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A SURVEY OF CARDINALITY BOUNDS ON HOMOGENEOUS TOPOLOGICAL SPACES

NATHAN CARLSON

ABSTRACT. In this survey we catalogue the many results of the past several decades concerning bounds on the cardinality of a topological space with homogeneous or homogeneous-like properties. These results include van Douwen's Theorem, which states $|X| \leq 2^{\pi w(X)}$ if X is a power homogeneous Hausdorff space [26], and its improvements $|X| \leq d(X)^{\pi\chi(X)}$ [44] and $|X| \leq 2^{c(X)\pi\chi(X)}$ [19] for spaces X with the same properties. We also discuss de la Vega's Theorem, which states that $|X| \leq 2^{t(X)}$ if X is a homogeneous compactum [25], as well as its recent improvements and generalizations to other settings. This reference document also includes a table of strongest known cardinality bounds on spaces with homogeneous-like properties. The author has chosen to give some proofs if they exhibit typical or fundamental proof techniques. Finally, a few new results are given, notably (1) $|X| \leq d(X)^{\pi n\chi(X)}$ if X is homogeneous and Hausdorff, and (2) $|X| \leq \pi\chi(X)^{c(X)q\psi(X)}$ if X is a regular homogeneous space. The invariant $\pi n\chi(X)$, defined in this paper, has the property $\pi n\chi(X) \leq \pi\chi(X)$ and thus (1) improves the bound $d(X)^{\pi\chi(X)}$ for homogeneous Hausdorff spaces. The invariant $q\psi(X)$, defined in [33], has the properties $q\psi(X) \leq \pi\chi(X)$ and $q\psi(X) \leq \psi_c(X)$ if X is Hausdorff, thus (2) improves the bound $2^{c(X)\pi\chi(X)}$ in the regular, homogeneous setting.

1. INTRODUCTION

A topological space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$. Roughly, X is homogeneous if the topology at every point is "identical" to that of every other point. X is *power homogeneous* if there exists a cardinal κ such that X^κ is homogeneous. Many commonly studied spaces are homogeneous (for example, $\mathbb{R}^2$, the unit circle, all connected manifolds in general, and topological groups) and as such are ubiquitous across fields of mathematics. In particular, those homogeneous

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