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by

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A LIST OF OPEN PROBLEMS FOR THE SUBGROUP COMMUTATIVITY DEGREE OF GROUPS

ENIOLA KAZEEM

Abstract. The present survey describes the state of art of the theory of the subgroup commutativity degree for finite and infinite groups. A list of open problems are extrapolated from the relevant literature and we add new ones, due to the recent approach in the infinite case.

1. Relevant Literature

P. Erdős and P. Turán [17] introduced the notion of *commutativity degree*

$$\mathrm{d}(G) = \frac{|\{(x, y) \in G \times G \mid xy = yx\}|}{|G|^2} \tag{1.1}$$

of a finite group G, in order to give a measure of the abelianess of G. In fact $\mathrm{d}(G) = 1$ if and only if G is abelian and $\mathrm{d}(G)$ can be interpreted as the probability that two randomly picked elements of G commute. It turns out that G can be classified via restrictions on $\mathrm{d}(G)$, as one can see from the main results of [3, 16, 18, 19, 20, 29, 30, 32, 39, 58, 59], but these contributions also show that generalizations are meaningful in the infinite case when compact groups are considered. Note that a parallel approach can be found by using character theory (see again [7, 28, 29, 54]).

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We report some recent works on the commutativity degree in geometric group theory. Antolín and others [6] defined a generalisation of the commutativity degree for finitely presented group G with finite generating set X. They introduced

$$dc_X(G) = \limsup_{n\to\infty} \frac{|\{(u,v) \in (\mathbb{B}_X(n))^2 \mid uv = vu\}|}{|\mathbb{B}_X(n)|^2} \tag{1.2}$$

where $\mathbb{B}_X(n)$ is the ball of radius n centered at 1 on the Cayley graph $\Gamma(G, X)$ of G with respect to X. Here some classical notions of growth in the sense of Gromov are involved (see [14, Chapter IV]). In fact it is possible to see that a finitely generated residually finite group of subexponential growth, that is, one with the exponential growth rate equals *one* (see [14, Chapter 6, Page 181]) has positive (1.2) if and only if the group has an abelian subgroup of finite index. With completely different techniques, this was noted in the case of compact groups in [39], where the abelian group of finite index turns out to be the center $Z(FC(G))$ of the FC-center of a compact group G where

$$FC(G) = \{x \in G \mid |G : C_G(x)| < \infty\}$$

with $C_G(x) = \{g \in G \mid xg = gx\}$. Cox [10] considers the first non-hyperbolic (see [14, Section V.D.]) examples that do not have polynomial growth, namely lamplighter groups of the form $F \wr \mathbb{Z}$, that is, the wreath product (see [14, Chapter VIII, Page 214]) of any finite group F and $\mathbb{Z}$. He then shows that its commutativity degree here is zero ([10, Theorem 2]). Ciobanu and others [11] introduced the conjugacy ratio of a finitely generated group G on a set X of generators, generalising (1.2). They showed that the conjugacy ratio for all finitely generated, residually finite groups of stable subexponential growth that are not virtually abelian is zero, with respect to all finite generating sets (see [11, Theorem 3.7]). Finally we mention recent work of Valiunas [69] who studied an infinite group G, and X, a finite generating set for G such that the growth series of G with respect to X is a rational function. Note that the growth series $s_{G,X}(t)$ of a finitely generated group G with respect to a finite generating set X is defined by

$$s_{G,X}(t) = \sum_{n=0}^{\infty} |S_{G,X}(n)|\, t^n$$

where $S_{G,X}(n)$ is the set of all group elements in G of length exactly n with respect to the finite generating set X ([14, Chapter VI, Definition 1]). In all these references [6, 10, 11, 69], the notion of the commutativity degree is used in geometric group theory and structures of groups are detected via this invariant.

From the point of view of lattice theory, Tărnăuceanu [66, 67] introduced the *subgroup commutativity degree*, defining the ratio

$$\mathrm{sd}(G) = \frac{|\{(H,K) \in \mathrm{L}(G) \times \mathrm{L}(G) \mid HK = KH\}|}{|\mathrm{L}(G) \times \mathrm{L}(G)|} \tag{1.3}$$

where $\mathrm{L}(G)$ denotes the subgroup lattice of G. He also computed sd(G) for some classes of groups like dihedral group and generalised quaternion group. Nonabelian simple groups and additional classes of finite groups were considered in [1, 2, 21, 22, 23, 24, 56, 61, 62].

Note that the number $\mathrm{sd}(G)$ also gives important information on how far G is from being abelian, describing the probability of commuting subgroups. In fact a finite group G has $\mathrm{sd}(G) = 1$ if and only if any two randomly chosen subgroups of G permute. This is a *quasihamiltonian group* in the sense of Iwasawa [48, 49], that is, a finite group G such that $HK = KH$ for all $H, K \in \mathrm{L}(G)$. Among quasihamiltonian finite groups we find abelian groups and the well-known quaternion group Q_8 of order 8. On the other hand, any subgroup of Q_8 is normal, but there are finite groups like the *modular group*

$$M = \langle a, b \mid a^2 = b^8 = 1, ba = ab^5 \rangle$$

of order 16, where $HK = KH$ for all $(H,K) \in \mathrm{L}(M) \times \mathrm{L}(M)$, but not necessarily all H and K are normal in M. In fact the prefix "quasi" is used to weaken the notion of *hamiltonian finite group*, that is, a finite group G such that $\mathrm{L}(G) = \mathrm{N}(G)$, where $\mathrm{N}(G)$ denotes the lattice of all normal subgroups of G. Abelian groups and Q_8 are hamiltonian, and in particular quasihamiltonian. Note that one can produce examples of finite p-groups (p odd prime) which are nonabelian quasihamiltonian via

$$M_n = \langle a, b \mid b^{2^n} = 1, b^{2^{n-1}} = a^2, bab^{-1} = a^{-1} \rangle.$$

In the infinite case, the first generalisations of quasihamiltonian groups were due to the same Iwasawa [49], but Kümmich [45, 46, 47] studied the behaviour of projective limits of finite quasihamiltonian groups. Strunkov [65] and Diaconis [15] also contributed to the theory of infinite quasihamiltonian groups, but from completely different perspectives. Recently, Herfort and others [33, 38] developed the theory of locally compact quasihamiltonian groups, making another important step in the area.

Observing the literature, we recognised three phases which are somehow cyclic.

The first phase is devoted to results of classification for extremal cases of probability equal to one. This means that one looks for theorems of structure in prescribed classes of groups. Concerning the subgroup commutativity degree, we think at classifications as in [48, 49, 33, 63].

A second phase is devoted to the development of a probability theory which describes upper and lower bounds, useful for numerical considerations on how far we are from the extremal case. This is pure probabilistic group theory: we think at results in [1, 2, 21, 22, 23, 66, 67].

A third case deals with variations on the original invariant, and consequently a new classification might be possible, so again one could go back to the first phase, weakening further the original classifications. We think at [50, 51, 52, 53, 56] about generalisations of the subgroup commutativity degree in this phase.

In each of the above three phases, fundamental differences appear when we want to move from the finite case to the infinite case and topology comes in at this stage, because the infinite case requires a formalisation in terms of measure spaces and appropriate topological spaces. Then we were motivated to study (1.3) for infinite groups. We illustrate a result on the subgroup commutativity degree of profinite groups. The main idea generalises an idea of Heyer in [36] and more details can be found in [40, 41, 42]. Let us recall some information on projective limits from [37, Page 17].

Definition 1.1 (See [37], Definition 1.25)**.** Let J be a directed set, that is, a set with a reflexive, transitive and antisymmetric relation $\leq$ such that every finite non empty subset has an upper bound. A *projective system* of topological groups over J is a family of morphisms

$$\{f_{jk} : G_k \to G_j \mid (j,k) \in J \times J, j \leq k\},$$

where G_j are topological groups for all $j \in J$ (i.e.: G_j is a topological group with a Hausdorff topology), satisfying the following conditions:

(i). $f_{jj} = \mathrm{id}_{G_j}$ for all $j \in J$

(ii). $f_{jk} \circ f_{kl} = f_{jl}$ for all $j,k,l \in J$ with $j \leq k \leq l$.

Consider the topological group $\prod_{j \in J} G_j$ with the product topology. Then for the projective system of topological groups, the set defined by

$$G = \{(g_j)_{j\in J} \in \prod_{j \in J} G_j \mid j \leq k \ (\forall j,k \in J) \Rightarrow f_{jk}(g_k) = g_j\}$$

is called the *projective limit* of the groups G_j and is written $G = \lim_{j \in J} G_j$. The mappings (morphisms) $f_j : G \to G_j$ are called *limit maps* and the morphisms $f_{jk} : G_k \to G_j$ are called *bonding maps.*

In particular, given a projective system of finite groups, *a profinite group* is a topological group which is obtained as the projective limit of finite groups, each endowed with the discrete topology. Note also that:

Proposition 1.2. *(See* [37, Lemma 1.26]*).*
In $G = \lim_{j \in J} G_j$ *with* J *directed set and* G_j *topological groups;*

(i). *If* $\mathrm{inc} : G \to \prod_{j \in J} G_j$ *denotes the inclusion and* $\mathrm{pr}_j : \prod_{j \in J} G_j \to G_j$ *the projection, then the function* $f_j = \mathrm{pr}_j \circ \mathrm{inc} : G \to G_j$ *is a morphism of topological groups for all* $j \in J$*, and for* $j \leq k$ *in* J *the relation* $f_j = f_{jk} \circ f_k$ *is satisfied.*

(ii). *If all groups* G_j *in the projective system are compact, then* $\prod_{j \in J} G_j$ *and* G *are compact groups.*

There is another important way to describe profinite groups and we give this below.

Theorem 1.3 (Approximation Theorem, see [27], Proposition 6)**.** *Suppose* G *is profinite and* $\mathcal{N}(G)$ *is the set of all open normal subgroups of* G*. Then* $\lim_{H \in \mathcal{N}(G)} G/H = G$*.*

Next, we describe the topology on the space of closed subgroups of a profinite group G denoted $\mathcal{SUB}(G)$ according to [33], using the fact that every topological group supports a variety of uniformities with which they are compatible. This topology is given via the local basis for the uniformity $\mathcal{U}$ at a point H in $\mathcal{SUB}(G)$

$$B(H, U) = \{K \in \mathcal{SUB}(G) \mid KU = HU\},$$

where U is an open normal subgroup. This is equivalent to giving the *Vietoris* topology on $\mathcal{SUB}(G)$, which is the unique topology compatible with the uniformity on $\mathcal{SUB}(G)$ (see [26, 27, 55]). It should be noted that $\mathcal{SUB}(G)$ may be constructed as a projective limit, when G is profinite.

Theorem 1.4 (See [27], Proposition 7)**.** *If* G *is a profinite group, then*

$$\mathcal{SUB}(G) = \lim_{H \in \mathcal{N}(G)} \mathcal{SUB}(G/H).$$

The above result is known in various forms and one can also refer to [37, Appendix 3, Proposition A3.44] and [33, Part 1, Chapter 1] for more details. The proof can also be found in [31, Main Theorem] and [9, Theorem 3]. Now some elementary notions of functional analysis must be recalled for the construction of the probability measure $\mathrm{sd}(G)$ and they can be found in classical textbooks as [4, 5, 36, 60, 70]. Take a locally compact topological space E and K, a compact subset of E. Define

$$\mathcal{K}_{\mathbb{R}}(E, K) = \{f : E \to \mathbb{R} \mid f \text{ is continuous and } \mathrm{supp}(f) \subseteq K\}$$

and write

$$\mathcal{K}_{\mathbb{R}}(E) = \bigcup_{\substack{K \subseteq E \\ K \text{ compact}}} \mathcal{K}_{\mathbb{R}}(E, K).$$

A Radon measure $\mu : \mathcal{K}_{\mathbb{R}}(E) \to \mathbb{R}$ can be seen as a functional $f \mapsto \mu(f) = \int_E f \mathrm{d}\mu$ such that $|\mu(f)| \leq M_K \|f\|$, where $\|f\| = \sup_{x\in E} |f|$ and M_K a constant. The set

$$\mathcal{M}_+(E) = \{\mu : \mathcal{K}_{[0,+\infty)}(E) \to [0, +\infty) \mid \mu \text{ is a Radon measure}\}$$

is the set of all positive Radon measures and we have

$$\mathcal{K}_{[0,+\infty)}(E) = \bigcup_{\substack{K\subseteq E\\ K\text{compact}}} \mathcal{K}_{[0,+\infty)}(E, K),$$

where we can briefly use the symbol $\mathcal{K}_+(E) = \mathcal{K}_{[0,+\infty)}(E)$.

We constructed a measure of probability on $\mathcal{SUB}(G)$ with G profinite, extending (1.3). We made a construction in [40, Lemma 3.1, Theorem 3.4] such that the resulting extension of $\mathrm{sd}(G)$ had properties of regularity in the usual sense of the Riesz Representation Theorem [60, Theorem 2.14].

We considered bonding maps $\pi_i : \mathcal{SUB}(G_{i+1}) \to \mathcal{SUB}(G_i)$ for the projective system $(\mathcal{SUB}(G_i), \pi_i, I)$ where $G = \lim_{i\in I} G_i$ and G_i finite groups. Lemma 1.2 and Definition 1.1 were then satisfied for this projective system. The following functions

$$\pi^*_{i-1} : \varphi_i \in \mathcal{K}_+(\mathcal{SUB}(G_i)) \longmapsto \pi^*_{i-1}(\varphi_i) = \varphi_i \circ \pi^{-1}_{i-1} \in \mathcal{K}_+(\mathcal{SUB}(G_{i-1})),$$
$$\phi^*_i : \varphi \in \mathcal{K}_+(\mathcal{SUB}(G)) \longmapsto \phi^*_i(\varphi) = \varphi_i \in \mathcal{K}_+(\mathcal{SUB}(G_i))$$

with $\varphi_i = \varphi_{|_{\mathcal{SUB}(G_i)}}$ from $\mathcal{SUB}(G_i)$ to $[0, +\infty]$ whose extension to $\mathcal{SUB}(G)$ gives φ, that is,

$$\varphi_i : H_i \in \mathcal{SUB}(G_i) \longmapsto \varphi_i(H_i) \in [0, +\infty)$$

were involved in our main result [40, Theorem 3.4]. The main ingredient of [40, Proof of Theorem 3.4] was a variation on a theme of Heyer in [36, Lemma 1.2.18]. See [40] and [41, Lemma 6.1]. The idea is that for a profinite group G, any positive function $f \in \mathcal{K}_+(\mathcal{SUB}(G))$ admits an approximating function $g \in \mathcal{K}_+(\mathcal{SUB}(G))$, which is constant on a large enough set and vanishes out of it. Note that $\mathcal{K}_{\mathbb{R}}(\mathcal{SUB}(G))$ must be endowed with the uniform norm, so $f \in \mathcal{K}_{\mathbb{R}}(\mathcal{SUB}(G))$ has $\|f\| = \sup_{H\in\mathcal{SUB}(G)} |f(H)|$. Note that $\mathcal{SUB}(G)$ is not compact in general when equipped with the Vietoris topology however it is always compact when we put the Chabauty topology on it. These two topologies coincide when G is compact. We also have the result below adapted from [36, Theorem 1.2.17].

Theorem 1.5. *Let $G = \lim_{i\in I} G_i$ be a profinite group. If $(\mathcal{SUB}(G_i), \pi_i, I)$ is the projective system realising $\mathcal{SUB}(G)$ and $\mu_i \in \mathcal{M}_+(\mathcal{SUB}(G_i))$ such that $\phi^*_{i-1} = \pi^*_{i-1} \circ \phi^*_i$ and $\mu_i = \pi^*_{i-1} \circ \mu_{i-1}$, then there exists exactly one $\mu \in \mathcal{M}_+(\mathcal{SUB}(G))$ such that $\mu|\mathcal{SUB}(G_i) = \mu_i$ and the following diagram is commutative.*

$$\begin{tikzcd}
 & & \mathcal{K}_+(\mathcal{SUB}(G)) \arrow[dll, dashed, "\phi^*_{i-1}"'] \arrow[dl] \arrow[dr] \arrow[drr, dashed, "\phi^*_i"] & & \\
\mathcal{K}_+(\mathcal{SUB}(G_0)) \arrow[drr, dashed] & \mathcal{K}_+(\mathcal{SUB}(G_{i-1})) \arrow[l, dashed] \arrow[dr, "\mu_{i-1}"'] & & \mathcal{K}_+(\mathcal{SUB}(G_i)) \arrow[ll, "\pi^*_{i-1}"'] \arrow[dl, "\mu_i"] & \cdots \arrow[l, dashed, "\pi^*_i"'] \arrow[dll, dashed] \\
 & & {[0,+\infty)} & &
\end{tikzcd}$$

As the reader has noted, we have just specialised $E = \mathcal{SUB}(G)$ in the previous notion of positive Radon measure, that is, we considered

$$\mathcal{M}_+(\mathcal{SUB}(G)) = \{\mu : \mathcal{K}_{[0,+\infty)}(\mathcal{SUB}(G_i)) \to [0,+\infty) \mid \mu \text{ is a Radon measure}\}$$

all positive Radon measures on the compact space $\mathcal{SUB}(G)$.

Applying Theorem 1.5, we were able to study the subgroup commutativity degree of infinite groups, that is, we showed that the following definition is a good generalisation of the subgroup commutativity degree of finite groups.

Definition 1.6. Assume $G = \lim_{i\in I} G_i$ is profinite and $[\{\mu_i \in \mathcal{M}_+(\mathcal{SUB}(G_i)) \mid i \in I\}]$ are given measures on $\mathcal{SUB}(G_i)$. The unique measure on G, ensured by Theorem 1.5, will be denoted by $\mu = \lim_{i\in I} \mu_i$. The subgroup commutativity degree $\mathrm{sd}(G)$ of G is the unique positive probability measure

$$\mathrm{sd}(G) = \lim_{i\in I} \mathrm{sd}(G_i) \tag{1.4}$$

extending the subgroup commutativity degree, $\mathrm{sd}(G_i)$ of a finite group G_i.

In fact we have:

Corollary 1.7. *If G is a finite group, then* (1.4) *becomes* (1.3).

Now one could observe an interesting fact, again motivated by evidences of the finite case. Russo [62] studied the *strong subgroup commutativity degree* in a finite group G, that is, the number

$$\mathrm{ssd}(H,K) = \frac{|\{(X,Y) \mid X \le H, Y \le K \text{ and } [X,Y]=1\}|}{|\{X \mid X \le H\}||\{Y \mid Y \le K\}|}.$$

This number again can be interpreted as a probability measure: it is in fact the probability that a randomly picked pair (H, K) of subgroups of a finite group G satisfies $[H, K] = 1$. This condition is stronger than $HK = KH$ and in fact relations are found between $\mathrm{ssd}(G) = \mathrm{ssd}(G, G)$ and $\mathrm{sd}(G)$. It looks like the *relative commutativity degree*

$$\mathrm{d}(H, K) = \frac{|\{(h, k) \in H \times K \mid [h, k] = 1\}|}{|H||K|}$$

of Erfanian and others [18, 19, 20], but at the level of subgroups, not at the level of elements. The interesting fact is that $\mathrm{ssd}(H, K)$ can be seen as a generalised character [62, Theorem 3.2]. Therefore the following problem is open:

Problem 1.8. Describe the subgroup commutativity degree (both for finite groups and profinite groups) via character theory.

2. The factorisation number of infinite groups

There is an equivalent way to describe the subgroup commutativity degree of a finite group in terms of the *factorisation number* so that it is enough to know the factorisation number of its subgroups, according to Farrokhi and others [21, 22, 23].

Given two subgroups H and K of a finite group G, we say that G is *factorised* by H and K, if $G = HK$, and this expression is called a *factorisation* of G. The number of all such factorisations is called the factorisation number of G, denoted $F_2(G)$. Of course, the consideration of S_3 shows that not all subgroups of a group may be involved in a factorisation. It is interesting to note that the relationship between $\mathrm{sd}(G)$ and $F_2(G)$ can be given by following formula

$$\mathrm{sd}(G) = \frac{1}{|\mathrm{L}(G)|^2} \sum_{H \in \mathrm{L}(G)} F_2(H). \tag{2.1}$$

There is a lot of work in recent years on (2.1), see [21, 22, 23, 24, 25].

On the other hand, it is possible to provide an alternative counting formula for (2.1). This can be found always in [66, 67]. For every subgroup H of G, we denote by $C(H)$ the set consisting of all subgroups of G which commute with H, that is,

$$C(H) = \{K \in \mathrm{L}(G) \mid HK = KH\}.$$

Then one can see that

$$\mathrm{sd}(G) = \frac{1}{|\mathrm{L}(G)|^2} \sum_{H \in \mathrm{L}(G)} |C(H)|, \tag{2.2}$$

concluding by (2.1) that it does not matter whether we count $F_2(H)$ or $|C(H)|$ with H variable in $\mathrm{L}(G)$. Both of them will give us the subgroup commutativity degree of a finite group, up to the factor of normalisation $1/|\mathrm{L}(G)|^2$. There is a third equivalent way, which seems to be more tractable for a generalisation to the infinite case.

We define the function $f : \mathrm{L}(G)^2 \to \{0,1\}$ on the finite group G by

$$f(H,K) = \begin{cases} 1, & KH = HK \\ 0, & HK \neq KH \end{cases} \quad \text{for all } H, K \in \mathrm{L}(G). \tag{2.3}$$

Then we have

$$|C(H)| = \sum_{K \in \mathrm{L}(G)} f(H,K)$$

for any $H \in \mathrm{L}(G)$ so that

$$\mathrm{sd}(G) = \frac{1}{|\mathrm{L}(G)|^2} \sum_{H,K \in \mathrm{L}(G)} f(H,K). \tag{2.4}$$

As we can see, (1.3) is equivalent to (2.1) but also to (2.4). As a result, note also that (1.3) can be formulated via characteristic functions and these are usually measurable in a Lebesgue space (see [60, Page 11]).

Let N be a fixed normal subgroup of G and consider the sets

$$\mathrm{A}_1 = \{H \in \mathrm{L}(G) \mid N \subseteq H\} \quad \text{and} \quad \mathrm{A}_2 = \{H \in \mathrm{L}(G) \mid H \subset N\}.$$

Since $\mathrm{A}_1 \cup \mathrm{A}_2 \subseteq \mathrm{L}(G)$, one can see that

$$\begin{aligned} \mathrm{sd}(G) &\geq \frac{1}{|\mathrm{L}(G)|^2} \sum_{H,K \in \mathrm{A}_1 \cup \mathrm{A}_2} f(H,K) \\ &= \frac{1}{|\mathrm{L}(G)|^2} \left(\sum_{H,K \in \mathrm{A}_1} f(H,K) + \sum_{H,K \in \mathrm{A}_2} f(H,K) \right. \\ &\left. +2 \sum_{H \in \mathrm{A}_1} \sum_{K \in \mathrm{A}_2} f(H,K) \right). \end{aligned}$$

In fact the following lower bound is known:

Proposition 2.1 (See [66], Proposition 2.4)**.** *For a finite group G with normal subgroup N,*

$$\begin{aligned} sd(G) \geq \frac{1}{|\mathrm{L}(G)|^2} &\left[\left(|\mathrm{L}(N)| + \left|\mathrm{L}\left(\frac{G}{N}\right)\right| - 1 \right)^2 \right. \\ &\left. +(sd(N)-1)|\mathrm{L}(N)|^2 + \left(sd\left(\frac{G}{N}\right) - 1 \right) \left|\mathrm{S}\left(\frac{G}{N}\right)\right|^2 \right]. \end{aligned}$$

Now one of the main open problems is to find an infinite version of Proposition 2.1. This of course should involve an appropriate study of the equations (2.1) and (2.4) in the infinite case.

Lazorec in [50, 51, 52] studied the subgroup commutativity degree, choosing only the lattice of subgroups of cyclic groups and he showed that this quantity can be computed using the sizes of the conjugacy classes of these groups. In fact [50, 51, 52] study the *cyclic* subgroup commutativity degree $\mathrm{csd}(G)$ of G by replacing the lattice of subgroups $\mathrm{L}(G)$ by $\mathrm{L}_1(G)$, the poset of cyclic subgroups of G, in the expression (1.3). As the name suggests, $\mathrm{csd}(G)$ measures the probability that two cyclic subgroups of G commute.

Farrokhi and others [21, 22, 23] computed the factorisation number for some classes of finite groups. However, Tărnăuceanu [68] improved these results, obtaining explicit expressions for the factorisation number of G, for two classes of finite abelian groups using the Möbius inversion formula. For instance, [21] considers the problem of determining $\mathrm{sd}(\mathrm{PSL}(2,p^n))$ where $\mathrm{PSL}(2,p^n)$ is the projective special linear group over a finite field of order p^n. The idea is to use some of the following results:

- the expression (2.1) above,
- the fact that all the subgroups of $\mathrm{PSL}(2,p^n)$ are well-known,
- the knowledge of a family of subgroups $\{H_i, i \in I\}$ such that $\bigcup_{i\in I} H_i = \mathrm{PSL}(2,p^n)$ and $H_i \cap H_j = \{1\}$ for all $i \neq j$.

In the same way that $F_2(G)$ corresponds to $\mathrm{sd}(G)$, Lazorec and others [52, 53] study the concept of *cyclic factorisation number* $CF_2(G)$ of a finite group G and this is the number of all pairs $(H,K) \in \mathrm{L}_1(G)^2$ satisfying $G = HK$, that is,

$$\mathrm{csd}(G) = \frac{1}{|\mathrm{L}_1(G)|^2} \sum_{H\in \mathrm{L}_1(G)} CF_2(H). \tag{2.5}$$

By using the Möbius inversion formula and other methods involving the cyclic subgroup structure, a relationship between $CF_2(G)$ and $\mathrm{csd}(G)$ is derived and $CF_2(G)$ is explicitly computed for some important classes of finite groups. A further step can be seen in [51] via the notion of *relative* subgroup commutativity degree of a subgroup H of a finite group G, that is, via the new notion

$$\mathrm{sd}(H,G) := \frac{|\{(A,B) \in \mathrm{L}(H) \times \mathrm{L}(G) \mid AB = BA\}|}{|\mathrm{L}(H) \times \mathrm{L}(G)|} \tag{2.6}$$

Here Lazorec and others [52, 53] consider the function

$$f : H \in \mathrm{L}(G) \to f(H) = \mathrm{sd}(H,G) \in [0,1]$$

and classify groups in the set

$$\mathcal{G} = \{G \text{ finite } : |\mathrm{Im} f| = 2\}.$$

They study groups for which the image of f consists of exactly two elements and show that there is an infinite number of finite groups with this property (that is, with two relative subgroup commutativity degrees). They also indicate conditions such that a finite group has at least three relative subgroup commutativity degrees.

As we have seen, there is work in the finite case, but some of the theory does not work in the infinite case since the challenge requires extensions of measures as in Theorem 1.5. It was therefore important to understand what happens in the profinite case.

We tried to solve this kind of problem for locally compact groups which are not necessarily profinite (see [42]) and one of the main difficulties was to understand the structure of the space of closed subgroups (see [8, 12]). There are problems which arise in this case when we try to study $\mathcal{SUB}(G)$ when G is locally compact and non-compact. For instance, $\mathcal{SUB}(\mathbb{R}^n)$ is difficult to describe concretely. Several authors tried to study this (see [12]). Take the additive group of the real numbers which is locally compact and non-compact. When $n = 1$, the Chabauty space (see [8]) of $\mathbb{R}$ is $[0, +\infty]$ which up to isomorphism is a circle which is a 1-variety and $\mathcal{SUB}(\mathbb{R})$ in this case is compact and connected. When $G \simeq \mathbb{R}^2$, $\mathcal{SUB}(G)$ is homeomorphic to a 4-variety which is compact and connected. However, the moment we take $G \simeq \mathbb{R}^3$, a precise description for $\mathcal{SUB}(\mathbb{R}^3)$ is not known; we have what is known as a *pseudovariety* which is locally connected and locally compact. Note also that [37, Theorem A1.12] describes the closed subgroups of $\mathbb{R}^n$ via decomposition into products of vector spaces. In conclusion, for $n \geq 3$, $\mathcal{SUB}(\mathbb{R}^n)$ is not a topological manifold and its structure is very delicate to understand and the reader can refer to [43, 44, 57] for more details. Note that in general, Theorem 27 in [26] provides a necessary and sufficient condition for $\mathcal{SUB}(G_1 \times G_2)$ to be homeomorphic to $\mathcal{SUB}(G_1) \times \mathcal{SUB}(G_2)$ under prescribed conditions for G_1 and G_2.

Questions 2.2. Let G be a locally compact group.

(1) Can we formulate $\mathrm{sd}(G)$ by the means of an equation as in (2.1)?

(2) What is the formulation of Proposition 2.1 in the infinite case? Even if G is profinite we do not have enough information.

(3) Motivated by [50], consider a finite group G and

$$\{\mathrm{sd}(G) \mid G \text{ is finite}\} := \mathcal{D}(G).$$

If $|\mathcal{D}(G)| \leq n$ with n arbitrary, what is the structure of G? We have satisfactory information for $n = 2$, but for larger n? This problem is open in the infinite case, that is, if G is a profinite infinite group with $|\mathcal{D}(G)| \leq n$, can we find a structural description for G? Here even the case $n = 2$ is completely open.

The moment we would like to consider locally compact groups which are noncompact, there is an additional problem of measure theory since we have studied pathologies in [42], that is, we noted that the absence of the Kolmogorov extension [5] may happen, and so one can have problems with Definition 1.6. Note that [36, Preliminaries], [64, Section 40], [13, Chapter 5], [34, Chapter II, Section 9] and [35, Chapter VII] give information on the structure of infinite locally compact nonabelian groups and we will then need to understand their Chabauty spaces carefully, when we would like to extend the subgroup commutativity degree to all locally compact groups.

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