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MUTIDIMENSIONAL SHIFTS AND FINITE MATRICES

PUNEET SHARMA AND DILEEP KUMAR

Abstract. Let X be a multi-dimensional subshift of finite type generated by a finite set of finite forbidden blocks. We give an algorithm for generating the elements of the shift space using a sequence of finite matrices (of increasing order). We prove that the sequence generated yields precisely the elements of the shift space X and hence characterizes the elements of the shift space X. In the process, we prove that elements of d-dimensional shift of finite type can be characterized by a sequence of finite matrices.

1. Introduction

Symbolic dynamics have been used in recent times to approximate the behavior of various physical and natural processes occurring around us. The subject has found applications not only in various branches of sciences and engineering but has also been used to study topological dynamics and ergodic theory. The simpler structure and easier visualization of the topic makes it attractive and has gained the attention of several researchers across the globe. Consequently, qualitative analysis in the subject has gained attention and many interesting results have been obtained. Such studies have not only enriched the literature but has also provided greater insight into the theory of ergodic theory and dynamical systems. While Jacques Hadamard applied the theory of symbolic dynamics to study the geodesic flows on surfaces of negative curvature [4], Claude Shennon used symbolic dynamics to develop the mathematical theory of communication systems [2]. In recent times, the topic has found applications in areas like data storage, data transmission, control networks and modelling of gene networks[3, 7, 11]. In recent times, multidimensional symbolic dynamics

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has been a topic of interest and has attracted the attention of several researchers across the globe. As many of the systems cannot be modelled using one dimensional symbolic systems, multidimensional symbolic systems can been used to model dynamically complex systems and investigate dynamical complexities of a system that cannot be investigated using one dimensional systems.

In one of the early works, Berger investigated multidimensional subshifts of finite type over a finite number of symbols. He proved that for a multidimensional subshift of finite type, it is algorithmically undecidable whether an allowed partial configuration can be extended to a point in the multidimensional shift space [9]. In [10], the author provides an example to prove that a multidimensional shift of finite type need not contain any periodic points. The investigations motivated further research in this area and many interesting results have been obtained [1, 5, 6]. In this paper, we address the problem of characterizing the elements of a multidimensional shift of finite type generated by a given set of forbidden patterns. In particular, given a multidimensional shift space characterized by a finite set of forbidden blocks, we provide an algorithm characterizing the elements of the shift space X using a sequence of finite matrices (of increasing order). The algorithm generates arbitrarily large cubes for the shift space X and hence generates an element valid for X (as a limit of arbitrarily large allowed cubes). Further, as any element of X can be visualized as a limit of arbitrarily large finite cubes, the algorithm generates all possible elements of the shift space using the matrices generated and hence determines the shift space completely. Before we move further, we first provide some of the basic definitions and notations used.

Let $A = \{a_i : i \in I\}$ be a finite set and let d be a positive integer. Let the set A be equipped with the discrete metric and let $A^{\mathbb{Z}^d}$, the collection of all functions $c : \mathbb{Z}^d \rightarrow A$ be equipped with the product topology. Any such function c is called a configuration over A. It may be noted that the product topology on $A^{\mathbb{Z}^d}$ is metrizable and is generated by the metric $\mathcal{D}(x, y) = \frac{1}{n+1}$, where n is the least non-negative integer such that $x \neq y$ in $R_n = [-n, n]^d$. For any $a \in \mathbb{Z}^d$, the map $\sigma_a : A^{\mathbb{Z}^d} \rightarrow A^{\mathbb{Z}^d}$ defined as $(\sigma_a(x))(k) = x(k + a)$ is a d-dimensional homeomorphism of $A^{\mathbb{Z}^d}$. A set $X \subseteq A^{\mathbb{Z}^d}$ is σ_a-invariant if $\sigma_a(X) \subseteq X$. Any set $X \subseteq A^{\mathbb{Z}^d}$ is shift-invariant if it is invariant under σ_a for all $a \in \mathbb{Z}^d$. A non-empty, closed shift invariant subset of $A^{\mathbb{Z}^d}$ is called a shift space. For any nonempty $S \subset \mathbb{Z}^d$, an element in A^S is called a pattern over S. A pattern is said to be finite if it is defined over a finite subset of $\mathbb{Z}^d$.

A pattern q over S is said to be an extension of the pattern p over T if $T \subset S$ and $q|_T = p$. The extension q is said to be proper extension if $T \cap Bd(S) = \phi$, where $Bd(S)$ denotes the boundary of S. Let $\mathcal{F}$ be a given set of finite patterns (possibly over different subsets of $\mathbb{Z}^d$) and let $X = \overline{\{x \in A^{\mathbb{Z}^d} : \text{any pattern from } \mathcal{F} \text{ does not appear in } x\}}$. Then, X defines a subshift of finite type generated by set of forbidden patterns $\mathcal{F}$. If the set $\mathcal{F}$ is a finite set of finite patterns, we say that the shift space X is a shift of finite type. We say that a pattern is allowed for X if it is not an extension of any pattern forbidden for X. We denote the shift space generated by the set of forbidden patterns $\mathcal{F}$ by $X_{\mathcal{F}}$. Two forbidden sets $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be equivalent if they generate the same shift space, i.e. $X_{\mathcal{F}_1} = X_{\mathcal{F}_2}$. A forbidden set $\mathcal{F}$ of patterns is called minimal for the shift space X if $\mathcal{F}$ is the set with least cardinality such that $X = X_{\mathcal{F}}$. It is worth mentioning that a shift space X is of finite type if and only if its minimal forbidden set is a finite set of finite patterns.

For any two distinct k-tuples $(a_1, a_2, \dots, a_k), (b_1, b_2, \dots, b_k)$ over $\mathbb{R}$, we say that $(a_1, a_2, \dots, a_k) < (b_1, b_2, \dots, b_k)$ if $a_r < b_r$ where $r = \min\{i : a_i \neq b_i\}$. The relation defines a total order on $\mathbb{R}^k$ and is known as the dictionary order on $\mathbb{R}^k$. Let $O_{\mathcal{H}}$ $(O_{\mathcal{V}})$ be the restriction of the dictionary order on set of 2×2 matrices, when any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is represented as (a, b, c, d) $((a, c, b, d)$ respectively). Analogously, let $O_{\mathcal{H}}$ and $O_{\mathcal{V}}$ be the orders defined on the set of $r \times s$ matrices obtained by restricting dictionary order, when any matrix is represented as a tuple by reading entries row-wise (left to right) and column-wise (top to bottom) respectively.

Let M be a $k^2 \times k^2$ matrix of the form

$$M = \begin{pmatrix} M_{11} & M_{12} & \dots & M_{1k} \\ M_{21} & M_{22} & \dots & M_{2k} \\ \vdots & \vdots & & \vdots \\ M_{k1} & M_{k2} & \dots & M_{kk} \end{pmatrix}$$

where each M_{ij} is a $k \times k$ matrix. For each pair of matrices (M_{ij}, M), assign a $1 \times k^4$ matrix $(M_{ij} \bigotimes M)$ as

$$(M_{ij} \bigotimes M)_{1r} = (M_{ij})_{\alpha\beta}(M_{\alpha\beta})_{\gamma\delta}$$

where $\alpha, \beta, \gamma, \delta \in \{1, 2, \dots, k\}$ and $(\alpha, \beta, \gamma, \delta)$ is the unique solution to the equation $r = \alpha k^3 + \beta k^2 + \gamma k + \delta - (k^3 + k^2 + k)$.

It may be noted that if the $1 \times k^4$ row vector is visualised as k^2 groups of k^2 elements, then the first k^2 entries of the resultant $(M_{ij} \bigotimes M)$ are obtained by multiplying $(M_{ij})_{11}$ with k^2 entries of M_{11} (entries read rowwise), next k^2 entries are obtained by multiplying $(M_{ij})_{12}$ with k^2 entries of M_{12} and so on. In general, for $n \in \{1, 2, \ldots, k^2\}$, the n-th group is determined by multiplying $(M_{ij})_{wz}$ by k^2 entries of M_{wz} where $(w, z) \in \{1, 2, \ldots, k\}^2$ is unique solution to the equation $n = (w-1)k + z$. Consequently, the operation $\bigotimes$ is well defined and assigns a k^4 row vector for any input pair (M_{ij}, M). It may be noted that although the operation is defined for the pair (M_{ij}, M) where M_{ij} is submatrix of M, the operation is well defined for any pair (P, Q) where P and Q are square matrices of order k and k^2 respectively.

Let X be a d dimensional shift space and let p be a pattern (over some subset S of $\mathbb{Z}^d$). Let l_p^i be the length of the pattern p in the i-th direction. Let $l_p = \max\{l_P^i : i = 1, 2, \ldots, d\}$ denote the length of the pattern p. Throughout this paper, any d-dimensional pattern defined over $\prod\limits_{i=1}^{d} I_i$ where $I_i = [r_i, r_i + n - 1]$ ($r_i \in \mathbb{Z}$ for $i = 1, 2, \ldots d$) is referred as an d-dimensional cube of size n.

2. Main Results

In this section, we investigate the problem of characterizing the elements of a finite dimensional shift space using a sequence of finite matrices. Note that if X is a shift of finite type, X is determined by a finite set of finite patterns forbidden for the shift space X. Further, extending these forbidden patterns to smallest possible squares yields an equivalent forbidden set of squares (possibly of different sizes) for the shift space X. As extension of these squares to squares of same size (where the common size is the maximum of the sizes across different squares) once again yields an equivalent forbidden set for X, every shift of finite type can be characterized by a finite set of forbidden cubes (of same size)[8]. Further, for the shift characterized by squares of same size, one may find the compatibility of the allowed squares in each of the d directions. Further, as any of such matrices generates infinite allowed strips in one of the directions, verifying their compatibility in the subsequent directions yields a point in the shift space X. Note that compatibility of the strips in the subsequent directions cannot be captured using a matrix of finite dimension. However, as any point in the shift space can be viewed as an extension of infinite strips, the shift space can be characterized using a matrix of infinite dimension[8]. For the sake of completion, we now state the results discussed below.

Proposition 1. [8] *X is a d-dimensional shift of finite type $\implies$ there exists a set $\mathcal{C}$ of d-dimensional cubes such that $X = X_{\mathcal{C}}$.*

Remark 1. The above result establishes a given shift of finite type to be necessarily generated by a forbidden set of squares (all of same size). Note that as the size of the squares is the maximum of lengths of the forbidden patterns (across different possible Euclidean directions), many of the squares included in the new forbidden set are proper extensions of the originally forbidden patterns. However, as dropping these proper extensions yields an equivalent forbidden set of reduced cardinality, the complexity of the system can be reduced by considering only those squares which are extensions but not proper extensions of the originally given forbidden patterns. It is worth mentioning that although such a consideration reduces the cardinality of the forbidden set, the set obtained is not minimal (and has cardinality more that the cardinality of the originally given forbidden set).

Proposition 2. [8] *If X is a d-dimensional shift of finite type, then the elements of X can be determined by an infinite square matrix.*

Remark 2. The result characterizes the elements of the shift space X using an infinite square matrix. However, the algorithm requires to compute compatibility of two infinite strips which in general can be a tedious task. It is worth mentioning that a similar approach can be used to verify compatibility of finite squares which can be used to generate larger blocks allowed for the shift space X. Consequently, repeating the process inductively generates arbitrarily large allowed blocks (for X) which generates a point in the shift space X (in the limiting case). Further, if all the forbidden patterns are squares of same size, the structure can be better utilized to generate a general element of the shift space X. Hence the shift space can be computed precisely from the modified set of forbidden patterns. We now establish our claims below.

Theorem 1. *For every 2-dimensional shift of finite type, there exists a sequence of finite matrices characterizing the elements of X.*

Proof. Let X be the shift space generated by forbidding given set of patterns. In this proof, we provide an algorithm for generating finite matrices (of increasing order) characterizing the existence of arbitrary large squares (rectangles) allowed for the shift space X. The sequence of matrices in turn characterizes the elements of the shift space X and thus addresses the non-emptiness problem for the shift space generated by a given set of forbidden blocks. We now describe the proposed algorithm below:

Step 1: Computation of Generating Matrices

Let X be a 2-dimensional shift of finite type and let $\mathcal{P}$ be the finite set of forbidden patterns characterizing the shift space X. By Lemma 1, there exists a set of forbidden squares $\mathcal{S}$ (all of same size, say l) generating the shift space X. Let $B_1, B_2, \ldots, B_k$ be the collection of all squares of size l. Let V^0 be the $k \times k$ matrix (indexed by $B_1, B_2, \ldots, B_k$) indicating the vertical compatibility of the squares B_i, as described below.

For notational convenience, let the index B_i be denoted by i, $\forall i = 1, 2, \ldots, k$. Then,

$$V^0_{ij} = \begin{cases} 1, & \begin{pmatrix} B_i \\ B_j \end{pmatrix} \text{ is allowed in } X \\ 0 & \text{otherwise} \end{cases}$$

Let H^0 be a matrix indexed by the set of $2l \times l$ rectangles of indicating their horizontal compatibility. As any $2l \times l$ rectangle is of the form $\begin{pmatrix} B_i \\ B_j \end{pmatrix}$, H^0 is a $k^2 \times k^2$ matrix indexed by rectangles of the form $\begin{pmatrix} B_i \\ B_j \end{pmatrix}$. For notational convenience, let the index $\begin{pmatrix} B_i \\ B_j \end{pmatrix}$ be denoted by (ij), $\forall i, j \in \{1, 2, \ldots, k\}$. Then,

$$H^0_{(ij)(rs)} = \begin{cases} 1, & \begin{pmatrix} B_i & B_r \\ B_j & B_s \end{pmatrix} \text{ is allowed in } X \\ 0 & \text{otherwise} \end{cases}$$

It may be noted that while entries of the matrix V^0 indicate the vertical compatibility of squares of length l, entries of the matrix H^0 characterize the existence of all $2l \times 2l$ squares. In the next step, we use the matrices computed to verify the validity of all possible $4l \times 2l$ and $4l \times 4l$ rectangles.

Step 2: One Step Extension: Computing V^1 and H^1

In the first step, we computed the matrices V^0 and H^0 characterizing existence of all possible $2l \times l$ and $2l \times 2l$ rectangles respectively. To add clarity to the structure of the matrix H^0, let the index set of H^0 follow the dictionary order, i.e., let the index set of H^0 be ordered as $\{(11), (12) \ldots, (1k), (21), (22), \ldots, (2k), \ldots, (k1), (k2), \ldots, (kk)\}$. Consequently, H^0 can be viewed as a block matrix of the form

$$H^0 = \begin{pmatrix} R_{11}^0 & R_{12}^0 & \dots & R_{1k}^0 \\ R_{21}^0 & R_{22}^0 & \dots & R_{2k}^0 \\ \vdots & \vdots & & \vdots \\ R_{k1}^0 & R_{k2}^0 & \dots & R_{kk}^0 \end{pmatrix}$$

where R_{ij}^0 is a $k \times k$ matrix whose entries characterize the squares of the form $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$. More precisely, (r, s)-th entry of R_{ij}^0 is 1 if and only if $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ is allowed. Equivalently, the entries of the matrices R_{ij}^0 characterize all $2l \times 2l$ squares whose top half (which is a $l \times 2l$ rectangle) is $(B_i \ B_j)$.

Let V^1 be $k^4 \times k^4$ matrix indexed by $2l \times 2l$ squares indicating their vertical compatibility. As any $2l \times 2l$ square is of the form $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$, V^1 is equivalently indexed by the squares of the form $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$. For notational convenience, let the index $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ be denoted by $(ijrs)$ $\forall 1 \le i, j, r, s \le k$. Consequently,

$$V^1_{(ijrs)(uvwz)} = \begin{cases} 1, & \begin{pmatrix} B_i & B_j \\ B_r & B_s \\ B_u & B_v \\ B_w & B_z \end{pmatrix} \text{ is allowed in } X; \\ 0 & \text{otherwise .} \end{cases}$$

For computational purposes, let the index set of V^1 be ordered using order $O_{\mathcal{H}}$. In particular, the index set of V^1 follows the following order:

$$\begin{array}{l}
(1111), (1112), \dots, (111k), (1121), \dots, (112k), \dots, (11k1), \dots, (11kk) \\
(1211), (1212), \dots, (121k), (1221), \dots, (122k), \dots, (12k1), \dots, (12kk) \\
\vdots \\
(1k11), (1k12), \dots, (1k1k), (1k21), \dots, (1k2k), \dots, (1kk1), \dots, (1kkk) \\
(2111), (2112), \dots, (211k), (2121), \dots, (212k), \dots, (21k1), \dots, (21kk) \\
(2211), (2212), \dots, (221k), (2221), \dots, (222k), \dots, (22k1), \dots, (22kk) \\
\vdots \\
(2k11), (2k12), \dots, (2k1k), (2k21), \dots, (2k2k), \dots, (2kk1), \dots, (2kkk) \\
\vdots
\end{array}$$

$(k111), (k112), \dots, (k11k), (k121), \dots, (k12k), \dots, (k1k1), \dots, (k1kk)$
$(k211), (k212), \dots, (k21k), (k221), \dots, (k22k), \dots, (k2k1), \dots, (k2kk)$
$\vdots$
$(kk11), (kk12), \dots, (kk1k), (kk21), \dots, (kk2k), \dots, (kkk1), \dots, (kkkk).$

It may be noted that for any index $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ for the matrix V^1, the corresponding row can be determined as follows:

(1) If $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ is forbidden, then the corresponding row is all zeros.

(2) If $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ is allowed then the corresponding row is $R^0_{rs} \bigotimes H^0$.

Finally let the index set of V^1 be ordered using the order $O_{\mathcal{V}}$.

Note that if $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ is not allowed then it cannot be vertically aligned with another square to obtain a allowed pattern and hence the corresponding row is the zero row. Further, as the size of the squares B_k is determined by maximum possible length or breadth among patterns from $\mathcal{S}$, any pattern from $\mathcal{S}$ cannot be spread beyond a $2l \times 2l$ square. Thus, validity of any given pattern can be verified by examining the validity of all the $2l \times 2l$ squares in the given pattern. Consequently, $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ is vertically compatible with $\begin{pmatrix} B_u & B_v \\ B_w & B_z \end{pmatrix}$ if and only if $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$, $\begin{pmatrix} B_r & B_s \\ B_u & B_v \end{pmatrix}$ and $\begin{pmatrix} B_u & B_v \\ B_w & B_z \end{pmatrix}$ are allowed. As $(R^0_{rs})_{uv}$ and $(R^0_{uv})_{wz}$ are 1 if and only if $\begin{pmatrix} B_r & B_s \\ B_u & B_v \end{pmatrix}$ and $\begin{pmatrix} B_u & B_v \\ B_w & B_z \end{pmatrix}$ are allowed, their product characterizes the validity of the block $\begin{pmatrix} B_i & B_j \\ B_r & B_s \\ B_u & B_v \\ B_w & B_z \end{pmatrix}$ under the validity of $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ and hence $(R^0_{rs}) \bigotimes H^0$ is the row corresponding to $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ (under the validity of $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$). Thus, the constructed matrix indeed characterizes the vertical compatibility of the $2l \times 2l$ squares (and hence characterizes all $4l \times 2l$ rectangles allowed for the shift space X).

Let us investigate the structure of V^1 in detail. Note that the index set of V^1 (now ordered using the order $O_{\mathcal{V}}$) can be viewed as the block matrix of the form

$$V^1 = \begin{pmatrix} S^0_{11} & S^0_{12} & \dots & S^0_{1k^2} \\ S^0_{21} & S^0_{22} & \dots & S^0_{2k^2} \\ \vdots & \vdots & & \vdots \\ S^0_{k^2 1} & S^0_{k^2 2} & \dots & S^0_{k^2 k^2} \end{pmatrix}$$

where each S^0_{ij} is a $k^2 \times k^2$ matrix. Note that for any $i \in \{1, 2, \dots, k^2\}$, there exists unique pair (p_i, q_i), $(p_i \in \{0, 2, \dots, k-1\},\ q_i \in \{1, 2, \dots, k\})$ such that $i = kp_i + q_i$. Identifying i with $(p_i + 1\ \ q_i)$ $\forall i = 1, 2, \dots, k^2$, the matrix V^1 can be represented as

$$V^1 = \begin{pmatrix}
S^0_{(11)(11)} & S^0_{(11)(12)} & \cdots & S^0_{(11)(1k)} & S^0_{(11)(21)} & S^0_{(11)(22)} & \cdots & S^0_{(11)(2k)} & \cdots & S^0_{(11)(k1)} & S^0_{(11)(k2)} & \cdots & S^0_{(11)(kk)} \\
S^0_{(12)(11)} & S^0_{(12)(12)} & \cdots & S^0_{(12)(1k)} & S^0_{(12)(21)} & S^0_{(12)(22)} & \cdots & S^0_{(12)(2k)} & \cdots & S^0_{(12)(k1)} & S^0_{(12)(k2)} & \cdots & S^0_{(12)(kk)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
S^0_{(1k)(11)} & S^0_{(1k)(12)} & \cdots & S^0_{(1k)(1k)} & S^0_{(1k)(21)} & S^0_{(1k)(22)} & \cdots & S^0_{(1k)(2k)} & \cdots & S^0_{(1k)(k1)} & S^0_{(1k)(k2)} & \cdots & S^0_{(1k)(kk)} \\
S^0_{(21)(11)} & S^0_{(21)(12)} & \cdots & S^0_{(21)(1k)} & S^0_{(21)(21)} & S^0_{(21)(22)} & \cdots & S^0_{(21)(2k)} & \cdots & S^0_{(21)(k1)} & S^0_{(21)(k2)} & \cdots & S^0_{(21)(kk)} \\
S^0_{(22)(11)} & S^0_{(22)(12)} & \cdots & S^0_{(22)(1k)} & S^0_{(22)(21)} & S^0_{(22)(22)} & \cdots & S^0_{(22)(2k)} & \cdots & S^0_{(22)(k1)} & S^0_{(22)(k2)} & \cdots & S^0_{(22)(kk)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
S^0_{(2k)(11)} & S^0_{(2k)(12)} & \cdots & S^0_{(2k)(1k)} & S^0_{(2k)(21)} & S^0_{(2k)(22)} & \cdots & S^0_{(2k)(2k)} & \cdots & S^0_{(2k)(k1)} & S^0_{(2k)(k2)} & \cdots & S^0_{(2k)(kk)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
S^0_{(k1)(11)} & S^0_{(k1)(12)} & \cdots & S^0_{(k1)(1k)} & S^0_{(k1)(21)} & S^0_{(k1)(22)} & \cdots & S^0_{(k1)(2k)} & \cdots & S^0_{(k1)(k1)} & S^0_{(k1)(k2)} & \cdots & S^0_{(k1)(kk)} \\
S^0_{(k2)(11)} & S^0_{(k2)(12)} & \cdots & S^0_{(k2)(1k)} & S^0_{(k2)(21)} & S^0_{(k2)(22)} & \cdots & S^0_{(k2)(2k)} & \cdots & S^0_{(k2)(k1)} & S^0_{(k2)(k2)} & \cdots & S^0_{(k2)(kk)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
S^0_{(kk)(11)} & S^0_{(kk)(12)} & \cdots & S^0_{(kk)(1k)} & S^0_{(kk)(21)} & S^0_{(kk)(22)} & \cdots & S^0_{(kk)(2k)} & \cdots & S^0_{(kk)(k1)} & S^0_{(kk)(k2)} & \cdots & S^0_{(kk)(kk)}
\end{pmatrix}$$

To understand the generated submatrices better, let the k^4 rows (columns) be divided into k^2 groups of k^2 rows (columns) each. Note that if $i = kp_i + q_i$ $(i \in \{1, 2, \dots, k^2\})$, then the rows (columns) of the i-th group are indexed by squares of the form $\begin{pmatrix} B_{p_i+1} & * \\ B_{q_i} & * \end{pmatrix}$. As S^0_{ij} verifies the vertical compatibility of i-th group with the j-th group, entries of any $S^0_{(pq)(rs)}$ characterize all $4l \times 2l$ rectangles whose left half (which is a $4l \times l$ rectangle) is $\begin{pmatrix} B_p \\ B_q \\ B_r \\ B_s \end{pmatrix}$. It is worth mentioning that for $i = kp_i + q_i,\ j = kp_j + q_j,\ S^0_{ij}$ is same as $S^0_{(p_i+1\ q_i)(p_j+1\ q_j)}$ and the expression above is another way of representing the same matrix.

Let H^1 be the matrix indexed by the set of $4l \times 2l$ rectangles indicating horizontal compatibility of the indices. As any $4l \times 2l$ rectangle is of the form $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$, H^1 is a $k^8 \times k^8$ matrix equivalently indexed by rectangles of the form $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$. For notational convenience, let the index $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$ be denoted by $(ijrsuvwz)$.

For computational purposes, let the index set of H^1 be ordered using order $O_{\mathcal{V}}$. It may be noted that for any index $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$ for the matrix H^1, the corresponding row can be determined as follows:

(1) If $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$ is forbidden, then the corresponding row is all zeros.

(2) If $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$ is allowed then the corresponding row is $S^0_{(uv)(wz)} \bigotimes V^1$.

Finally let the index set of H^1 be ordered using the order $O_{\mathcal{H}}$.

Note that if a $4l \times 2l$ block is not allowed then it cannot be horizontally aligned with another $4l \times 2l$ block to obtain an allowed pattern and hence the corresponding row is the zero row. For any given pattern $\mathcal{P}$, as establishing the validity of all $2l \times 2l$ squares (or rectangles or squares which are extension of these) embedded in $\mathcal{P}$ is sufficient to establish the validity of $\mathcal{P}$ for X, $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$ is horizontally compatible with $\begin{pmatrix} B_{i'} & B_{u'} \\ B_{j'} & B_{v'} \\ B_{r'} & B_{w'} \\ B_{s'} & B_{z'} \end{pmatrix}$ if and only if $\begin{pmatrix} B_u & B_{i'} \\ B_v & B_{j'} \\ B_w & B_{r'} \\ B_z & B_{s'} \end{pmatrix}$ and $\begin{pmatrix} B_{i'} & B_{u'} \\ B_{j'} & B_{v'} \\ B_{r'} & B_{w'} \\ B_{s'} & B_{z'} \end{pmatrix}$ are allowed in X (under allowedness of $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$). Finally, as entries of any submatrix $S^0_{(i'j')(r's')}$ characterizes all allowed $4l \times 2l$ rectangles with fixed half $\begin{pmatrix} B_{i'} \\ B_{j'} \\ B_{r'} \\ B_{s'} \end{pmatrix}$, $S^0_{(uv)(wz)} \bigotimes V^1$ indeed is the row (of H^1) corresponding to $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$ and the computation of H^1 is complete.

Step 3: Computation of V^{n+1} and H^{n+1}

Let us assume that the matrices $V^0, H^0, V^1, H^1, \dots, V^n, H^n$ have been computed. Then, the matrix H^n can be visualized as

$$H^n = \begin{pmatrix} R^n_{11} & R^n_{12} & \dots & R^n_{1k^{4^n}} \\ R^n_{21} & R^n_{22} & \dots & R^n_{2k^{4^n}} \\ \vdots & \vdots & & \vdots \\ R^n_{k^{4^n}1} & R^n_{k^{4^n}2} & \dots & R^n_{k^{4^n}k^{4^n}} \end{pmatrix}$$

where each R^n_{ij} is a $k^{4^n} \times k^{4^n}$ matrix. As H^n establishes the horizontal compatibility of $2^{n+1}l \times 2^n l$ rectangles, entries of R^n_{ij} establishes the validity of $2^{n+1}l \times 2^{n+1}l$ squares (which are indices of V^{n+1}) with a fixed top half. Consequently, $R_{ij} \bigotimes H^n$ determines the rows of V^{n+1} (under allowedness of the index), when the index set of V^{n+1} is ordered using order $O_{\mathcal{H}}$. Further, note that the matrix V^{n+1} (ordered using order $O_{\mathcal{V}}$ after computation) can be visualized as:

$$V^{n+1} = \begin{pmatrix} S^{n+1}_{11} & S^{n+1}_{12} & \dots & S^{n+1}_{1k^{2.4^n}} \\ S^{n+1}_{21} & S^{n+1}_{22} & \dots & S^{n+1}_{2k^{2.4^n}} \\ \vdots & \vdots & & \vdots \\ S^{n+1}_{k^{2.4^n}1} & S^{n+1}_{k^{2.4^n}2} & \dots & S^{n+1}_{k^{2.4^n}k^{2.4^n}} \end{pmatrix}$$

where each S^{n+1}_{ij} is a $k^{2\cdot4^n} \times k^{2\cdot4^n}$ matrix. As V^{n+1} establishes the vertical compatibility of $2^{n+1}l$ squares, entries of S^{n+1}_{ij} establishes the validity of all allowed $2^{n+2}l \times 2^{n+1}l$ rectangles (which is an index of H^n) with a fixed left half. Consequently, $S^{n+1}_{ij} \bigotimes V^{n+1}$ determines the rows of H^{n+1} (when the index set of H^{n+1} is ordered using order $O_{\mathcal{V}}$) and thus computation of V^{n+1} and H^{n+1} is complete.

Finally, as entries of V^{n+1} (H^{n+1}) characterize all allowed $2^{n+2}l \times 2^{n+1}l$ ($2^{n+2}l \times 2^{n+2}l$ rectangles (squares) respectively), the shift space X is non-empty if and only if each V^i and H^i are non-zero for all $i \in \mathbb{N}$. Further, as any element of X can be viewed as a limit of finite blocks arising from V^i(or H^i), elements of the shift space are precisely the limits of the blocks arising from V^i(or H^i). □

Remark 3. The above result provides an iterative procedure to generate arbitrarily large blocks for a shift of finite type. Note that the algorithm verifies validity of $2^{n+1}l \times 2^n l$ rectangles and $2^{n+1}l \times 2^{n+1}l$ squares (which are indices of H^{n+1} and V^{n+1} respectively) using the matrices generated in the previous iteration. It is worth mentioning that although the algorithm generates larger valid rectangles using all squares generated in the previous iteration, the matrices V^n can be indexed by a smaller collection of all allowed squares to generate the same set and hence the order of the matrices V^n can be reduced. Further, as any compatible vertical alignment of indices of V^n is an index for H^n, the number of 1's in V^n characterizes the order of H^n (while working with matrices of reduced order). Note that while the matrices V^n are computed with the indices being ordered using $O_{\mathcal{H}}$, the indices are re-ordered after computation using $O_{\mathcal{V}}$. Such an arrangement is useful as such an ordering of the index set helps represent the generated matrix in the form of a block matrix where each block characterizes rectangles of larger order with a fixed left half which is useful to compute the rows of H^n (via the product $S^n_{ij} \bigotimes V^n$). A similar set of observations ensures indexing and simplified computation of H^n by smaller collection for generation of squares valid for the shift space X and hence reduces the order of the matrices H^n generated. Note that while the ability to construct arbitrarily large blocks for the space X guarantees non-emptiness of the shift space X, any element of the shift space is a limit of arbitrarily large blocks allowed for X.

Consequently, the shift space X is non-empty if and only if the matrices V^i and H^i are non-zero at each iteration.

Example 1. Let Σ_2 be the two dimensional full shift over two symbols $\{0, 1\}$ and let X be the shift of finite type generated by the forbidden set $\mathcal{F}_1 = \left\{ \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}, \ \begin{smallmatrix} 1 & 1 \end{smallmatrix} \right\}$. As already observed, X is also generated by the forbidden set

$$\mathcal{F}_2 = \left\{ \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right\},$$

and any element of X is a sequential (two dimensional) arrangement of blocks in $\mathcal{A}_2$ where

$$\mathcal{A}_2 = \left\{ \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right\}.$$

Constructing the matrices $V^i(H^i)$ (of reduced order) for the shift space Σ_2, note that V^0 is a 7×7 square matrix where the indices are precisely the set $\mathcal{A}_2$ (with same order). Also, as V^0 contains eight zeros, H^0 is a 41×41 square matrix (with indices ordered using the order $O_{\mathcal{H}}$). Further, it may be observed that the order of the matrices V^i (H^i) in subsequent iterations increase exponentially, the collection of finite blocks vertically (horizontally) compatible with each other is infinite. Further, as finite sets vertically (horizontally) compatible with each other are generated at each of the iterations, the shift space possesses infinitely many periodic points with finite orbits. Alternatively, note that as V^0 generates infinite strips of height two, verification of vertical compatibility of any such two strips allows the configuration to be extended over $\mathbb{Z}^2$ and hence generates an element valid for the shift space X. Further, as any element of X can be visualized in such a manner, the elements of the shift space can be characterized using the infinite strips generated by V^0. In particular, any sequential arrangement of $\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}$ and $\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}$ avoids adjacent placement of 1's and hence is an allowed infinite strip for X. Consequently, for any such strip H, sequential vertical arrangement of H generates a periodic point for X (and hence addresses the nonemptiness problem in finite iterations). It may be noted that, as infinite strips (of fixed height) can be generated (from V^i (H^i)) at any iteration, points with infinite orbits can be generated (by verifying the vertical compatibility of the strips generated). Consequently, the nonemptiness problem for the generated shift space can be addressed in finite iterations (without using the periodic points available) in this case.

Remark 4. The above example discusses the structure of matrices $V^i(H^i)$ generated at each of the iterations. It is worth mentioning that as vertical (horizontal) compatibility of finite blocks can be verified at each iteration, the information can be used to generate periodic points for the shift space (if any). Further, if the order of the matrices V^i (H^i) is non-decreasing (at each iteration), patterns compatible in vertical and horizontal directions are generated at each of the iterations which in turn generate a point of X in the limiting case (which need not be periodic). Further, for a d-dimensional shift space, a similar algorithm yields a sequence of matrices $M_1^1, M_1^2, \dots M_1^d, M_2^1, M_2^2, \dots M_2^d \dots, M_n^1, M_n^2, \dots M_n^d \dots$ (where M_n^i characterises the possible extensions in the i-th direction at n-th iteration) such that the d-dimensional shift space is non-empty if and only if M_n^i ($i = 1, 2, \dots, d$) are non-zero at each iteration. Thus we get the following corollary.

Corollary 1. *For every d-dimensional shift of finite type, there exists a sequence of finite matrices characterizing the elements of X.*

Proof. The proof is a natural extension of the 2 dimensional case and follows directly by first extending the allowed d-dimensional cubes in each of the d directions and iteratively generating cubes (cuboids) of arbitrarily large size. For the sake of clarity, we provide an outline of the proof below.

Computing Generating Matrices

Let X be a d-dimensional shift space generated by a finite set of forbidden patterns. By Lemma 1, there exists a set of forbidden cubes $\mathcal{S}$ (all of same size, say l) generating the shift space X. Let $e_1, e_2, \dots, e_d$ be the set of d mutually orthogonal directions (along the directions of standard basis vectors of $\mathbb{R}^d$) and let $B_1, B_2, \dots, B_k$ be the the collection of all allowed cubes of size l. Let M_1^1 be the $k \times k$ matrix (indexed by $B_1, B_2, \dots, B_k$) indicating compatibility of the cubes B_i in e_1 direction. Consequently, entries of M_1^1 characterizes the validity of all $2l \times \underbrace{l \times l \times \dots \times l}_{d-1 \ times}$ blocks for the space X. Similarly, if M_1^2 be the matrix indexed by all allowed $2l \times \underbrace{l \times l \times \dots \times l}_{d-1 \ times}$ blocks indicating the compatibility of the indices in e_2 direction then, entries of M_1^2 characterizes the validity of all $2l \times 2l \times \underbrace{l \times l \times \dots \times l}_{d-2 \ times}$ blocks for the space X. Inductively, for $i = 3, 4, \dots, d$, if M_1^i is indexed by all allowed $\underbrace{2l \times 2l \times \dots \times 2l}_{i-1 \ times} \times \underbrace{l \times l \times \dots \times l}_{d-i+1 \ times}$ blocks indicating the compatibility of the indices in i-th direction, entries of M_1^i characterize the validity of all $\underbrace{2l \times 2l \times \dots \times 2l}_{i \ times} \times \underbrace{l \times l \times \dots \times l}_{d-i \ times}$ blocks for the space X. In particular, M_1^d is a matrix indexed by all allowed $\underbrace{2l \times \dots 2l}_{d-1 \ times} \times l$

blocks indicating the compatibility of the indices in the direction e^d and hence characterizes all allowed cubes of size $2l$.

Application of Induction and Generating Cubes of Arbitrarily Large Size

Let us assume that $M_1^1, M_1^2, \ldots M_1^d, M_2^1, M_2^2, \ldots M_2^d \ldots, M_n^1, M_n^2, \ldots M_n^d$ have been computed. Note that as M_n^i is indexed by all allowed $\underbrace{2^n l \times 2^n l \times \ldots \times 2^n l}_{i-1\ times} \times \underbrace{2^{n-1} l \times 2^{n-1} l \times \ldots \times 2^{n-1} l}_{d-i+1\ times}$ blocks, entries of M_n^i characterize all allowed $\underbrace{2^n l \times 2^n l \times \ldots \times 2^n l}_{i\ times} \times \underbrace{2^{n-1} l \times 2^{n-1} l \times \ldots \times 2^{n-1} l}_{d-i\ times}$ blocks for the space X.

Further, in order for the proof of Theorem 1 to be extendable to the d-dimensional case, while the indices of M_n^i need to be ordered using $(i-1)$-th direction (mod d) while computing M_n^i, they need to be ordered using $(i+1)$-th direction (mod d) after computation, to facilitate the computation of the next matrix. Finally, note that as the sequence M_n^i generates arbitrarily large cubes (and cuboids), the process yields an element of the shift if each M_n^i is non-empty. Further, as any element of sequence can be viewed as a limit of arbitrarily large cubes (or cuboids), the elements of the shift space are characterized by the sequence generated. □

Remark 5. The above corollary extends the algorithm given in Theorem 1 for generating elements of d-dimensional shift space. The algorithm generates cubes (cuboids) of arbitrarily large size by iteratively extending the allowed blocks in each of the d directions. Note that as compatibility of two blocks in one of the directions cannot indicate its extension in any other direction, the matrices $M_1^1, M_1^2, \ldots, M_1^d$ are independent of each other (cannot be determined from each other). Further, as the order of the matrices $B_1, B_2, \ldots, B_k$ is the maximum possible length (in any of the directions) of the generating set of forbidden patterns $\mathcal{P}$, to examine the validity of any given pattern in $\mathbb{R}^d$, it is sufficient to examine the validity of all cubes of size $2l$ (or larger cubes/cuboids). Finally, as any element of X can be visualized as a limit of finite cubes (cuboids) and generation of arbitrarily large cubes (cuboids) yields an element of X, the sequence generated characterizes the elements of X. To realize the convergence mathematically, one may visualize the allowed cube as an element which is obtained by placing the cube (center of the cube) at the origin and assigning 0 at all the other places. Consequently, a sequence of arbitrarily large blocks yields a cauchy sequence in the full shift without forbidden blocks in the central cube (which is of arbitrarily large size) and hence converges to an element of X.

Example 2. Let $n \geq 2$ and $A = \{0, 1, 2, \cdots, n-1\}$ denote the set of n distinct colours. Let X be the collection of all three dimensional configurations in which adjacent lattice points have different colours. It may be noted that for $n = 2$, each M_i^j is a 2×2 identity matrix. Consequently, each index of M_i^j is a complementary set which generates a point with finite orbit and hence the shift space contains precisely two points in this case. For the case $n \geq 3$, as any three dimensional configuration can be visualized as a sequential arrangement of two dimensional configurations, the orders of matrices M_i^j increase exponentially (generating a nonperiodic point in the limiting case). However, as the matrices generated contain 1 on the diagonal, periodic points with finite orbits are generated in finite iterations. Further, as the sequence of finite matrices generates a complementary set in the limiting case, all the points with infinite orbits are generated as limits of sequences spread across elements generated by indices of (M_i^j). Such an observation helps characterizing the periodic points for the shift space X and can be used to generate any general point for the shift space X.

Remark 6. The above example discusses generation of arbitrarily large blocks in a three dimensional shift space. It may be observed that to generate a point in the shift space, one need not use the entire set of matrices generated (as one can observe the structure of the matrices generated at each step and appropriate sub-matrices can be used to generate a point in the shift space). While periodic points with finite orbits are guaranteed to be generated in finite iterations, analysis of vertical (horizontal) compatibility can be used to generate points in finite steps (which need not be periodic). Consequently, the structure of the matrices can be used to generate the points valid for the shift space X.

3. Conclusion and Some Open Questions

In this work, we have provided an algorithmic approach to generate the elements of a multi-dimensional shift space. In the process, we give an algorithm to generate the elements of multidimensional shift space using a sequence of finite matrices (of increasing order). The algorithm not only addresses the non-emptiness problem of the multi-dimensional shift space but also characterizes the elements of the shift space in question. Consequently, the work provides the characterization of a shift of finite type using a sequence of finite matrices (recall that in the one dimensional case any shift of finite type is characterized by a finite matrix).

Although the work addresses some of the questions for the multi-dimensional shift space, some curious questions arising from this work are yet to be answered. For example, how do the sequence of matrices evolve with increasing time? Under what conditions (regarding structure of original d matrices for the d-dimensional case), the non-emptiness problem for the multidimensional shift space be answered affirmatively? Under what conditions, does a multi-dimensional shift space exhibit periodic points? How can the structure of the original d matrices be used to answer some of the existing questions for the multi-dimensional shift space? The questions raised are quite natural and can be investigated to answer many of the existing questions for the multi-dimensional shift space.

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