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FILTERS ON ω AND IRREDUNDANT FAMILIES

ANDRÉS MILLÁN

Abstract. In this note we generalize a result by J. Cancino, O. Guzmán and A. Miller by proving that no irredundant family on any countable set can generate a finite intersection of prime ideals. We prove among other results that either there are at most $2^{\mathfrak{d}}$-many Q-points or there are $2^{\mathfrak{c}}$-many Rudin-Keisler incomparable Q-points. We also give another characterization of the reaping number $\mathfrak{r}$ in terms of partitions of ω.

1. Preliminaries

We will use standard set theoretic notation. If $A, B \in [\omega]^\omega$ we write $A \subseteq^* B$ provided that $A \setminus B$ is finite. Letters $\mathcal{A}$, $\mathcal{I}$ and $\mathcal{R}$ will denote families of subsets of ω. A family $\mathcal{A}$ has the strong finite intersection property provided every finite subfamily of $\mathcal{A}$ has an infinite intersection; this property will be denoted in the text as SFIP. Given a family of sets we denote $\langle\mathcal{A}\rangle$, depending on the context, either the ideal or the filter generated by $\mathcal{A}$. A family $\mathcal{G} \subseteq \omega^\omega$ is *dominating* provided for every $h \in \omega^\omega$ there is a $g \in \mathcal{G}$ such that $\forall^\infty n < \omega\ h(n) < g(n)$. Letter $\mathfrak{d}$ denotes the minimum cardinality of a dominating family on ω^ω. A family $\mathcal{R} \subseteq [\omega]^\omega$ is reaping provided for every $A \in [\omega]^\omega$ there is a $R \in \mathcal{R}$ such that either $R \subseteq^* A$ or $R \subseteq^* \omega \setminus A$. A family $I \subseteq [\omega]^\omega$ is independent provided for every $\mathcal{A}, \mathcal{B} \in [I]^{<\omega}$ with $\mathcal{A} \cap \mathcal{B} = \emptyset$

$$|\bigcap_{A\in\mathcal{A}} A \cap \bigcap_{B\in\mathcal{B}} (\omega \setminus B)| = \omega.$$

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A set $S \in [\omega]^\omega$ *splits* a family $\mathcal{A} \subseteq [\omega]^\omega$ provided $|A \cap S| = |A \setminus S| = \omega$ for every $A \in \mathcal{A}$. Letter $\mathfrak{r}$ denotes the minimum cardinality of a reaping family. It is known that $\omega_1 \leq \mathfrak{r} \leq \mathfrak{i} \leq \mathfrak{c}$, where $\mathfrak{i}$ denotes the minimum cardinality of a maximal independent family. Letter $\mathscr{F}$ will always denote a filter on ω containing the cofinite filter $\mathscr{F}_{\mathsf{cof}}$. Similarly, $\mathscr{G}$ will denote an ideal on ω containing $[\omega]^{<\omega}$ and $\mathscr{I}$ will always be a prime ideal. If $\mathscr{F}$ is a filter then, $I_{\mathscr{F}}$ is its dual ideal and if I is an ideal $I^+ = \mathcal{P}(\omega)\setminus I$ denotes the set of I-positive subsets of ω. If $\mathscr{F}$ is a filter $\chi(\mathscr{F})$, denotes the minimum cardinality of a basis of $\mathscr{F}$. Ultrafilters are denoted either by $\mathscr{U}$ or $\mathscr{V}$. All ultrafilters considered in this paper will be non principal ultrafilters. The ultrafilter number is defined as $\mathfrak{u} = \min\{\chi(\mathscr{U})\colon \mathscr{U} \in \omega^*\}$. We denote ω^* the remainder of the Stone-Čech compactification of ω that we identify with the set of non principal ultrafilters on ω. This space has a basis $\mathcal{B} = \{V(A)\colon A \in [\omega]^\omega\}$ where $V(A) = \{\mathscr{U} \in \omega^*\colon A \in \mathscr{U}\}$. Given $\mathscr{U}$ and $f \in \omega^\omega$ the ultrafilter $f(\mathscr{U})$ is defined by $f(\mathscr{U}) = \{Y \subseteq \omega\colon f^{-1}(Y) \in \mathscr{U}\}$. The Rudin-Keisler partial ordering of ultrafilters is defined by $\mathscr{U} \leq_{RK} \mathscr{V}$ provided there is a $f \in \omega^\omega$ such that $\mathscr{U} = f(\mathscr{V})$. It is a celebrated theorem by M. E. Rudin and S. Shelah [10] that in ZFC there are, $2^{\mathfrak{c}}$-many RK-incomparable ultrafilters on ω. A non principal ultrafilter $\mathscr{U}$ on ω is a P-point provided for every $\mathcal{A} \in [\mathscr{U}]^\omega$ there exists an $U \in \mathscr{U}$ such that $U \subseteq^* A$ for every $A \in \mathcal{A}$. Given $f \in \omega^\omega$ and $U \in [\omega]^\omega$ we say that U is f-rare provided $\forall n, m \in U$ if $m < n$ then $f(m) < n$. A family $\mathcal{A} \subseteq [\omega]^\omega$ is rare if $\mathcal{A}$ contains an f-rare set for every $f \in \omega^\omega$. A Q-point is an ultrafilter that contains a rare family. A rapid ultrafilter is an ultrafilter $\mathscr{U}$ such that for every $f \in \omega^\omega$ there is an $U \in \mathscr{U}$ such that $|U \cap f(n)| \leq n$ for every $n < \omega$. Every Q-point is a rapid ultrafilter but not conversely. Let $\mathscr{F}$ be a filter, $\mathcal{P} = \{P_n\colon n < \omega\}$ a partition of ω and let $\langle \mathscr{U}_n\colon n < \omega\rangle$ be a sequence of ultrafilters such that $P_n \in \mathscr{U}_n$ for every $n < \omega$. The limit of the sequence $\langle \mathscr{U}_n\colon n < \omega\rangle$ along $\mathscr{F}$ is the filter

$$\lim_{n\to\mathscr{F}} \mathscr{U}_n = \{X \subseteq \omega\colon \{n < \omega\colon X \cap P_n \in \mathscr{U}_n\} \in \mathscr{F}\}.$$

Also, if $\mathscr{F}$ is a filter on ω and $\mathscr{U}$ is an ultrafilter on ω then, $\mathscr{F} \otimes \mathscr{U}$ is the filter on $\omega \times \omega$ defined by

$$\mathscr{F} \otimes \mathscr{U} = \{A \subseteq \omega \times \omega\colon \{n < \omega\colon (A)_n \in \mathscr{U}\} \in \mathscr{F}\}.$$

Where $(A)_n = \{m \in \omega\colon (n, m) \in A\}$. In the case when $\mathscr{F}$ is and ultrafilter say, $\mathscr{V}$ both, $\lim_{n\to\mathscr{V}} \mathscr{U}_n$ and $\mathscr{V} \otimes \mathscr{U}$ are ultrafilters on ω and $\omega \times \omega$ respectively. The next Proposition is folklore. It will be useful later.

Proposition 1.1. *Let $\mathscr{F}$, $\mathscr{U}$ be as above and let $\langle \mathscr{U}_n\colon n < \omega\rangle$, be a sequence of ultrafilters and $\mathcal{P} = \{P_n\colon n < \omega\}$ a partition of ω such that $P_n \in \mathscr{U}_n$ for every $n < \omega$, then*

(i) *The ultrafilters extending* $\lim_{n\to\mathscr{F}}\mathscr{U}_n$ *are of the form* $\lim_{n\to\mathscr{V}}\mathscr{U}_n$ *for some ultrafilter* $\mathscr{V}$ *extending* $\mathscr{F}$.

(ii) *The ultrafilters extending* $\mathscr{F}\otimes\mathscr{U}$ *are of the form* $\mathscr{V}\otimes\mathscr{U}$ *for some ultrafilter* $\mathscr{V}$ *extending* $\mathscr{F}$.

Proof. We only check (i) since (ii) is similar. Let $\mathscr{W}$ be any ultrafilter extending $\lim_{n\to\mathscr{F}}\mathscr{U}_n$. If $W\in\mathscr{W}$ the set $\mathsf{supp}(W)=\{n<\omega\colon W\in\mathscr{U}_n\}$ is always infinite. Moreover, if $W_1,W_2\in\mathscr{W}$ then, $\mathsf{supp}(W_1\cap W_2)\subseteq\mathsf{supp}(W_1)\cap\mathsf{supp}(W_2)$ so, the family $\{\mathsf{supp}(W)\colon W\in\mathscr{W}\}$ can be extended to an ultrafilter $\mathscr{V}$. If $F\in\mathscr{F}$ then, $X_F=\bigcup\{P_n\colon n\in F\}\in\lim_{n\to\mathscr{F}}\mathscr{U}_n\subseteq\mathscr{W}$ and $F=\mathsf{supp}(X_F)\in\mathscr{V}$. Thus, $\mathscr{F}\subseteq\mathscr{V}$. The identity $\lim_{n\to\mathscr{V}}\mathscr{U}_n=\mathscr{W}$ follows because for every $W\in\mathscr{W}$ the family $\lim_{n\to\mathscr{V}}\mathscr{U}_n\cup\{W\}$ has the SFIP. But since $\lim_{n\to\mathscr{V}}\mathscr{U}_n$ is an ultrafilter it is maximal (with) respect to SFIP. Thus, $W\in\lim_{n\to\mathscr{V}}\mathscr{U}_n$ and $\mathscr{W}\subseteq\lim_{n\to\mathscr{V}}\mathscr{U}_n$. Since $\mathscr{W}$ is itself an ultrafilter the equality $\mathscr{W}=\lim_{n\to\mathscr{V}}\mathscr{U}_n$ holds. □

A co-partition of ω is a family formed by the complements of a partition of ω.

2. Irredundant families

Definition 2.1. We say that $\mathcal{I}\subseteq[\omega]^\omega$ is irredundant provided for every $I\in\mathcal{I}$ and $\mathcal{I}_0\in[\mathcal{I}\setminus\{I\}]^{<\omega}$ $I\not\subseteq^*\bigcup\mathcal{I}_0$.

Almost disjoint and independent families on ω are both examples of irredundant families. However, there are other examples.

Proposition 2.2 (Cancino. J., Guzmán, O., Miller, A. [2]). *An irredundant family cannot generate a prime ideal.*

In particular, Proposition 2.2 implies that no maximal independent family is reaping [1]

Proposition 2.3. *No independent family on ω is reaping.*

Proof. If $\mathcal{I}\subseteq[\omega]^\omega$ is independent so is $\mathcal{I}^*=\{\omega\setminus I\colon I\in\mathcal{I}\}$. Thus, both $\mathcal{I}$ and $\mathcal{I}^*$ are irredundant. Also, if $\mathcal{I}$ were reaping then, the filter $\mathscr{F}=\langle\mathcal{I}\rangle$ generated by $\mathcal{I}$ would be an ultrafilter and the dual ideal $\mathscr{I}_{\mathscr{F}}=\langle\mathcal{I}^*\rangle$ would be a prime ideal. This contradicts Proposition 2.2. □

[1] In page 85 of [1] we find the following statement "A maximal independent family is easily seen to be reaping and, hence, $\mathfrak{r}\leq\mathfrak{i}$." As Proposition 2.2 shows this is not true. What the author of [1] meant is the following: “The family of **boolean combinations** of members of a maximal independent family is easily seen reaping and, hence, we see $\mathfrak{r}\leq\mathfrak{i}$.”

3. Primary Filters and Ideals

In this next section we generalize Proposition 2.2.

Definition 3.1. We call a filter $\mathscr{F}$ primary if it can be written as the intersection of finitely many ultrafilters and it is called non primary otherwise. Also, we call an ideal $\mathscr{I}$ primary or non primary provided its dual filter $\mathscr{F}_{\mathscr{I}}$ is primary.

Lemma 3.2. *Let $n < \omega$ and $\{\mathscr{U}_i \colon i \leq n\}$ be finite set of ultrafilters and $\mathscr{F} = \cap_{i \leq n} \mathscr{U}_i$ then, the set of ultrafilters extending $\mathscr{F}$ is exactly $\{\mathscr{U}_i : i \leq n\}$. Also, if $\{\mathscr{U}_i \colon i < \omega\}$ is a countable family of P-points and $\mathscr{F} = \cap_{i<\omega} \mathscr{U}_i$ then, the set of P-points extending $\mathscr{F}$ is exactly $\{\mathscr{U}_i \colon i < \omega\}$.*

Proof. We prove only the second statement since the first one is even easier. Let $\mathscr{V}$ be a P-point extending $\mathscr{F}$. If $\mathscr{V} \notin \{\mathscr{U}_i \colon i < \omega\}$ pick $V_i \in \mathscr{V} \setminus \mathscr{U}_i$ for every $i < \omega$ and let $V \in \mathscr{V}$ such that $V \subseteq^* V_i$ for every $i < \omega$. Then, $V \in \mathscr{V} \setminus \bigcup_{i<\omega} \mathscr{U}_i$ and $\omega \setminus V \in \bigcap_{i<\omega} \mathscr{U}_i = \mathscr{F} \subseteq \mathscr{V}$ which is impossible. □

Lemma 3.3. *Let $n < \omega$ and $\{\mathscr{U}_i \colon i \leq n\}$ and $\mathscr{F}$ be a finite family of ultrafilters and a filter respectively $\mathscr{F}$ on ω. The following statements are equivalent:*

(a) $\mathscr{F} = \bigcap_{i \leq n} \mathscr{U}_i$.

(b) *There is a partition $\{P_i \colon i \leq n\}$ of ω so that $P_i \in \mathscr{U}_i$ and $\mathscr{U}_i = \langle \mathscr{F} \cup \{P_i\} \rangle$ for every $i \leq n$.*

Proof. To prove (a)⇒(b) suppose that $\mathscr{F} = \bigcap_{i \leq n} \mathscr{U}_i$. Since ultrafilters $\mathscr{U}_i$ are pairwise different, we find an $U_i \in \mathscr{U}_i \setminus \bigcup_{j \leq n, j \neq i} \mathscr{U}_j$ for every $i \leq n$. Define $P_0 = U_0$ and $P_i = U_i \setminus \bigcup_{j<i} U_j$ for every $i \leq n$. It is clear that if $n < \omega$, the family $\{P_i \colon i \leq n\}$ is pairwise disjoint, and that $P_i \in \mathscr{U}_i$ for every $i \leq n$. We can make it a partition of ω by adding if necessary, the missing points to P_0. Therefore, $\langle \mathscr{F} \cup \{P_i\} \rangle \subseteq \mathscr{U}_i$ for every $i \leq n$. But $\langle \mathscr{F} \cup \{P_i\} \rangle$ has to be an ultrafilter because otherwise, the set of ultrafilters extending $\mathscr{F}$ is strictly larger than $\{\mathscr{U}_i \colon i \leq n\}$ contradicting Lemma 3.2. To prove that (b)⇒(a) first notice that $\mathscr{F} \subseteq \bigcap_{i \leq n} \langle \mathscr{F} \cup \{P_i\} \rangle$. Pick any $X \in \bigcap_{i \leq n} \langle \mathscr{F} \cup \{P_i\} \rangle$. Then there are $F_i \in \mathscr{F}$ such that $F_i \cap P_i \subseteq X$ for every $i \leq n$. Put $F = \bigcap_{i \leq n} F_i$. Thus, $F \in \mathscr{F}$ and $F = \bigcup_{i \leq n} F \cap P_i \subseteq X$. Hence, $X \in \mathscr{F}$. □

Corollary 3.4. Let $\mathscr{I}_0, \ldots, \mathscr{I}_n$ and $\mathscr{G}$ be prime ideals and an ideal on ω. The following statements are equivalent,

(a) $\mathscr{G} = \bigcap_{i \leq n} \mathscr{I}_i$.

(b) There is a finite co-partition $\{Q_i \colon i \leq n\}$ of ω so that $Q_i \in \mathscr{I}_i$ and $\mathscr{I}_i = \langle \mathscr{G} \cup \{Q_i\} \rangle$ for every $i \leq n$.

The following generalizes Proposition 3 from [2].

Theorem 3.5. *An infinite irredundant family cannot generate a primary ideal.*

Proof. Let $\mathcal{I} \subseteq [\omega]^\omega$ be an infinite irredundant family and suppose that the ideal generated by $\mathcal{I}$ is primary. Then, there is a finite family of prime ideals $\{\mathscr{I}_i : i \leq n\}$ on ω such that $\langle \mathcal{I} \rangle = \bigcap_{i \leq n} \mathscr{I}_i$. Moreover, $\mathscr{I}_i = \langle \mathcal{I} \cup \{Q_i\} \rangle$ for some finite co-partition $\mathcal{Q} = \{Q_i : i \leq n\}$ of ω. Pick $p > n+1$ and p-many pairwise disjoint countably infinite subfamilies $\{\{I^r_j : j < \omega\} : r < p\}$ of $\mathcal{I}$. For every $r < p$ and $k < \omega$ define sets A^r_k as follows.

(a) $A^0_0 = I^0_0$ and $A^0_k = I^0_{pk} \setminus \bigcup_{s<p} \bigcup_{j<pk} I^s_j$.

(b) $A^r_k = I^r_{pk+r} \setminus \bigcup_{s<p} \bigcup_{j<pk+r} I^s_j$ for $0 < r < p$.

Then, put $A_r = \bigcup_{k<\omega} A^r_k$ for every $r < p$. The sets $A_0, \dots, A_{p-1}$ are pairwise disjoint and we can ensure that they form a partition of ω either by adding if necessary, the missing points to A_0 if they are finitely many or adding a single point to each A^r_k otherwise. Also, for every $i \leq n$ there is an $r_i < p$ such that $\bigcup_{r \in p \setminus \{r_i\}} A_r \in \langle \mathcal{I}_i \cup \{Q_i\} \rangle$. Thus, if $r \in \bigcap_{i \leq n} (p \setminus \{r_i\})$ then, $A_r \in \bigcap_{i \leq n} \langle \mathcal{I} \cup \{Q_i\} \rangle$ because ideals are downward closed. For each $i \leq n$ find $\mathcal{I}_i \in [\mathcal{I}]^{<\omega}$ such that $A_r \subseteq^* \bigcup \mathcal{I}_i \cup Q_i$. Then $A_r \setminus \bigcup_{i \leq n} \bigcup \mathcal{I}_i \subseteq^* \bigcap_{i \leq n} Q_i = \emptyset$. If $k < \omega$ is large enough to ensure that $I^r_{pk+r} \notin \bigcup_{i \leq n} \mathcal{I}_i$ then,

$$I^r_{pk+r} \subseteq^* \left(\bigcup_{i \leq n} \bigcup \mathcal{I}_i \cup \bigcup_{s<p} \bigcup_{j<pk+r} I^s_j \right)$$

which contradicts the irredundancy of $\mathcal{I}$. □

Corollary 3.6. The dual filter of an ideal generated by an irredundant family is not primary.

Corollary 3.7. The ideal on ω generated by an almost disjoint family is non primary. Neither the ideal nor the filter generated by an independent family is primary.

Proof. The first statement is immediate. If $\mathcal{I}$ is an independent family so is the dual family $\mathcal{I}^* = \{\omega \setminus I : I \in \mathcal{I}\}$. Thus, both families are irredundant. By Theorem 3.5 neither $\mathcal{I}$ nor $\mathcal{I}^*$ can generate a primary ideal. Thus, $\mathcal{I}$ cannot generate a primary filter. □

Let $C(\mathscr{F}) = \{\mathscr{U} \in \omega^* : \mathscr{F} \subseteq \mathscr{U}\}$. It is clear that $C(\mathscr{F})$ is always closed in ω^*.

Theorem 3.8. *The following statements are equivalent for a filter $\mathscr{F}$*

(a) *$\mathscr{F}$ is non primary.*
(b) *$C(\mathscr{F})$ is an infinite closed subset of ω^*.*
(c) *$\mathscr{F}$ can be extended to $2^{\mathfrak{c}}$-many RK-incomparable ultrafilters.*
(d) *There is a partition $\mathcal{P}$ of ω into infinitely many pieces such that $\forall P \in \mathcal{P}\ \forall F \in \mathscr{F}\ |F \cap P| = \omega$.*

Proof. $(a) \Leftrightarrow (b)$ If $\mathscr{F}$ is primary, $\mathscr{F}$ is the intersection of finitely many ultrafilters which are, by Lemma 3.2 its unique ultrafilter extensions. Thus, $C(\mathscr{F})$ is finite. If $C(\mathscr{F})$ is finite let $\mathscr{G}$ be the intersection of the members of $C(\mathscr{F})$. Then, $\mathscr{F} \subseteq \mathscr{G}$. If $\mathscr{F} \neq \mathscr{G}$, then $I^+_{\mathscr{G}} \subseteq I^+_{\mathscr{F}}$ and $I^+_{\mathscr{G}} \neq I^+_{\mathscr{F}}$. Pick $A \in I^+_{\mathscr{G}} \setminus I_{\mathscr{F}}{}^+$. Then, $\mathscr{F} \cup \{\omega \setminus A\}$ has the SFIP and $\mathscr{F}$ can be extended to an ultrafilter $\mathscr{V}_A \in C(\mathscr{F})$. Thus, $A \in I^+_{\mathscr{G}}$ which is impossible. To see that $(b) \Leftrightarrow (c)$ notice that $(c) \Rightarrow (b)$ is trivial and $(b) \Rightarrow (c)$ is precisely Corollary 4.7 in [4]. To see that $(d) \Rightarrow (a)$ let $\mathcal{P} = \{P_n \colon n \in \omega\}$. Notice that if $X \notin \mathscr{F}$ then, $\mathscr{F} \cup \{\omega \setminus X\}$ has the SFIP. If $\mathscr{U}_X$ is any ultrafilter extending $\mathscr{F} \cup \{\omega \setminus X\}$ then $\mathscr{F} = \bigcap_{X \notin \mathscr{F}} \mathscr{U}_X$. But the family $\{\mathscr{U}_X : X \notin \mathscr{F}\}$ contains also the ultrafilters $\mathscr{U}_{\omega \setminus P_n}$ which are pairwise distinct since $P_n \in \mathscr{U}_{\omega \setminus P_n}$ and $\omega \setminus P_n \notin \mathscr{F}$. Thus, $\mathscr{F}$ is non primary. To prove direction $(a) \Rightarrow (d)$, suppose that $\mathscr{F} = \bigcap_{i \in I} \mathscr{U}_i$ such that I is infinite and define inductively sequences $\langle \mathscr{F}_n : n < \omega \rangle$, $\langle R_n \in [I]^\omega : n < \omega \rangle$, and $\langle A_n \colon n < \omega \rangle$ such that for every $n < \omega$,

(i) $\mathscr{F}_0 = \mathscr{F}$,
(ii) $R_{n+1} \in [R_n]^\omega$, and
(iii) $\mathscr{F}_n = \langle \mathscr{F} \cup \{A_i \colon i \leq n\} \rangle \subseteq \bigcap_{i \in R_n} \mathscr{U}_i$.

To see that this construction is possible, suppose that we have defined $\mathscr{F}_n$ and R_n in such a way that conditions (i), (ii) and (iii) are satisfied. Since $\mathscr{F}_n \subseteq \bigcap_{i \in R_n} \mathscr{U}_i$, $\mathscr{F}_n$ is not an ultrafilter. Thus, there is an infinite and coinfinite set A_n so that both $\mathscr{F}_n \cup \{A_n\}$ and $\mathscr{F}_n \cup \{\omega \setminus A_n\}$ have the SFIP. Then, there is an $R_{n+1} \in [R_n]^\omega$ such that $A_n \in \bigcap_{i \in R_{n+1}} \mathscr{U}_i$. Thus, $\langle \mathscr{F}_n \cup \{A_n\} \rangle \subseteq \bigcap_{i \in R_{n+1}} \mathscr{U}_i$. Put $\mathscr{F}_{n+1} = \langle \mathscr{F}_n \cup \{A_n\} \rangle$. This completes the inductive construction. Now define $P_0 = \omega \setminus A_0$ and for every $n \geq 1$ put $P_n = (\omega \setminus A_n) \cap \bigcap_{k<n} A_k$. By the inductive construction, $\mathcal{P} = \{P_n : n < \omega\}$ is a pairwise disjoint family of infinite sets and we can make it a partition by adding if necessary, the remaining points to P_0. It is clear from the construction that every set in $\mathscr{F}$ intersects every P_n at an infinite set. $\square$

Corollary 3.9. A filter generated by $< \mathfrak{u}$-many sets is non primary. Meager and measurable filters are also non primary.

Proof. Suppose that $\chi(\mathscr{F}) < \mathfrak{u}$ and $\mathscr{F}$ is primary. By lemma 3.3 there is a partition $\mathcal{P} = \{P_i \colon i < n\}$ of ω such that $\langle \mathscr{F} \cup \{P_i\} \rangle$ is an ultrafilter for every $i < n$. But $\chi(\langle \mathscr{F} \cup \{P_i\}\rangle) = \chi(\mathscr{F}) < \mathfrak{u}$ which is a contradiction. Meager filters are non primary because it is a result by S. Plewik (See [9], page 8) that the intersection of $< \mathfrak{c}$ ultrafilters is non meager. Measurable filters are non primary because by a result of Proposition 7 in [11], the intersection of countably many non measurable filters is again non measurable. $\square$

Definition 3.10. An ultrafilter $\mathscr{U}$ is a bad point if there exist a countable family of ultrafilters S such that $\mathscr{U}$ is in the ω^*-closure of $S \setminus \{\mathscr{U}\}$. An ultrafilter which is not a bad point is a weak P-point.

The next Proposition is well known. I include a proof for the sake of completeness.

Proposition 3.11. *If $C \subseteq \omega^*$ is nonempty and closed then, there exists a filter $\mathscr{F}$ such that $C = C(\mathscr{F})$.*

Proof. Given $C \subseteq \omega^*$ closed and nonempty let $\mathscr{F} = \{A \subseteq \omega \colon C \subseteq V(A)\}$. Notice that $\mathscr{F} \neq \emptyset$ since $\omega \in \mathscr{F}$. Moreover, since we are dealing with non principal ultrafilters, $\mathscr{F}_{\mathsf{cof}} \subseteq \mathscr{F}$. It is straightforward to check that $\mathscr{F}$ is a filter. Also, $C \subseteq \bigcap_{A \in \mathscr{F}} V(A) = C(\mathscr{F})$. If this contention is strict then, there is an $\mathscr{U} \in C(\mathscr{F}) \setminus C$. Since C is closed there exists an open basic set $V(A)$ such that $A \in \mathscr{U}$ and $V(A) \cap C = \emptyset$. Then, $C \subseteq V(\omega \setminus A)$. This implies that $\omega \setminus A \in \mathscr{F}$ but, this is impossible because A and $\omega \setminus A$ cannot be both in $\mathscr{U}$. Hence, $C = C(\mathscr{F})$. $\square$

Bad points are extremely abundant in ω^* as the next well known proposition shows.

Proposition 3.12. *Every infinite closed subset of ω^* contains an infinite closed subset consisting of just bad points.*

Proof. If $C \subseteq \omega^*$ is closed and infinite there exists by Proposition 3.11 a non primary filter $\mathscr{F}$ such that $C = C(\mathscr{F})$ and also a partition $\mathcal{P} = \{P_n \colon n < \omega\}$ such that $|F \cap P_n| = \omega$ for every $F \in \mathscr{F}$ and $n < \omega$. Put $\mathscr{F}_n = \langle \{X \cap P_n \colon X \in \mathscr{F}\}\rangle$ and let $\mathscr{U}_n$ be an ultrafilter on ω extending $\mathscr{F}_n$. If $\mathscr{F}^*$ is the filter defined by

$$\mathscr{F}^* = \{X \subseteq \omega \colon \{n < \omega \colon X \cap P_n \in \mathscr{U}_n\} \in \mathscr{F}_{\mathsf{cof}}\}$$

then, $C(\mathscr{F}^*) \subseteq C(\mathscr{F})$ and $C(\mathscr{F}^*) = \{\lim_{n \to \mathscr{V}} \mathscr{U}_n \colon \mathscr{V} \in \omega^*\}$ by Proposition 1.1. $\square$

Therefore, there is not hope for finding an infinite closed set in ω^* containing just weak P-points. However K. Kunen, by using nothing beyond ZFC showed in [6] that there are $2^{\mathfrak{c}}$-many weak P-points. It is possible to show that many infinite closed subsets of ω^* contain P-points, weak P-points and other types of ultrafilters in various models of ZFC. See for example Theorem 1.2 in [5] and Theorems 2 and 3 in [3].

Despite Proposition 3.12 we can still prove in ZFC that there are quite a few closed sets containing a lot of weak P-points.

Theorem 3.13. *There exists a disjoint family of $2^{\mathfrak{c}}$-many closed sets of ω^* where each member contains $2^{\mathfrak{c}}$-many weak P-points.*

Proof. By Theorem 2.10 in [7] every meager filter can be extended to $2^{\mathfrak{c}}$-many $\mathfrak{c}$-OK points which in particular, are weak P-points by Lemma 1.3 in [6]. Thus, we only need to show that there are $2^{\mathfrak{c}}$-many pairwise different meager filters. Fix an independent family $\mathcal{J}$ on ω with $|\mathcal{J}| = \mathfrak{c}$ and let $\mathcal{P} = \{P_n : n < \omega\}$ be a partition of ω into infinite sets. For every $J \in \mathcal{J}$ let $X_J = \bigcup\{P_n : n \in J\}$ and if $g \in 2^{\mathcal{J}}$ consider the family $\mathcal{J}_g = \{X_J : g(J) = 1\} \cup \{\omega \setminus X_J : g(J) = 0\}$. Then, each $\mathcal{J}_g$ is an independent family with $|\mathcal{J}_g| = \mathfrak{c}$. If $h \in H = \prod_{n<\omega}[P_n]^{<\omega}$ put $E^h_n = \bigcup_{k>n}(P_k \setminus h(n))$ for every $n < \omega$ and $h \in H$. Let $\mathscr{E} = \{E^h_n : h \in H, n < \omega\}$. Define the filter

$$\mathscr{F}_g = \left\langle \left\{ \mathscr{F}_{\mathsf{cof}} \cup \mathcal{J}_g \cup \left\{ \omega \setminus \bigcap \mathcal{B} : \mathcal{B} \in [\mathcal{J}_g]^{\geq\omega} \right\} \cup \mathscr{E} \right\} \right\rangle .$$

Notice that each $\mathscr{F}_g$ is meager because every member of it almost-contain a member of the countable family $\mathcal{P}$. Also, the ultrafilter extensions cannot be neither $< \mathfrak{c}$-generated nor P-points. Also, if $g_1, g_2 \in 2^{\mathcal{J}}$ and $g_1 \neq g_2$ then, $C(\mathscr{F}_{g_1}) \cap C(\mathscr{F}_{g_2}) = \emptyset$. Thus, the family $\{C(\mathscr{F}_g) : g \in 2^{\mathcal{J}}\}$ is the one we seek. □

With rapid ultrafilters and Q-points we can do better.

Theorem 3.14. *If there is a rapid ultrafilter then, there is an infinite closed subset of ω^* containing rapid ultrafilters only. If there is a Q-point $\mathscr{U}$ such that $\mathfrak{d} < \chi(\mathscr{U})$ then, there is an infinite closed set containing Q-points only.*

Proof. Let $\mathscr{U}$ be a rapid ultrafilter and consider the filter $\mathscr{F} = \mathscr{F}_{\mathsf{cof}} \otimes \mathscr{U}$ on $\omega \times \omega$. The ultrafilter extensions of $\mathscr{F}$ are precisely the ultrafilters of the form $\mathscr{V} \otimes \mathscr{U}$ where $\mathscr{V}$ is any ultrafilter on ω and all these ultrafilters are rapid by Theorem 4 in [8]. Thus, $\mathscr{F}$ is non primary, $C(\mathscr{F})$ is infinite and contains rapid ultrafilters only.

Let $\mathscr{U}$ be a Q-point such that $\mathfrak{d} < \chi(\mathscr{U})$, and let $\{f_\xi : \xi < \mathfrak{d}\}$ be a dominant family on ω^ω. For each $\xi < \mathfrak{d}$ pick $U_\xi \in \mathscr{U}$ such that U_ξ is f_ξ-rare. If $\mathscr{F} = \langle\{U_\xi : \xi < \mathfrak{d}\}\rangle$ then, $\chi(\mathscr{F}) \leq \mathfrak{d} < \chi(\mathscr{U})$. The filter $\mathscr{F}$ cannot be primary because otherwise, by Lemma 3.3 there exists $n < \omega$ and a partition $\{P_i : i \leq n\}$ of ω such that $\mathscr{F} = \bigcap_{i\leq n}\langle \mathscr{F} \cup \{P_i\}\rangle$ and $\langle \mathscr{F} \cup \{P_i\}\rangle$ is an ultrafilter for every $i \leq n$. Since $\mathscr{U}$ is an ultrafilter there is an $i_0 \leq n$ such that $P_{i_0} \in \mathscr{U}$. Therefore, $\mathscr{U} = \langle \mathscr{F} \cup \{P_{i_0}\}\rangle$ but this is impossible because $\chi(\langle \mathscr{F} \cup \{P_{i_0}\}\rangle) = \chi(\mathscr{F}) = \mathfrak{d} < \chi(\mathscr{U})$. By Theorem 3.8, $C(\mathscr{F})$ is an infinite closed subset of ω^* containing Q-points only. □

The next two corollaries follow from Theorem 3.14 and Rudin-Shelah theorem previously used.

Corollary 3.15. If there is a rapid ultrafilter then, there are $2^{\mathfrak{c}}$-many RK-incomparable rapid ultrafilters.

Corollary 3.16. If there exists a Q-point then either there are $2^{\mathfrak{c}}$-many RK-incomparable Q-points or there are at most $2^{\mathfrak{d}}$-many Q-points.

Proof. Suppose that Q-points exist. If there exists a Q-point $\mathscr{U}$ such that $\mathfrak{d} < \chi(\mathscr{U})$ then we can apply Theorem 3.14 to obtain an infinite closed set consisting of Q-points only by Theorem 3.14. Thus, there are $2^{\mathfrak{c}}$-many RK-incomparable ultrafilters in this set by Theorem 3.8. Hence, there are $2^{\mathfrak{c}}$-many RK-incomparable Q-points. If for every Q-point $\mathscr{U}$, the inequality $\chi(\mathscr{U}) \leq \mathfrak{d}$ holds then $\chi(\mathscr{U}) = \mathfrak{d}$ because every Q-point contains a rare base. Since there are at most $2^{\mathfrak{d}}$-many of such bases there are at most $2^{\mathfrak{d}}$-many Q-points. □

Also, we can apply Lemma 3.3 to obtain

Corollary 3.17. If there exists a P-point then, there exists an infinite closed subset of ω^* containing exactly ω-many P-points.

4. Partitions and the Reaping number

Recall that a a family $\mathcal{R} \subseteq [\omega]^\omega$ is a reaping family provided if for every $A \subseteq \omega$ there is an $R \in \mathcal{R}$ such that either $R \subseteq^* A$ or $R \subseteq^* \omega \setminus A$. For example, if $\mathcal{I} \subseteq [\omega]^\omega$ is a maximal independent family then the family

$$\mathcal{R}(\mathcal{I}) = \left\{ \bigcap_{A\in H} A \cap \bigcap_{B \in K} (\omega \setminus B) \colon H, K \in [\mathcal{I}]^{<\omega} \ \&\ H \cap K = \emptyset \right\}$$

is reaping and $|\mathcal{R}(\mathcal{I})| = |\mathcal{I}|$. The reaping number is defined by

$$\mathfrak{r} = \min\{|\mathcal{R}| \colon \mathcal{R} \subseteq [\omega]^\omega \ \&\ \mathcal{R} \text{ is reaping }\}.$$

Then, $\omega_1 \leq \mathfrak{r} \leq \mathfrak{i} \leq \mathfrak{c}$ where $\mathfrak{i}$ is the minimal size of a maximal independent family on ω.

Definition 4.1. $\mathcal{A} \subseteq [\omega]^\omega$ is robust if every member is co-infinite and for each partition $\mathcal{P} = \{P_n \colon n < \omega\}$ of ω there exists an $A \in \mathcal{A}$ such that

$$\text{either} \quad \exists^\infty n < \omega\ P_n \cap A = \emptyset \quad \text{or} \quad \exists^\infty n < \omega\ P_n \subseteq A.$$

Theorem 4.2.

$$\mathfrak{r} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \ \&\ \mathcal{R} \textit{ is robust}\,\}$$

Proof. Let κ be the minimum cardinality of a robust family. Every reaping family can be refined to get a reaping family of the same size but formed by coinfinite sets. Let $\mathcal{A} \subseteq [\omega]^\omega$ be such refined reaping family and let $\mathcal{P} = \{P_n : n < \omega\}$ be a partition of ω. There is an $A \in \mathcal{A}$ such that either $A \subseteq^* \bigcup_{k<\omega} P_{2k}$ or $A \subseteq^* \bigcup_{k<\omega} P_{2k+1}$. In both cases there are infinitely many $n < \omega$ such that $A \cap P_n = \emptyset$ and we are done. Therefore, $\kappa \leq \mathfrak{r}$. Now, we prove that if $\mathcal{A} \subseteq [\omega]^\omega$ and $|\mathcal{A}| < \mathfrak{r}$ then $\mathcal{A}$ is not robust. This will be accomplished by constructing a partition $\mathcal{P} = \{P_n : n < \omega\}$ such that for every $A \in \mathcal{A}$ and every $n < \omega$, $|P_n \cap A| = |P_n \setminus A| = \omega$. Since $|\mathcal{A}| < \mathfrak{r}$ there exists a $R_0 \in [\omega]^\omega$ that splits $\mathcal{A} \cup \{\omega \setminus A \colon A \in \mathcal{A}\}$. Put $P_0 = R_0$. If we have defined P_i and R_i for $i < n$ such that $\{A \setminus \bigcup_{i<n} R_i \colon A \in \mathcal{A}\} \cup \{(\omega \setminus A) \setminus \bigcup_{i<n} R_i \colon A \in \mathcal{A}\} \subseteq [\omega]^\omega$. Since this family has size $< \mathfrak{r}$ there exists an $R_n \in [\omega]^\omega$ that splits it. Put $P_n = R_n \setminus \bigcup_{i<n} R_i$. It is clear that $\mathcal{P} = \{P_n : n < \omega\}$ is disjoint and it can be made a partition by adding if necessary, the remaining points of ω to P_0. Also, the construction shows that every set from $\mathcal{A} \cup \{\omega \setminus A \colon A \in \mathcal{A}\}$ intersects every P_n at an infinite set. Therefore, $\mathfrak{r} \leq \kappa$ and $\kappa = \mathfrak{r}$. □

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