). Since $C \subseteq B$, C is a chain. Therefore, we can conclude that $x^* \notin \overline{C}$. Thus, $(X, \tau^{\mathbb{P}})$ is not countably tight. □

Forcing may not be able to destroy strong Fréchet-Urysohn property of a non-first-countable space: Let $X^* = \{\infty\} \cup X$ be the one-point compactification of an uncountable discrete space X. The point ∞ is not a point of first countability. Every sequence in X converges to ∞ so X^* is strongly Fréchet-Urysohn. The neighborhood filter of ∞, $\{\{\infty\}\cup(X\setminus F) : F \in [X]^{<\aleph_0}\}$, does not change in any forcing extension. So we have

Example 2.2. The one-point compactification of an uncountable discrete space is a non-first-countable space with the strong Fréchet-Urysohn property that remains strongly Fréchet-Urysohn in every forcing extension.

In Example 2.1, the forcing destroys not only strong Fréchet-Urysohn property but also countable tightness. So we ask if forcing can destroy strong Fréchet-Urysohn property while preserving countable tightness. Assuming CH, we answer this question positively in Example 2.8. In order to provide Example 2.8, for the rest of this section, we let X and Y be such that

- (X, μ) is a separable metric space, and
- $(Y, \tau) = (C_p(X), \tau)$, where $C_p(X)$ is the set of all continuous real-valued functions on X, endowed with the topology of pointwise convergence.

We will show that if X is an uncountable γ-set (in 2^ω) then $Y = C_p(X)$ is a strongly Fréchet-Urysohn space that after adding one Hechler real is no longer sequential but is still countably tight. Let us give some definitions.

Definition 2.3. An open cover $\mathcal{U}$ of X is *an ω-cover* if for every finite set $F \subseteq X$, there is $U \in \mathcal{U}$ such that $F \subseteq U$.

An open cover $\mathcal{U} = \{U_n : n \in \omega\}$ of X is *a γ-cover* if $X \subseteq \bigcup_{n\in\omega} \bigcap_{k\geq n} U_k$.

X is *a γ-set* if X is a separable metric space and every ω-cover of X contains a γ-cover of X.

We can show that the results in [7] and [8] by Gerlits and Nagy still hold in forcing extensions.

Proposition 2.4 ([7], [8])**.** *Let (X, μ) be a separable metric space and let $(Y, \tau) = (C_p(X), \tau)$. For any forcing $\mathbb{P}$, the following are equivalent:*

(1) *$(Y, \tau^{\mathbb{P}})$ is strongly Fréchet-Urysohn.*
(2) *$(Y, \tau^{\mathbb{P}})$ is Fréchet-Urysohn.*
(3) *$(Y, \tau^{\mathbb{P}})$ is sequential.*
(4) *$(X, \mu^{\mathbb{P}})$ is a γ-set.*

We prove that adding a Hechler real destroys uncountable γ-sets (Proposition 2.7) by extracting a portion of the proof of Theorem 1 in [17] by Miller. Let us give a definition.

Definition 2.5. ([17, proof of Theorem 1]) For $f \in {}^\omega\omega$, define $\mathcal{U}_f = \{C_F : (\exists n \in \omega)(F \subseteq 2^{f(n)} \text{ and } |F| \leq n)\}$, where $C_F = \{x \in 2^\omega : x\restriction f(n) \in F\}$.

For every $f \in {}^\omega\omega$, $\mathcal{U}_f$ is always an ω-cover of $2^\omega \cap \mathbf{V}$ and if h is a Hechler real over $\mathbf{V}$, then $\mathcal{U}_h$ does not contain a γ-cover of $2^\omega \cap \mathbf{V}$ and this is how the sequentiality of Y will be destroyed. Lemma 5 in [17] and the comment after it say the following:

Lemma 2.6. ([17, Lemma 5]) *Let $\mathbb{P}$ be the Hechler forcing and let $h \in {}^\omega\omega \cap \mathbf{V}^\mathbb{P}$ be a Hechler real. Then for every $\mathcal{V} \in [\mathcal{U}_h]^{\aleph_0}$, $(\bigcap \mathcal{V}) \cap \mathbf{V}$ is countable.*

Note that $\mathcal{U}_h$ does not contain a γ-cover of any uncountable subset of $2^\omega \cap \mathbf{V}$.

Proposition 2.7. ([17, Theorem 1]) *Let (X, μ) be an uncountable γ-set. Let $\mathbb{P}$ be the Hechler forcing. Then $(X, \mu^\mathbb{P})$ is not a γ-set.*

Proof. γ-sets are zero-dimensional so we may assume that $X \subseteq 2^\omega$. Let $\mathbb{P}$ be the Hechler forcing and let $h \in {}^\omega\omega \cap \mathbf{V}^\mathbb{P}$ be a Hechler real. Let $\mathcal{U}_h$ be as in Definition 2.5. We have seen that $\mathcal{U}_h$ is an ω-cover of X, and by Lemma 2.6, $\mathcal{U}_h$ does not contain a γ-cover of X. □

Finally, we provide the example.

Example 2.8. Assume CH. There are a topological space (Y, τ) and a forcing $\mathbb{P}$ such that

(1) (Y, τ) is a strongly Fréchet-Urysohn space, and
(2) $(Y, \tau^\mathbb{P})$ is a non-sequential, countably tight space.

Proof. CH implies that there is an uncountable γ-set (X, μ) ([11]). We will show that the space $(Y, \tau) = (C_p(X), \tau)$ with the Hechler forcing is such an example. By Proposition 2.4 with $\mathbb{P} = \emptyset$, (Y, τ) is strongly Fréchet-Urysohn. Let $\mathbb{P}$ be the Hechler forcing. By Proposition 2.7, $(X, \mu^\mathbb{P})$ is not a γ-set. By Proposition 2.4, $(Y, \tau^\mathbb{P})$ is not sequential. We show that $(Y, \tau^\mathbb{P})$ is hereditarily separable, which implies that it is countably tight. Let A be an arbitrary subspace of $(Y, \tau^\mathbb{P})$. Let $\mathcal{B}$ be a countable base for (X, μ) and let $\mathcal{O}$ be the set of all open intervals of the real line with rational endpoints. For each $\mathcal{E} \in [\mathcal{B} \times \mathcal{O}]^{<\aleph_0}$, say $\mathcal{E} = \{(B_i, O_i) : i < n\}$, pick $f_\mathcal{E} \in A$ such that $f_\mathcal{E}(B_i) \subseteq O_i$ for each $i < n$ if such a function exists in A; otherwise, let $f_\mathcal{E}$ be an arbitrary member of A. Then $\{f_\mathcal{E} : \mathcal{E} \in [\mathcal{B} \times \mathcal{O}]^{<\aleph_0}\}$ is a countable dense subset of A. □

3. Destroying Fréchet-Urysohn property

In this section, we prove that adding a Cohen real destroys Fréchet-Urysohn property of non-strongly Fréchet-Urysohn Hausdorff spaces (Theorem 3.4). The following spaces are useful for our study.

Definition 3.1. For an infinite cardinal κ, define a topological space

$$S(\kappa) = \{x^*\} \cup (\kappa \times \omega),$$

where each point in $\kappa \times \omega$ is isolated and a neighborhood of x^* has the form $\{x^*\} \cup \{(i,j) : i \in \kappa \text{ and } j > f(i)\}$ for some $f \in {}^{\kappa}\omega$. $S(\kappa)$ is called the sequential fan with κ-many spines ([5]). $S(\kappa)$ is a Fréchet-Urysohn, non-strongly Fréchet-Urysohn space.

Define a topological space

$$S_2 = \{x^*\} \cup (\omega \times (\omega + 1)),$$

where $\omega \times (\omega+1)$ has the product topology and a neighborhood of x^* has the form $\{x^*\} \cup \{(i,\omega) : i > n\} \cup \{(i,j) : i > n \text{ and } j > f(i)\}$ for some $n \in \omega$ and $f \in {}^{\omega}\omega$. S_2 is called Arens' space ([1]). S_2 is a sequential, non-Fréchet-Urysohn space.

Here are properties of a Cohen real. $\mathrm{Fn}(\omega,\omega)$ is the set of all finite partial functions from ω to ω ([16, VII Definition 5.1]).

Lemma 3.2. *Let $\mathbb{P} = \mathrm{Fn}(\omega,\omega)$. Let G be a $\mathbb{P}$-generic filter and let $g = \bigcup G$. Then $g \in {}^{\omega}\omega \cap \mathbf{V}^{\mathbb{P}}$ and the following hold:*

1. *$(\forall f \in {}^{\omega}\omega \cap \mathbf{V})(\{n \in \omega : f(n) < g(n)\}$ is infinite);*
2. *$(\forall D \in [\omega]^{\aleph_0} \cap \mathbf{V}^{\mathbb{P}})(\exists f \in {}^{\omega}\omega \cap \mathbf{V})(\{n \in D : g(n) < f(n)\}$ is infinite).*

Proof. We can prove (1) by a standard density argument. To prove (2), assume on the contrary that there is a set $D \in [\omega]^{\aleph_0} \cap \mathbf{V}^{\mathbb{P}}$ such that for every $f \in {}^{\omega}\omega \cap \mathbf{V}$, $f(n) < g(n)$ for all but finitely many $n \in D$; that is, ${}^{\omega}\omega \cap \mathbf{V}$ is bounded on D. By [3, Fact 3.4], ${}^{\omega}\omega \cap \mathbf{V}$ is bounded in ${}^{\omega}\omega \cap \mathbf{V}^{\mathbb{P}}$; that is, there is a function $h \in {}^{\omega}\omega \cap \mathbf{V}^{\mathbb{P}}$ such that for every $f \in {}^{\omega}\omega \cap \mathbf{V}$, $f(n) < h(n)$ for all but finitely many $n \in \omega$. This contradicts [2, Lemma 3.1.2(2)]. □

Here is what a Cohen real does to $S(\omega)$ and S_2.

Lemma 3.3. *Let $g \in {}^{\omega}\omega$ be as in Lemma 3.2. Regard g as a subset of $\omega \times \omega$. In $S(\omega)$ and S_2, we have $x^* \in \overline{g}$, where $\overline{g}$ is the closure of g, and no subsequence of $\langle (n, g(n)) : n < \omega \rangle$ converges to x^*.*

Proof. We prove the case for $S(\omega)$. A proof for the case for S_2 is similar. Since $\{n \in \omega : f(n) < g(n)\}$ is infinite for all $f \in {}^{\omega}\omega \cap \mathbf{V}$, we have $x^* \in \overline{g}$. Let $h \subseteq g$ and regard h as a subsequence of $\langle (n, g(n)) : n < \omega \rangle$. Let D be the domain of h. By Lemma 3.2(2), there is a function $f \in {}^{\omega}\omega \cap \mathbf{V}$ such that $\{n \in D : g(n) < f(n)\}$ is infinite. $\{x^*\} \cup \{(i,j) : i \in \omega \text{ and } j > f(i)\}$ is a neighborhood of x^* which misses infinitely many points of h. Thus, the sequence h does not converge to x^*. □

Using Lemma 3.3, let us prove the main result of this paper: Adding a Cohen real destroys Fréchet-Urysohn property, even sequentiality, of non-strongly Fréchet-Urysohn Hausdorff spaces.

Theorem 3.4. *Suppose that* (X, τ) *is a Fréchet-Urysohn, non-strongly Fréchet-Urysohn Hausdorff space. Let* $\mathbb{P} = \mathrm{Fn}(\omega, \omega)$*. Then* $(X, \tau^{\mathbb{P}})$ *is a non-sequential, countably tight space.*

Proof. According to [19, Proposition 1.11], every Fréchet-Urysohn, non-strongly Fréchet-Urysohn space contains a copy of $S(\omega)$. Let $Y = \{x^*\} \cup (\omega \times \omega)$ be such a subspace of (X, τ). Let $g \in {}^\omega\omega \cap \mathbf{V}^{\mathbb{P}}$ be as in Lemma 3.2 and let $g \subseteq Y$. Then $x^* \in \overline{g}$ by Lemma 3.3, where the closure is taken in $(X, \tau^{\mathbb{P}})$. Assume on the contrary that $(X, \tau^{\mathbb{P}})$ remains sequential. Then there is a sequence $\{a_n : n \in \omega\} \subseteq \overline{g} \setminus \{x^*\}$ that converges to x^*. By Lemma 3.3, no subsequence of $\langle (n, g(n)) : n < \omega \rangle$ converges to x^*. Therefore, we may assume that $g \cap \{a_n : n \in \omega\} = \emptyset$. Using Hausdorffness, for each $n \in \omega$, take an open set U_n such that $a_n \in U_n$ and $x^* \notin \overline{U_n}$. Let $C_i = \{i\} \times \omega$, the i^{th}-column of $\omega \times \omega$. Since $x^* \notin \overline{U_n}$, $U_n \cap C_i$ is finite for all $i \in \omega$. Let $E = \bigcup_{n \in \omega} (U_n \cap \bigcup_{i \geq n} C_i)$.

Claim. $x^* \in \overline{E}$.

Proof. Since $a_n \in \overline{g}$ and U_n is an open set containing a_n, we have $a_n \in \overline{g \cap U_n}$ for each $n \in \omega$. We also have, for each $n \in \omega$,

$$\begin{aligned} (g \cap U_n) \setminus E &\subseteq (g \cap U_n) \setminus (U_n \cap \textstyle\bigcup_{i \geq n} C_i) \\ &= (g \cap U_n) \setminus \textstyle\bigcup_{i \geq n} C_i \\ &\subseteq ((\omega \times \omega) \cap U_n) \setminus \textstyle\bigcup_{i \geq n} C_i \\ &= U_n \cap \textstyle\bigcup_{i < n} C_i, \text{ which is finite.} \end{aligned}$$

Since $a_n \notin g$, we can conclude that $a_n \in \overline{E}$ for each $n \in \omega$. Since $\langle a_n : n < \omega \rangle$ converges to x^*, we have $x^* \in \overline{E}$. ⊣ (Claim)

For each $k \in \omega$, $E \cap C_k = [\bigcup_{n \in \omega} (U_n \cap \bigcup_{i \geq n} C_i)] \cap C_k = \bigcup_{n \leq k} (U_n \cap C_k)$, which is finite. So we can choose a function $f \in {}^\omega\omega$ such that $E \subseteq \{(i, j) : i \in \omega \text{ and } j < f(i)\}$. Then $\{x^*\} \cup \{(i, j) : i \in \omega \text{ and } j > f(i)\}$ is a neighborhood of x^* missing E. This contradicts the claim. Hence, we conclude that $(X, \tau^{\mathbb{P}})$ is not sequential. By [4, Lemma 5.2], adding Cohen reals preserves countable tightness. Therefore, $(X, \tau^{\mathbb{P}})$ remains countably tight. □

4. Destroying sequentiality

In this section, we destroy sequentiality of non-Fréchet-Urysohn spaces. If a space is regular and each point of the space is G_δ, then adding a Cohen real does this.

Theorem 4.1. *Suppose that (X, τ) is a sequential, non-Fréchet-Urysohn regular space in which every point is G_δ. Let $\mathbb{P} = \mathrm{Fn}(\omega, \omega)$. Then $(X, \tau^{\mathbb{P}})$ is a non-sequential, countably tight space.*

Proof. According to [19, Proposition 1.20], every sequential, non-Fréchet-Urysohn regular space where every point is G_δ contains a closed copy of S_2. Let $Y = \{x^*\} \cup (\omega \times (\omega + 1))$ be such a subspace of X. Let $g \in {}^\omega\omega$ be as in Lemma 3.2 and let $g \subseteq Y$. By Lemma 3.3, we have $x^* \in \overline{g}$, where the closure is taken in $(X, \tau^{\mathbb{P}})$. Since Y is a closed subspace of X, we have $\overline{g} \subseteq Y$, and so $\overline{g} = \{x^*\} \cup g$. Again by Lemma 3.3, no subsequence of $\langle (n, g(n)) : n < \omega \rangle$ converges to x^*. Thus, $(X, \tau^{\mathbb{P}})$ is not a sequential space. $(X, \tau^{\mathbb{P}})$ remains countably tight by [4, Lemma 5.2]. $\square$

We give an example of a non-Fréchet-Urysohn space whose sequentialiy is preserved by any forcing (Example 4.4). To do so, let us give some definitions. We say that $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ is *an almost disjoint family of subsets of* ω, provided whenever $A, B \in \mathcal{A}$ with $A \neq B$, $A \cap B$ is finite. If $\mathcal{A}$ is an infinite almost disjoint family of subsets of ω and for every $B \in [\omega]^{\aleph_0}$, there is $A \in \mathcal{A}$ such that $A \cap B$ is infinite, then we say $\mathcal{A}$ is *a maximal almost disjoint family of subsets of* ω.

Definition 4.2. ([9, 5I]) Let $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ be an infinite almost disjoint family of subsets of ω. Define a topological space

$$\Psi_{\mathcal{A}} = \omega \cup \mathcal{A},$$

where each point in ω is isolated and a neighborhood of $A \in \mathcal{A}$ has the form $\{A\} \cup (A \setminus n)$ for some $n \in \omega$. $\Psi_{\mathcal{A}}$ is a non-compact, locally compact space. Let

$$\Psi^*_{\mathcal{A}} = \{\infty\} \cup \Psi_{\mathcal{A}}$$

be the one-point compactification of $\Psi_{\mathcal{A}}$.

Lemma 4.3. *$\Psi^*_{\mathcal{A}}$ is a sequential space.*

Proof. Let $E \subseteq \Psi^*_{\mathcal{A}}$ be a sequentially closed set. We will prove that E is a closed subset of $\Psi^*_{\mathcal{A}}$ by showing $\overline{E} = E$. Take $x \in \overline{E}$. If $x \in \omega$ or $x \in \mathcal{A}$, then it is not difficult to prove that $x \in E$. So assume that $x = \infty$.

First suppose that $A \cap E$ is infinite for infinitely many $A \in \mathcal{A}$. For each $n \in \omega$, take $A_n \in \mathcal{A}$ such that $|A_n \cap E| = \aleph_0$. Since a sequence formed by any enumeration of $A_n \cap E$ converges to A_n, we have $A_n \in E$ for each n. Since $\langle A_n : n < \omega \rangle$ converges to ∞, we have $\infty \in E$.

Next suppose that $A \cap E$ is infinite for only finitely many $A \in \mathcal{A}$, say $\{A \in \mathcal{A} : |A \cap E| = \aleph_0\} = \{A_i : i < n\}$. Let $E' = E \setminus \bigcup_{i<n} A_i$. If E' is finite, then $\overline{E} = E \cup \{A_i : i < n\}$, and so $\infty \in E$. So suppose that E' is infinite.

If $E' \cap \omega$ is infinite, then a sequence formed by any enumeration of $E' \cap \omega$ converges to ∞, and so $\infty \in E$. If $E' \cap \mathcal{A}$ is infinite, then a sequence consisting of distinct points in $E' \cap \mathcal{A}$ converges to ∞, and so $\infty \in E$. $\square$

Now we provide an example where sequentiality of a non-Fréchet-Urysohn space is preserved by any forcing.

Example 4.4. There is a sequential, non-Fréchet-Urysohn, compact Hausdorff space (X, τ) such that for any forcing $\mathbb{P}$, $(X, \tau^{\mathbb{P}})$ remains sequential.

Proof. Let $\mathcal{A}$ be a maximal almost disjoint family of subsets of ω. We will show that $(\Psi^*_{\mathcal{A}}, \tau)$ in Definition 4.2 is such an example. By Lemma 4.3, $(\Psi^*_{\mathcal{A}}, \tau)$ is sequential. We have $\infty \in \overline{\omega}$, but no sequence in ω converges to ∞ because of the maximality of $\mathcal{A}$. Therefore, $(\Psi^*_{\mathcal{A}}, \tau)$ is not Fréchet-Urysohn. Forcing preserves compactness of compact scattered spaces ([15, Lemma 7]). Since $(\Psi^*_{\mathcal{A}}, \tau)$ is a compact scattered space, $(\Psi^*_{\mathcal{A}}, \tau^{\mathbb{P}})$ remains compact for any forcing $\mathbb{P}$, and we can use the proof of Lemma 4.3 to show that $(\Psi^*_{\mathcal{A}}, \tau^{\mathbb{P}})$ is sequential. $\square$

Remark 4.5. The point ∞ in $\Psi^*_{\mathcal{A}}$ is not G_δ in the proof of Example 4.4. Therefore, Theorem 4.1 is not applicable to $\Psi^*_{\mathcal{A}}$ in the proof of Example 4.4.

Remark 4.6. Sequentiality of any Fréchet-Urysohn Hausdorff space that is not strongly Fréchet-Urysohn is destroyed by adding a Cohen real (Theorem 3.4), but sequentiality of a non-Fréchet-Urysohn Hausdorff space may not be destroyed by any forcing (Example 4.4).

5. Destroying countable tightness

In Example 2.1, the forcing $\mathrm{Fn}(\omega_1, 2, \omega_1)$ destroys countable tightness of the space $X = \{x^*\} \cup {}^{<\omega_1}2$. For an uncountable regular cardinal κ, the forcing $\mathrm{Fn}(\kappa, \omega, \kappa)$ destroys countable tightness of $S(\kappa)$ ([18, Example 8]). [1] $\mathrm{Fn}(\omega_1, 2, \omega_1)$ and $\mathrm{Fn}(\kappa, \omega, \kappa)$ are not ccc forcings, and adding Cohen reals preserves countable tightness ([4, Lemma 5.2]). So we ask whether a ccc forcing can destroy countable tightness. We show this is possible.

Example 5.1. There are a topological space (X, τ) and a ccc forcing $\mathbb{P}$ such that

(1) (X, τ) is Fréchet-Urysohn, and
(2) $(X, \tau^{\mathbb{P}})$ is not countably tight.

[1] In [18, Example 8], it is shown that $\mathrm{Fn}(\omega_1, \omega, \omega_1)$ destroys countable tightness of $S(\omega_1)$. It is not difficult to see that for any uncountable regular cardinal κ, $\mathrm{Fn}(\kappa, \omega, \kappa)$ destroys countable tightness of $S(\kappa)$.

Proof. The space $(S(\omega_1),\tau)$ in Definition 3.1 is Fréchet-Urysohn, and we will show that adding $\aleph_1$-many random reals destroys countable tightness of $(S(\omega_1),\tau)$. Let ω^{ω_1} be the product space of ω_1-many copies of ω. For each $n\in\omega$, $\{n\}$ has the measure $2^{-(n+1)}$ and ω^{ω_1} carries the product measure. Let $\mathcal{B}$ be the σ-algebra generated by the basic clopen subsets of ω^{ω_1}. Let $\mathbb{P}=\mathcal{B}/\mathcal{N}$, where $\mathcal{N}$ is the ideal of measure zero sets. Forcing with the Boolean algebra $\mathbb{P}$ adds $\aleph_1$-many random reals. Let $\mathcal{S}$ be the set of all finite partial functions from ω_1 to ω. For each $s\in\mathcal{S}$, let $B_s=\{f\in\omega^{\omega_1}: s\subseteq f\}$. For a $\mathbb{P}$-generic filter G, let $g=\bigcup\{s:[B_s]\in G\}$, where $[B_s]$ is the equivalence class containing B_s. Then we have $g\in\omega^{\omega_1}\cap\mathbf{V}^{\mathbb{P}}$.

Claim: g is unbounded in $\omega^{\omega_1}\cap\mathbf{V}$; that is, for every $f\in\omega^{\omega_1}\cap\mathbf{V}$ and every $\alpha<\omega_1$, there is an ordinal $\gamma>\alpha$ such that $f(\gamma)<g(\gamma)$.

Proof. Fix $f\in\omega^{\omega_1}\cap\mathbf{V}$, $\alpha<\omega_1$ and $p\in\mathbb{P}$. Let $\dot{g}$ be a $\mathbb{P}$-name for g. We will find $q<p$ such that for some $\gamma>\alpha$, $q\Vdash f(\gamma)<\dot{g}(\gamma)$. Pick $A_p\in p$. Since $\mathcal{B}$ is σ-generated by basic clopen subsets of ω^{ω_1}, there are an ordinal $\beta<\omega_1$ and a subset A'_p of ω^β such that $A_p=A'_p\times\omega^{\omega_1\setminus\beta}$. Take an ordinal $\gamma>\max\{\alpha,\beta\}$ and define

$$E=A'_p\times\omega^{\gamma\setminus\beta}\times\{n\in\omega:n>f(\gamma)\}\times\omega^{\omega_1\setminus(\gamma+1)}.$$

Take $q\in\mathbb{P}$ such that $E\in q$. Then $q<p$ and $q\Vdash f(\gamma)<\dot{g}(\gamma)$. $\dashv$ (Claim)

The claim implies that $x^*\in\overline{g}$, where $\overline{g}$ is the closure of $g\subseteq\omega_1\times\omega$ in the space $(S(\omega_1),\tau^{\mathbb{P}})$. To show that g witnesses the fact that $(S(\omega_1),\tau^{\mathbb{P}})$ is not countably tight, take a countable set $C\subseteq g$. Pick $\alpha<\omega_1$ such that $C\subseteq g\restriction\alpha$. Since the random forcing does not add unbounded reals ([14, Lemma 15.30(i)]), there is $f\in\omega^{\omega_1}\cap\mathbf{V}$ such that for all $\xi<\alpha$, $g(\xi)<f(\xi)$. $\{x^*\}\cup\{(i,j):i\in\omega_1$ and $j>f(i)\}$ is a neighborhood of x^* missing C. Thus, we have $x^*\notin\overline{C}$. Hence, $(S(\omega_1),\tau^{\mathbb{P}})$ is not countably tight. $\square$

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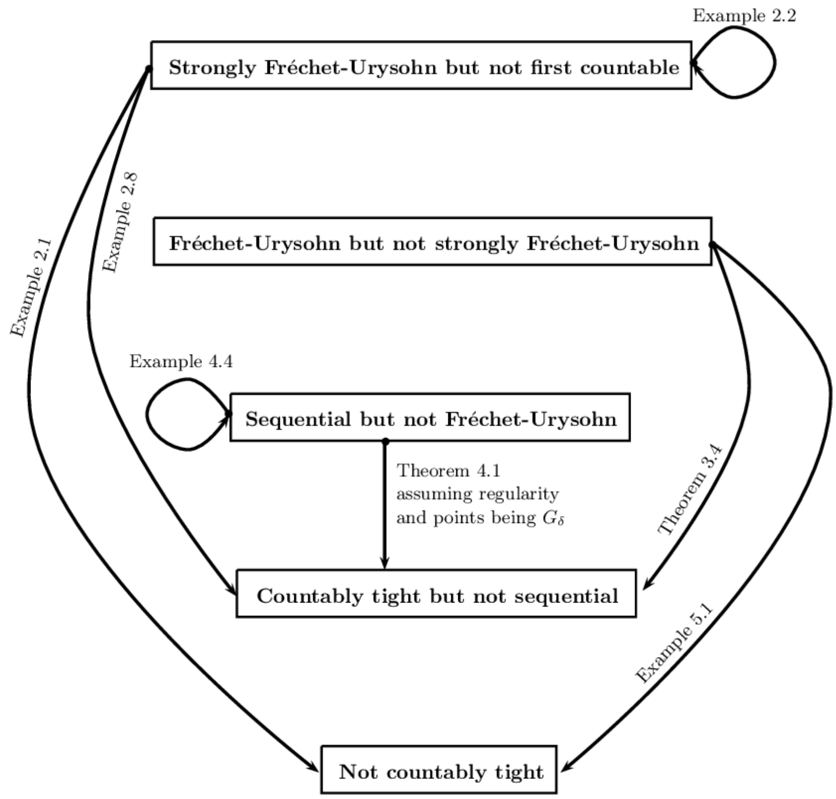
Diagram

This diagram indicates how topological spaces loose or keep properties in examples and theorems.



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