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A SIMPLE CONSTRUCTION OF NON-PLANAR ONE-DIMENSIONAL CONTINUA AS GENERALIZED INVERSE LIMITS

HAYATO IMAMURA

Abstract. Inverse limits of continuous maps on intervals are always one-dimensional planer continua. On the other hand, for every $n = 1, 2, \dots, \infty$, there exists generalized inverse limit with upper semi-continuous functions on intervals whose dimension is equal to n. In this paper, we give a simple construction of generalized inverse limits on intervals which are non-planar one-dimensional continua.

1. Introduction

Inverse sequences of continuous functions on compacta (= compact metric spaces) and inverse limits are fundamental tools of describing complicated continua and investigating dynamical systems of continuous functions. In order to study continua in 2004, Mahavier [11] introduced generalized inverse limits with set-valued functions on intervals. Later Ingram and Mahavier [6] generalized the notation to set-valued functions on compacta as follows.

Definition 1.1. For any $n \in \mathbb{N}$, let X_n be a compactum and let 2^{X_n} be the collection of all nonempty closed sets of X_n. Let $f_n : X_{n+1} \to 2^{X_n}$. An *inverse sequence* is defined as a sequence of pairs X_n and f_n, which is denoted by $\{X_n, f_n\}_{n\in\mathbb{N}}$. The *generalized inverse limit* $\varprojlim \{X_n, f_n\}$ of the inverse sequence $\{X_n, f_n\}_{n\in\mathbb{N}}$ is defined by

$$\varprojlim \{X_n, f_n\} := \left\{ (x_1, x_2, \dots) \in \prod_{n=1}^{\infty} X_n \;\middle|\; x_n \in f_n(x_{n+1}) \text{ for any } n \in \mathbb{N} \right\}.$$

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$\varprojlim \{X_n, f_n\}$ is a subspace of $\prod_{n=1}^{\infty} X_n$ endowed with the product topology. Thus we can assume that $\varprojlim \{X_n, f_n\}$ is a metric space with a metric function d defined by

$$d(\mathbf{x}, \mathbf{x}') = \sum_{n=1}^{\infty} 2^{-n} d_{X_n}(x_n, x'_n) \quad \text{for } \mathbf{x}, \mathbf{x}' \in \varprojlim \{X_n, f_n\}.$$

In the case where $X_n = X$ and $f_n = f$ for every $n \in \mathbb{N}$, we write the inverse limit by $\varprojlim \{X, f\}$.

The topological structure of generalized inverse limits is much more complicated than one of (usual) inverse limits. For example, non-degenerate chainable continua (= inverse limits of intervals) are always one-dimensional and embeddable in the plane, but for any $n = 1, 2, \ldots, \infty$, there exists a generalized inverse limit of intervals whose dimension is equal to n. Thus, we have non-planar examples of generalized inverse limits of intervals.

In this paper, we will give a simple construction of generalized inverse sequence with single bonding function on an interval whose limit is one-dimensional but not embeddable into the plane.

2. Main Results

Our main result is Theorem 2.5. To prove that we use Proposition 2.1, 2.2 and 2.4.

Proposition 2.1. ([4, Theorem 2.7]) *Suppose $\{X_i\}_{i\in\mathbb{N}}$ is a sequence of subintervals of $[0,1]$ and $\{f_i : X_{i+1} \to 2^{X_i}\}_{i\in\mathbb{N}}$ is a sequence of upper semi-continuous functions such that for each $i \in \mathbb{N}$ and any $x \in X_{i+1}$, $f_i(x)$ is a continuum. Then $\varprojlim \{X_i, f_i\}$ is a continuum.*

Proposition 2.2. ([10, Theorem 5.4]) *If X_1 is a continuum such that every non-degenerate subcontinuum K of X_1 contains a countable set that separates K, and for each i, X_i is compact and $f_i : X_{i+1} \to 2^{X_i}$ is upper-semi continuous, and for each $y \in X_i$, $\dim(\{x \in X_{i+1} \mid y \in f_i(x)\}) = 0$, then $\dim\left(\varprojlim \{X_i, f_i\}\right) \leq 1$.*

Definition 2.3. A continuum M will be said to be *triodic* if there exist subcontinua $M_1, M_2, M_3 \subseteq M$ such that $\bigcap_{i=1}^{3} M_i$ is a non-empty continuum, and for any $j, k \in \{1, 2, 3\}$ with $j \neq k$, $\bigcap_{i=1}^{3} M_i = M_j \cap M_k$.

Proposition 2.4. ([9]) *If, in the plane, $\mathcal{G}$ is an uncountable family of bounded triodic continua, there exists an uncountable subfamily $\mathcal{H}$ of $\mathcal{G}$ such that every two continua of the set $\mathcal{H}$ have a point in common.*

In the proof of Theorem 2.5, we use the following notation: Let $\mathbb{I}$ be a closed interval. For any $m, n \in \mathbb{N}$ with $m < n$, let $\pi_{<m,\ldots,n>} : \prod_{i=1}^{\infty} \mathbb{I} \to \prod_{i=m}^{n} \mathbb{I}$ be the projection. For a map $f : X \to Y$, $G(f) := \{(x,y) \in X \times Y \mid y = f(x)\}$ be the graph of f. For any set-valued function $g : X \to 2^Y$, the set $\{(x,y) \in X \times Y \mid y \in g(x)\}$ is also called the *graph* of g and denoted by $G(g)$.

Theorem 2.5. *Fix $n \in \mathbb{N}_{\geq 2}$. Let $g : \mathbb{I} \to \mathbb{I}$ be a surjective piecewise monotonic continuous function. Suppose that g has a periodic point p of prime period n and L is a closed interval such that $p, g(p) \in L$. If $f : \mathbb{I} \to 2^{\mathbb{I}}$ satisfies that $G(g) \cup (\{p\} \times L) \subseteq G(f)$, then $\varprojlim \{\mathbb{I}, f\}$ is non-planar. Moreover, if $G(g) \cup (\{p\} \times L) = G(f)$ then $\varprojlim \{\mathbb{I}, f\}$ is a one-dimensional continuum.*

Proof. We can assume that $G(f) = G(g) \cup (\{p\} \times L)$ and $p < g(p)$. We prove that $\varprojlim \{\mathbb{I}, f\}$ is non-planar. Let Q, Q_1, Q_2, Q_3 be the points of $\prod_{j=1}^{3n} \mathbb{I}$ defined by the following:

$$\begin{aligned}
Q &:= \left(g^{n-1}(p), \ldots, g(p), p, g^{n-1}(p), \ldots, g(p), p, g^{n-1}(p), \ldots, g(p), p\right), \\
Q_1 &:= \left(g^{n-2}(p), \ldots, g(p), p, p,\right. \\
&\qquad \left. g^{n-1}(p), \ldots, g^2(p), g(p), p, g^{n-1}(p), \ldots, g^2(p), g(p), p\right), \\
Q_2 &:= \left(g^{n-2}(p), \ldots, g(p), p, g^{n-1}(p), g^{n-2}(p), \ldots, g(p), p, p,\right. \\
&\qquad \left. g^{n-1}(p), \ldots, g^2(p), g(p), p\right), \\
Q_3 &:= \left(g^{n-2}(p), \ldots, g(p), p, g^{n-1}(p), g^{n-2}(p), \ldots, g(p), p,\right. \\
&\qquad \left. g^{n-1}(p), g^{n-2}(p), \ldots, g(p), p, p\right).
\end{aligned}$$

Moreover let A_1, A_2, A_3 be the subsets of $\prod_{j=1}^{3n} \mathbb{I}$ defined by

$$\begin{aligned}
A_1 &:= \left\{ \mathbf{x} \in \prod_{j=1}^{n-1} \mathbb{I} \;\middle|\; x_j = g(x_{j+1})\,(j \leq n-2), x_{n-1} \in [p, g(p)] \right\} \\
&\qquad \times \left\{ \left(p, g^{n-1}(p), \ldots, g(p), p, g^{n-1}(p), \ldots, g(p), p\right) \right\}, \\
A_2 &:= \left\{ \mathbf{x} \in \prod_{j=1}^{2n-1} \mathbb{I} \;\middle|\; x_j = g(x_{j+1})\,(j \leq 2n-2), x_{2n-1} \in [p, g(p)] \right\} \\
&\qquad \times \left\{ \left(p, g^{n-1}(p), \ldots, g(p), p\right) \right\}, \\
A_3 &:= \left\{ \mathbf{x} \in \prod_{j=1}^{3n} \mathbb{I} \;\middle|\; x_j = g(x_{j+1})\,(j \leq 3n-2), x_{3n-1} \in [p, g(p)], x_{3n} = p \right\}.
\end{aligned}$$

Then each A_i is an arc having end points Q and Q_i. If $i \neq j$, $A_i \cap A_j = \{Q\}$. Thus $A = \bigcup_{i=1}^{3} A_i$ is a triodic continuum in $\prod_{j=1}^{3n} \mathbb{I}$.

For each $k \in \mathbb{N}$, we define $J_{k,1}$ and $J_{k,2}$ by

$$J_{k,1} := L \times \{p\} \times \left\{g^{n-1}(p)\right\} \times \cdots \times \{g(p)\} \subseteq \prod_{i=3n+(k-1)(n+1)}^{3n+k(n+1)} \mathbb{I},$$

$$J_{k,2} := L \times \{p\} \times \cdots \times \{p\} \times \{p\} \subseteq \prod_{i=3n+(k-1)(n+1)}^{3n+k(n+1)} \mathbb{I}.$$

For each $s = (s_1, s_2, \dots) \in \prod_{j=1}^{\infty} \{1, 2\}$, let us define

$$B_s := \left\{ \mathbf{x} \in \prod_{j=1}^{\infty} \mathbb{I} \;\middle|\; \begin{array}{l} \pi_{<1,3n>}(\mathbf{x}) \in A, \\ \pi_{<3n+(k-1)(n+1),3n+k(n+1)>}(\mathbf{x}) \in J_{k,s_k} \text{ for each } k \in \mathbb{N} \end{array} \right\}.$$

Then B_s is a triodic continuum in $\varprojlim \{\mathbb{I}, f\}$. Moreover we can easily see that $B_s \cap B_{s'} = \emptyset$ if $s \neq s'$. Thus $\varprojlim \{\mathbb{I}, f\}$ has uncountably many mutually disjoint triodic subcontinua. Therefore $\varprojlim \{\mathbb{I}, f\}$ is non-planar by Proposition 2.4.

Since f has continuum value on each point on $\mathbb{I}$, $\varprojlim \{\mathbb{I}, f\}$ is a continuum from Proposition 2.1. Moreover any non-degenerate subcontinuum K of $\mathbb{I}$ is a closed interval and any point p of Int (K) separates K. Thus $\dim\left(\varprojlim \{\mathbb{I}, f\}\right) \leq 1$ from Proposition 2.2. Since $\varprojlim \{\mathbb{I}, g\} \subseteq \varprojlim \{\mathbb{I}, f\}$, $\dim\left(\varprojlim \{\mathbb{I}, f\}\right) = 1$. $\square$

Applying Theorem 2.5, we give an example of non-planar continuum. The example is named "monster" by Banič and studied by several authors [1], [2], [3], [4], [7] and [8].

Example 2.6. Let $f : [0,1] \to 2^{[0,1]}$ be given by $f(0) = [0,1]$ and $f(x) = \{1 - x\}$ for $0 < x \leq 1$. Then $\varprojlim \{[0,1], f\}$ is a non-planar one-dimensional continuum.

Proof. Let $g : [0,1] \to [0,1]$ be defined by $g(x) = 1 - x$. Then g is strictly monotonic and $p = 0$ is the periodic point of this function. Then $f(p) = [p, g(p)] = [0,1]$. Therefore, by Theorem 2.5, $\varprojlim \{[0,1], f\}$ is a non-planar one-dimensional continuum. $\square$

It is known that a periodic point whose period is not a power of 2 induces a complicated usual inverse limit on an interval (See [5, Theorem 32]). On the other hand, Theorem 2.5 says that the period of a periodic point is not essential. We note that the condition, $[p, g(p)] \subseteq f(p)$, is essential.

Example 2.7. ([4, Example 2.19]) Let $b \in (0,1)$ and let $f : [0,1] \to 2^{[0,1]}$ be defined by $f(0) = [b,1]$ and $f(x) = \{1-x\}$ for $0 < x \leq 1$. Then $\varprojlim \{[0,1], f\}$ is embeddable in the plane.

Let

$$A_0 = \left\{ \mathbf{x} \in \prod_{j=1}^{\infty} \mathbb{I} \;\middle|\; x_j = 1 - x_{j+1} \text{ for any } j \in \mathbb{N} \right\}$$

and for each $n \in \mathbb{N}$, let

$$A_{2n-1} := \left\{ \mathbf{x} \in \prod_{j=1}^{\infty} \mathbb{I} \;\middle|\; x_{2n-1} \in [b,1], x_{2n} = 0, x_j = 1 - x_{j+1} \text{ for } j \neq 2n-1 \right\},$$

$$A_{2n} := \left\{ \mathbf{x} \in \prod_{j=1}^{\infty} \mathbb{I} \;\middle|\; x_{2n} \in [b,1], x_{2n+1} = 0, x_j = 1 - x_{j+1} \text{ for } j \neq 2n \right\}.$$

Then we can easily see that

$$\varprojlim \{[0,1], f\} = \bigcup_{n=0}^{\infty} A_n.$$

Let O be the origin $(0,0)$ and let O_1 be the point $(0,1)$ in the plane $\mathbb{R}^2$. For each $n \in \mathbb{N}$ and $s \in [0,1]$, let K_n^s and L_n^s be the points $(1/n, s)$ and $(-1/n, s)$ in $\mathbb{R}^2$ respectively. For each $b \in (0,1)$, define

$$X_b := \left(\bigcup_{n \in \mathbb{N}} K_n^0 K_n^{1-b} \right) \cup OO_1 \cup \left(\bigcup_{n \in \mathbb{N}} L_n^1 L_n^b \right),$$

here for points $P, Q \in \mathbb{R}^2$ the segment joining P and Q is denoted by PQ. Then we define the quotient space Y_b of X_b identifying the sets $\{(1,0), (1/2,0), \cdots, (0,0)\}$ and $\{(-1,1), (-1/2,1), \cdots, (0,1)\}$ to the points S and N respectively. Clearly Y_b is a planar continuum. Put

$$Y_b^+ := p\left(\left(\bigcup_{n \in \mathbb{N}} K_n^0 K_n^{1-b} \right) \cup OO_1 \right) \text{ and } Y_b^- := p\left(OO_1 \cup \left(\bigcup_{n \in \mathbb{N}} L_n^1 L_n^b \right) \right),$$

where $p : X_b \to Y_b$ is the natural projection. For each $(z,t) \in X_b$ the point $p(z,t)$ is denoted by $[z,t]$.

We define $h_1 : Y_b^+ \to A_0 \cup \bigcup_{n \in \mathbb{N}} A_{2n}$ as follows:
for $(z,t) \in \left(\bigcup_{n \in \mathbb{N}} K_n^0 K_n^{1-b} \right) \cup OO_1$, put $h_1([z,t]) = \mathbf{x} \in \prod_{j=1}^{\infty} \mathbb{I}$ such that

(i) If $(z,t) \in K_n^0 K_n^{1-b}$, then

$$x_1 = t,\ x_{2n+1} = 0 \text{ and } x_j = 1 - x_{j+1} \ \text{ for } j \neq 2n.$$

(ii) If $(z,t) \in OO_1$, then

$$x_1 = t, \text{ and } x_j = 1 - x_{j+1} \ \text{ for each } j \geq 1.$$

$$i.e., h_1([1/n, t]) = \mathbf{x} = (t, 1-t, t, \cdots, 1-t, \overset{2n+1}{0}, 1, 0, 1, \cdots) \in A_{2n}.$$

$$h_1([0,t]) = \mathbf{x} = (t, 1-t, t, \cdots, t, 1-t, \cdots) \in A_0.$$

Since $h_1([z,0]) = (0,1,0,\cdots)$ for any $z \in Q$, h_1 is well-defined. Moreover, since $\lim_{\substack{n\to\infty \\ s\to t}} h_1([1/n,s]) = h_1([0,t])$ for any $t \in [0,1-b]$, we can see that h_1 is continuous. Thus, h_1 is a homeomorphism.

Similarly we have a homeomorphism $h_2 : Y_b^- \to A_0 \cup \bigcup_{n\in\mathbb{N}} A_{2n-1}$ such that

$$h_2([0,t]) = (t, 1-t, t, \cdots, t, 1-t, \cdots) \in A_0 \text{ for all } t \in [0,1].$$

Hence we can have a homeomorphism $h : Y_b = Y_b^+ \cup Y_b^- \to \bigcup_{n=0}^{\infty} A_n$ by $h|_{Y_b^+} = h_1$ and $h|_{Y_b^-} = h_2$. Therefore, $\varprojlim \{[0,1], f\}$ is embeddable in the plane.

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